Local spectral synthesis on Abelian groups

Miklós Laczkovich

Eötvös University, Budapest

4^{th} Workshop on Fourier Analysis
Budapest, Aug. 29, 2013

Let G be a topological Abelian group.

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions :

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions : $p = P(a_1, \ldots, a_n)$, where $P \in \mathbb{C}[x_1, \ldots, x_n]$ and $a_1, \ldots, a_n : G \to \mathbb{C}$ are continuous and additive.

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions : $p = P(a_1, \ldots, a_n)$, where $P \in \mathbb{C}[x_1, \ldots, x_n]$ and $a_1, \ldots, a_n : G \to \mathbb{C}$ are continuous and additive.
$m \in C(G)$ is an exponential if $m \neq 0$ and m is multiplicative:

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions : $p = P(a_1, \ldots, a_n)$, where $P \in \mathbb{C}[x_1, \ldots, x_n]$ and $a_1, \ldots, a_n : G \to \mathbb{C}$ are continuous and additive. $m \in C(G)$ is an exponential if $m \neq 0$ and m is multiplicative:

$m(x + y) = m(x) \cdot m(y)$.

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions : $p = P(a_1, \ldots, a_n)$, where $P \in \mathbb{C}[x_1, \ldots, x_n]$ and $a_1, \ldots, a_n : G \to \mathbb{C}$ are continuous and additive. $m \in C(G)$ is an exponential if $m \neq 0$ and m is multiplicative:

$m(x + y) = m(x) \cdot m(y)$.

Theorem

Let G be a top. Abelian group, and let $f \in C(G)$. The translates of f generate a finite dimensional linear space if and only if f is an exponential polynomial:

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions : $p = P(a_1, \ldots, a_n)$, where $P \in \mathbb{C}[x_1, \ldots, x_n]$ and $a_1, \ldots, a_n : G \to \mathbb{C}$ are continuous and additive.
$m \in C(G)$ is an exponential if $m \neq 0$ and m is multiplicative:

$m(x + y) = m(x) \cdot m(y)$.

Theorem

Let G be a top. Abelian group, and let $f \in C(G)$. The translates of f generate a finite dimensional linear space if and only if f is an exponential polynomial:

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions : $p = P(a_1, \ldots, a_n)$, where $P \in \mathbb{C}[x_1, \ldots, x_n]$ and $a_1, \ldots, a_n : G \to \mathbb{C}$ are continuous and additive. $m \in C(G)$ is an exponential if $m \neq 0$ and m is multiplicative:

$m(x + y) = m(x) \cdot m(y)$.

Theorem

Let G be a top. Abelian group, and let $f \in C(G)$. The translates of f generate a finite dimensional linear space if and only if f is an exponential polynomial: $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a polynomial and m_i is an exponential $(i = 1, \ldots, n)$.

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions : $p = P(a_1, \ldots, a_n)$, where $P \in \mathbb{C}[x_1, \ldots, x_n]$ and $a_1, \ldots, a_n : G \to \mathbb{C}$ are continuous and additive. $m \in C(G)$ is an exponential if $m \neq 0$ and m is multiplicative:

$m(x + y) = m(x) \cdot m(y)$.

Theorem

Let G be a top. Abelian group, and let $f \in C(G)$. The translates of f generate a finite dimensional linear space if and only if f is an exponential polynomial: $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a polynomial and m_i is an exponential $(i = 1, \ldots, n)$.

Loewner 1959: $G = \mathbb{R}^n$.

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions : $p = P(a_1, \ldots, a_n)$, where $P \in \mathbb{C}[x_1, \ldots, x_n]$ and $a_1, \ldots, a_n : G \to \mathbb{C}$ are continuous and additive.
$m \in C(G)$ is an exponential if $m \neq 0$ and m is multiplicative:

$m(x + y) = m(x) \cdot m(y)$.

Theorem

Let G be a top. Abelian group, and let $f \in C(G)$. The translates of f generate a finite dimensional linear space if and only if f is an exponential polynomial: $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a polynomial and m_i is an exponential $(i = 1, \ldots, n)$.

Loewner 1959: $G = \mathbb{R}^n$. J.J. Stone 1960: in general.

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions : $p = P(a_1, \dots, a_n)$, where $P \in \mathbb{C}[x_1, \dots, x_n]$ and $a_1, \dots, a_n : G \to \mathbb{C}$ are continuous and additive. $m \in C(G)$ is an exponential if $m \neq 0$ and m is multiplicative:

$m(x + y) = m(x) \cdot m(y)$.

Theorem

Let G be a top. Abelian group, and let $f \in C(G)$. The translates of f generate a finite dimensional linear space if and only if f is an exponential polynomial: $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a polynomial and m_i is an exponential $(i = 1, \dots, n)$.

Loewner 1959: $G = \mathbb{R}^n$. J.J. Stone 1960: in general.

Anselone, Korevar 1964: $G = \mathbb{R}$.

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions : $p = P(a_1, \ldots, a_n)$, where $P \in \mathbb{C}[x_1, \ldots, x_n]$ and $a_1, \ldots, a_n : G \to \mathbb{C}$ are continuous and additive.
$m \in C(G)$ is an exponential if $m \neq 0$ and m is multiplicative:

$m(x + y) = m(x) \cdot m(y)$.

Theorem

Let G be a top. Abelian group, and let $f \in C(G)$. The translates of f generate a finite dimensional linear space if and only if f is an exponential polynomial: $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a polynomial and m_i is an exponential $(i = 1, \ldots, n)$.

Loewner 1959: $G = \mathbb{R}^n$. J.J. Stone 1960: in general.

Anselone, Korevar 1964: $G = \mathbb{R}$.

M. Engert 1970: for loc. compact, σ-compact G.

Let G be a topological Abelian group.

$C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$

$p : G \to \mathbb{C}$ is a polynomial if p belongs to the algebra generated by the continuous additive functions : $p = P(a_1, \ldots, a_n)$, where $P \in \mathbb{C}[x_1, \ldots, x_n]$ and $a_1, \ldots, a_n : G \to \mathbb{C}$ are continuous and additive. $m \in C(G)$ is an exponential if $m \neq 0$ and m is multiplicative:

$m(x + y) = m(x) \cdot m(y)$.

Theorem

Let G be a top. Abelian group, and let $f \in C(G)$. The translates of f generate a finite dimensional linear space if and only if f is an exponential polynomial: $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a polynomial and m_i is an exponential $(i = 1, \ldots, n)$.

Loewner 1959: $G = \mathbb{R}^n$. J.J. Stone 1960: in general.

Anselone, Korevar 1964: $G = \mathbb{R}$.

M. Engert 1970: for loc. compact, σ-compact G.

L. Székelyhidi 1982: in general.

$V \subset C(\mathbb{R})$ is a variety if V is a closed linear subspace of $C(G)$ and is closed under translations.

$V \subset C(\mathbb{R})$ is a variety if V is a closed linear subspace of $C(G)$ and is closed under translations.

Spectral synthesis holds on G if every variety on G is spanned by exp. polynomials.

$V \subset C(\mathbb{R})$ is a variety if V is a closed linear subspace of $C(G)$ and is closed under translations.

Spectral synthesis holds on G if every variety on G is spanned by exp. polynomials.

Theorem (L. Schwartz 1948)

Spectral synthesis holds on $\mathbb{R}$.

$V \subset C(\mathbb{R})$ is a variety if V is a closed linear subspace of $C(G)$ and is closed under translations.

Spectral synthesis holds on G if every variety on G is spanned by exp. polynomials.

Theorem (L. Schwartz 1948)

Spectral synthesis holds on $\mathbb{R}$.

$V \subset C(\mathbb{R})$ is a variety if V is a closed linear subspace of $C(G)$ and is closed under translations.

Spectral synthesis holds on G if every variety on G is spanned by exp. polynomials.

Theorem (L. Schwartz 1948)

Spectral synthesis holds on $\mathbb{R}$.

Theorem (D.I. Gurevič 1975)

Spectral synthesis does not hold on $\mathbb{R}^2$.

$V \subset C(\mathbb{R})$ is a variety if V is a closed linear subspace of $C(G)$ and is closed under translations.

Spectral synthesis holds on G if every variety on G is spanned by exp. polynomials.

Theorem (L. Schwartz 1948)

Spectral synthesis holds on $\mathbb{R}$.

Theorem (D.I. Gurevič 1975)

Spectral synthesis does not hold on $\mathbb{R}^2$.

$V \subset C(\mathbb{R})$ is a variety if V is a closed linear subspace of $C(G)$ and is closed under translations.

Spectral synthesis holds on G if every variety on G is spanned by exp. polynomials.

Theorem (L. Schwartz 1948)

Spectral synthesis holds on $\mathbb{R}$.

Theorem (D.I. Gurevič 1975)

Spectral synthesis does not hold on $\mathbb{R}^2$.

What can we say if G is a discrete Abelian group?

$V \subset C(\mathbb{R})$ is a variety if V is a closed linear subspace of $C(G)$ and is closed under translations.

Spectral synthesis holds on G if every variety on G is spanned by exp. polynomials.

Theorem (L. Schwartz 1948)

Spectral synthesis holds on $\mathbb{R}$.

Theorem (D.I. Gurevič 1975)

Spectral synthesis does not hold on $\mathbb{R}^2$.

What can we say if G is a discrete Abelian group?

M. Lefranc 1958: True for $\mathbb{Z}^n$.

$V \subset C(\mathbb{R})$ is a variety if V is a closed linear subspace of $C(G)$ and is closed under translations.

Spectral synthesis holds on G if every variety on G is spanned by exp. polynomials.

Theorem (L. Schwartz 1948)

Spectral synthesis holds on $\mathbb{R}$.

Theorem (D.I. Gurevič 1975)

Spectral synthesis does not hold on $\mathbb{R}^2$.

What can we say if G is a discrete Abelian group?

M. Lefranc 1958: True for $\mathbb{Z}^n$.

R.J. Elliot 1965: True for *every* discrete Abelian group.

$V \subset C(\mathbb{R})$ is a variety if V is a closed linear subspace of $C(G)$ and is closed under translations.

Spectral synthesis holds on G if every variety on G is spanned by exp. polynomials.

Theorem (L. Schwartz 1948)

Spectral synthesis holds on $\mathbb{R}$.

Theorem (D.I. Gurevič 1975)

Spectral synthesis does not hold on $\mathbb{R}^2$.

What can we say if G is a discrete Abelian group?

M. Lefranc 1958: True for $\mathbb{Z}^n$.

R.J. Elliot 1965: True for *every* discrete Abelian group.

Z. Gajda $\sim$1980: ???

$V \subset C(\mathbb{R})$ is a variety if V is a closed linear subspace of $C(G)$ and is closed under translations.

Spectral synthesis holds on G if every variety on G is spanned by exp. polynomials.

Theorem (L. Schwartz 1948)

Spectral synthesis holds on $\mathbb{R}$.

Theorem (D.I. Gurevič 1975)

Spectral synthesis does not hold on $\mathbb{R}^2$.

What can we say if G is a discrete Abelian group?

M. Lefranc 1958: True for $\mathbb{Z}^n$.

R.J. Elliot 1965: True for *every* discrete Abelian group.

Z. Gajda $\sim$1980: ???

L. Székelyhidi 2004: False!

Székelyhidi's example 2004:

Székelyhidi's example 2004:

Let F_ω be the free Abelian group of rank ω :

Székelyhidi's example 2004:

Let F_ω be the free Abelian group of rank ω :

$$F_\omega = \{(x_1, x_2, \ldots) : x_i \in \mathbb{Z},\ x_i = 0\ (i \geq i_0)\}.$$

Székelyhidi's example 2004:

Let F_ω be the free Abelian group of rank ω :

$F_\omega = \{(x_1, x_2, \dots) : x_i \in \mathbb{Z},\ x_i = 0\ (i \geq i_0)\}.$

$P(x) = \sum_{i=1}^{\infty} x_i^2 \qquad (x = (x_1, x_2, \dots) \in F_\omega)$

Székelyhidi's example 2004:

Let F_ω be the free Abelian group of rank ω :

$F_\omega = \{(x_1, x_2, \ldots) : x_i \in \mathbb{Z},\ x_i = 0\ (i \geq i_0)\}.$

$P(x) = \sum_{i=1}^{\infty} x_i^2 \qquad (x = (x_1, x_2, \ldots) \in F_\omega)$

$c \in F_\omega, \quad c = (c_1, \ldots, c_n, 0, 0 \ldots),$

Székelyhidi's example 2004:

Let F_ω be the free Abelian group of rank ω :

$F_\omega = \{(x_1, x_2, \ldots) : x_i \in \mathbb{Z},\ x_i = 0\ (i \geq i_0)\}$.

$P(x) = \sum_{i=1}^{\infty} x_i^2 \qquad (x = (x_1, x_2, \ldots) \in F_\omega)$

$c \in F_\omega, \quad c = (c_1, \ldots, c_n, 0, 0 \ldots)$,

$$P(x + c) = P(x) + \sum_{i=1}^{n} 2c_i x_i + d$$

Székelyhidi's example 2004:

Let F_ω be the free Abelian group of rank ω :

$F_\omega = \{(x_1, x_2, \dots) : x_i \in \mathbb{Z},\ x_i = 0\ (i \geq i_0)\}$.

$P(x) = \sum_{i=1}^{\infty} x_i^2 \qquad (x = (x_1, x_2, \dots) \in F_\omega)$

$c \in F_\omega, \quad c = (c_1, \dots, c_n, 0, 0 \dots)$,

$$P(x+c) = P(x) + \sum_{i=1}^{n} 2c_i x_i + d$$

$$V = \{\alpha P + a + d : \alpha \in \mathbb{C},\ a \text{ is } \ \text{add.},\ d \in \mathbb{C}\}.$$

Székelyhidi's example 2004:

Let F_ω be the free Abelian group of rank ω :

$F_\omega = \{(x_1, x_2, \ldots) : x_i \in \mathbb{Z},\ x_i = 0\ (i \geq i_0)\}.$

$P(x) = \sum_{i=1}^{\infty} x_i^2 \qquad (x = (x_1, x_2, \ldots) \in F_\omega)$

$c \in F_\omega, \quad c = (c_1, \ldots, c_n, 0, 0 \ldots),$

$$P(x + c) = P(x) + \sum_{i=1}^{n} 2c_i x_i + d$$

$$V = \{\alpha P + a + d : \alpha \in \mathbb{C},\ a \text{ is } \text{ add.},\ d \in \mathbb{C}\}.$$

$m \in V \Longrightarrow m \equiv 1.$

Székelyhidi's example 2004:

Let F_ω be the free Abelian group of rank ω :

$F_\omega = \{(x_1, x_2, \ldots) : x_i \in \mathbb{Z},\ x_i = 0\ (i \geq i_0)\}$.

$P(x) = \sum_{i=1}^{\infty} x_i^2 \qquad (x = (x_1, x_2, \ldots) \in F_\omega)$

$c \in F_\omega, \quad c = (c_1, \ldots, c_n, 0, 0 \ldots)$,

$$P(x+c) = P(x) + \sum_{i=1}^{n} 2c_i x_i + d$$

$$V = \{\alpha P + a + d : \alpha \in \mathbb{C},\ a \text{ is add.},\ d \in \mathbb{C}\}.$$

$m \in V \Longrightarrow m \equiv 1. \qquad p \in V$ is a polynomial $\Longrightarrow p = a + d$.

Székelyhidi's example 2004:

Let F_ω be the free Abelian group of rank ω :

$F_\omega = \{(x_1, x_2, \ldots) : x_i \in \mathbb{Z},\ x_i = 0\ (i \geq i_0)\}.$

$P(x) = \sum_{i=1}^{\infty} x_i^2 \qquad (x = (x_1, x_2, \ldots) \in F_\omega)$

$c \in F_\omega, \quad c = (c_1, \ldots, c_n, 0, 0 \ldots),$

$$P(x+c) = P(x) + \sum_{i=1}^{n} 2c_i x_i + d$$

$$V = \{\alpha P + a + d : \alpha \in \mathbb{C},\ a \text{ is add.},\ d \in \mathbb{C}\}.$$

$m \in V \Longrightarrow m \equiv 1. \qquad p \in V$ is a polynomial $\Longrightarrow p = a + d.$

$\{a + d\}$ is not dense in V : $P \neq a + d.$

Székelyhidi's example 2004:

Let F_ω be the free Abelian group of rank ω :

$F_\omega = \{(x_1, x_2, \dots) : x_i \in \mathbb{Z},\ x_i = 0\ (i \geq i_0)\}.$

$P(x) = \sum_{i=1}^{\infty} x_i^2 \qquad (x = (x_1, x_2, \dots) \in F_\omega)$

$c \in F_\omega, \quad c = (c_1, \dots, c_n, 0, 0 \dots),$

$$P(x+c) = P(x) + \sum_{i=1}^{n} 2c_i x_i + d$$

$$V = \{\alpha P + a + d : \alpha \in \mathbb{C},\ a \text{ is add.},\ d \in \mathbb{C}\}.$$

$m \in V \implies m \equiv 1. \qquad p \in V$ is a polynomial $\implies p = a + d.$

$\{a + d\}$ is not dense in V : $P \neq a + d.$

Definition

$r_0(G)$ = the cardinality of a maximal independent system of elements of infinite order (the torsion free rank of G).

Theorem (L. Székelyhidi, M.L. 2007)

Spectral synthesis holds on a discrete Abelian group G if and only if $r_0(G)$ is finite.

Theorem (L. Székelyhidi, M.L. 2007)

Spectral synthesis holds on a discrete Abelian group G if and only if $r_0(G)$ is finite.

Theorem (L. Székelyhidi, M.L. 2007)

Spectral synthesis holds on a discrete Abelian group G if and only if $r_0(G)$ is finite.

$$\Delta_h f(x) = f(x+h) - f(x) \qquad (f : G \to \mathbb{C},\ h \in G)$$

Theorem (L. Székelyhidi, M.L. 2007)

Spectral synthesis holds on a discrete Abelian group G if and only if $r_0(G)$ is finite.

$$\Delta_h f(x) = f(x+h) - f(x) \qquad (f : G \to \mathbb{C},\ h \in G)$$

$$P(x) = \sum_{i=1}^{\infty} x_i^2; \qquad \Delta_{h_1} P = a + c; \qquad \Delta_{h_1}\Delta_{h_2} P \equiv c.$$

Theorem (L. Székelyhidi, M.L. 2007)

Spectral synthesis holds on a discrete Abelian group G if and only if $r_0(G)$ is finite.

$$\Delta_h f(x) = f(x+h) - f(x) \qquad (f : G \to \mathbb{C},\ h \in G)$$

$P(x) = \sum_{i=1}^{\infty} x_i^2; \qquad \Delta_{h_1} P = a + c; \qquad \Delta_{h_1}\Delta_{h_2} P \equiv c.$

$\Delta_{h_1}\Delta_{h_2}\Delta_{h_2} P \equiv 0.$

Theorem (L. Székelyhidi, M.L. 2007)

Spectral synthesis holds on a discrete Abelian group G if and only if $r_0(G)$ is finite.

$$\Delta_h f(x) = f(x+h) - f(x) \qquad (f : G \to \mathbb{C},\ h \in G)$$

$$P(x) = \sum_{i=1}^{\infty} x_i^2; \qquad \Delta_{h_1} P = a + c; \qquad \Delta_{h_1}\Delta_{h_2} P \equiv c.$$

$$\Delta_{h_1}\Delta_{h_2}\Delta_{h_2} P \equiv 0.$$

f is a generalized polynomial if $\exists\ n$, $\Delta_{h_1} \ldots \Delta_{h_n} f \equiv 0\ \forall\ h_1, \ldots, h_n \in G$.

Theorem (L. Székelyhidi, M.L. 2007)

Spectral synthesis holds on a discrete Abelian group G if and only if $r_0(G)$ is finite.

$\Delta_h f(x) = f(x+h) - f(x) \qquad (f : G \to \mathbb{C},\ h \in G)$

$P(x) = \sum_{i=1}^{\infty} x_i^2; \qquad \Delta_{h_1} P = a + c; \qquad \Delta_{h_1}\Delta_{h_2} P \equiv c.$

$\Delta_{h_1}\Delta_{h_2}\Delta_{h_2} P \equiv 0.$

f is a generalized polynomial if $\exists\ n$, $\Delta_{h_1} \ldots \Delta_{h_n} f \equiv 0\ \forall\ h_1, \ldots, h_n \in G$.

Generalized exponential polynomial:

Theorem (L. Székelyhidi, M.L. 2007)

Spectral synthesis holds on a discrete Abelian group G if and only if $r_0(G)$ is finite.

$\Delta_h f(x) = f(x+h) - f(x) \qquad (f : G \to \mathbb{C},\ h \in G)$

$P(x) = \sum_{i=1}^{\infty} x_i^2; \qquad \Delta_{h_1} P = a + c; \qquad \Delta_{h_1}\Delta_{h_2} P \equiv c.$

$\Delta_{h_1}\Delta_{h_2}\Delta_{h_2} P \equiv 0.$

f is a generalized polynomial if $\exists\ n$, $\Delta_{h_1} \ldots \Delta_{h_n} f \equiv 0\ \forall\ h_1, \ldots, h_n \in G$.

Generalized exponential polynomial: $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a gen. polynomial and m_i is an exponential $(i = 1, \ldots, n)$.

Theorem (L. Székelyhidi, M.L. 2007)

Spectral synthesis holds on a discrete Abelian group G if and only if $r_0(G)$ is finite.

$\Delta_h f(x) = f(x+h) - f(x) \qquad (f : G \to \mathbb{C},\ h \in G)$

$P(x) = \sum_{i=1}^{\infty} x_i^2; \qquad \Delta_{h_1} P = a + c; \qquad \Delta_{h_1}\Delta_{h_2} P \equiv c.$

$\Delta_{h_1}\Delta_{h_2}\Delta_{h_2} P \equiv 0.$

f is a generalized polynomial if $\exists\, n$, $\Delta_{h_1} \ldots \Delta_{h_n} f \equiv 0\ \forall\ h_1, \ldots, h_n \in G$.

Generalized exponential polynomial: $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a gen. polynomial and m_i is an exponential $(i = 1, \ldots, n)$.

Generalized spectral synthesis holds on G if every variety on G is spanned by gen. exp. polynomials.

Theorem (L. Székelyhidi, M.L. 2007)

Spectral synthesis holds on a discrete Abelian group G if and only if $r_0(G)$ is finite.

$\Delta_h f(x) = f(x+h) - f(x) \qquad (f : G \to \mathbb{C},\ h \in G)$

$P(x) = \sum_{i=1}^{\infty} x_i^2; \qquad \Delta_{h_1} P = a + c; \qquad \Delta_{h_1}\Delta_{h_2} P \equiv c.$

$\Delta_{h_1}\Delta_{h_2}\Delta_{h_2} P \equiv 0.$

f is a generalized polynomial if $\exists\, n$, $\Delta_{h_1} \dots \Delta_{h_n} f \equiv 0\ \forall\ h_1, \dots, h_n \in G$.

Generalized exponential polynomial: $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a gen. polynomial and m_i is an exponential $(i = 1, \dots, n)$.

Generalized spectral synthesis holds on G if every variety on G is spanned by gen. exp. polynomials.

Theorem (G. Kiss, M.L. 2012)

Generalized spectral synthesis does not hold on F_ω.

$$Q(x) = \sum_{n=1}^{\infty} x_n^n \qquad (x \in F_\omega).$$

$Q(x) = \sum_{n=1}^{\infty} x_n^n \qquad (x \in F_\omega).$

Let V be the set of functions $c + \sum_{n=1}^{\infty} p_n(x_n)$ such that $p_n \in \mathbb{C}[x]$, $\deg p_n \leq n$, $p_n(0) = 0$, and either

$Q(x) = \sum_{n=1}^{\infty} x_n^n \qquad (x \in F_\omega).$

Let V be the set of functions $c + \sum_{n=1}^{\infty} p_n(x_n)$ such that
$p_n \in \mathbb{C}[x],\ \deg p_n \leq n,\ p_n(0) = 0$, and either

$(\forall\ n)\ \deg p_n = n$ and the leading coefficients of p_n are the same, or

$Q(x) = \sum_{n=1}^{\infty} x_n^n \qquad (x \in F_\omega).$

Let V be the set of functions $c + \sum_{n=1}^{\infty} p_n(x_n)$ such that
$p_n \in \mathbb{C}[x],\ \deg p_n \leq n,\ p_n(0) = 0$, and either

$(\forall\ n)\ \deg p_n = n$ and the leading coefficients of p_n are the same, or

$(\forall\ n)\ \deg p_n < n$.

$$Q(x) = \sum_{n=1}^{\infty} x_n^n \qquad (x \in F_\omega).$$

Let V be the set of functions $c + \sum_{n=1}^{\infty} p_n(x_n)$ such that
$p_n \in \mathbb{C}[x],\ \deg p_n \leq n,\ p_n(0) = 0$, and either

$(\forall\ n) \deg p_n = n$ and the leading coefficients of p_n are the same, or

$(\forall\ n) \deg p_n < n$.

Then V is a variety and $Q \in V$.

$Q(x) = \sum_{n=1}^{\infty} x_n^n \qquad (x \in F_\omega).$

Let V be the set of functions $c + \sum_{n=1}^{\infty} p_n(x_n)$ such that $p_n \in \mathbb{C}[x],\ \deg p_n \le n,\ p_n(0) = 0$, and either

$(\forall\ n)\ \deg p_n = n$ and the leading coefficients of p_n are the same, or

$(\forall\ n)\ \deg p_n < n.$

Then V is a variety and $Q \in V$. If $m \in V$, then $m \equiv 1$.

$Q(x) = \sum_{n=1}^{\infty} x_n^n \qquad (x \in F_\omega).$

Let V be the set of functions $c + \sum_{n=1}^{\infty} p_n(x_n)$ such that $p_n \in \mathbb{C}[x],\ \deg p_n \le n,\ p_n(0) = 0$, and either

$(\forall\ n)$ $\deg p_n = n$ and the leading coefficients of p_n are the same, or

$(\forall\ n)$ $\deg p_n < n$.

Then V is a variety and $Q \in V$. If $m \in V$, then $m \equiv 1$.

Lemma

If f is a generalized polynomial on F_ω then $\exists\ n$,

$$f(x) = \sum c_{i_1 \dots i_k} \cdot x_1^{i_1} \cdots x_k^{i_k},$$

where $c_{i_1 \dots i_k} = 0$ if $i_1 + \ldots + i_k > n$.

$Q(x) = \sum_{n=1}^{\infty} x_n^n \qquad (x \in F_\omega).$

Let V be the set of functions $c + \sum_{n=1}^{\infty} p_n(x_n)$ such that $p_n \in \mathbb{C}[x]$, $\deg p_n \leq n$, $p_n(0) = 0$, and either

$(\forall\ n)$ $\deg p_n = n$ and the leading coefficients of p_n are the same, or

$(\forall\ n)$ $\deg p_n < n$.

Then V is a variety and $Q \in V$. If $m \in V$, then $m \equiv 1$.

Lemma

If f is a generalized polynomial on F_ω then $\exists\ n$,

$$f(x) = \sum c_{i_1 \dots i_k} \cdot x_1^{i_1} \cdots x_k^{i_k},$$

where $c_{i_1 \dots i_k} = 0$ if $i_1 + \ldots + i_k > n$.

$Q(x) = \sum_{n=1}^{\infty} x_n^n \qquad (x \in F_\omega).$

Let V be the set of functions $c + \sum_{n=1}^{\infty} p_n(x_n)$ such that $p_n \in \mathbb{C}[x],\ \deg p_n \le n,\ p_n(0) = 0$, and either

$(\forall\ n)\ \deg p_n = n$ and the leading coefficients of p_n are the same, or

$(\forall\ n)\ \deg p_n < n$.

Then V is a variety and $Q \in V$. If $m \in V$, then $m \equiv 1$.

Lemma

If f is a generalized polynomial on F_ω then $\exists\ n$,

$$f(x) = \sum c_{i_1 \dots i_k} \cdot x_1^{i_1} \cdots x_k^{i_k},$$

where $c_{i_1 \dots i_k} = 0$ if $i_1 + \ldots + i_k > n$.

The set of gen'd polynomials is not dense in V; Q is not in the closure.

$Q(x) = \sum_{n=1}^{\infty} x_n^n$ has the property that $Q|_H$ is a polynomial for every finitely generated subgroup H of G.

$Q(x) = \sum_{n=1}^{\infty} x_n^n$ has the property that $Q|_H$ is a polynomial for every finitely generated subgroup H of G.

Definition

$p : G \to \mathbb{C}$ is a local polynomial if $f|_H$ is a polynomial for every finitely generated subgroup H of G.

$Q(x) = \sum_{n=1}^{\infty} x_n^n$ has the property that $Q|_H$ is a polynomial for every finitely generated subgroup H of G.

Definition

$p : G \to \mathbb{C}$ is a local polynomial if $f|_H$ is a polynomial for every finitely generated subgroup H of G.

$Q(x) = \sum_{n=1}^{\infty} x_n^n$ has the property that $Q|_H$ is a polynomial for every finitely generated subgroup H of G.

Definition

$p : G \to \mathbb{C}$ is a local polynomial if $f|_H$ is a polynomial for every finitely generated subgroup H of G.

Theorem

For every $f : G \to \mathbb{C}$,
f is a polynomial $\implies$ f is a generalized polynomial $\implies$ f is a local polynomial.

$Q(x) = \sum_{n=1}^{\infty} x_n^n$ has the property that $Q|_H$ is a polynomial for every finitely generated subgroup H of G.

Definition

$p : G \to \mathbb{C}$ is a local polynomial if $f|_H$ is a polynomial for every finitely generated subgroup H of G.

Theorem

For every $f : G \to \mathbb{C}$,
f is a polynomial $\implies$ f is a generalized polynomial $\implies$ f is a local polynomial.

$Q(x) = \sum_{n=1}^{\infty} x_n^n$ has the property that $Q|_H$ is a polynomial for every finitely generated subgroup H of G.

Definition

$p : G \to \mathbb{C}$ is a local polynomial if $f|_H$ is a polynomial for every finitely generated subgroup H of G.

Theorem

For every $f : G \to \mathbb{C}$,
f is a polynomial $\implies$ f is a generalized polynomial $\implies$ f is a local polynomial.

If G is finitely generated, then
f is a polynomial $\iff$ f is a generalized polynomial $\iff$ f is a local polynomial.

$Q(x) = \sum_{n=1}^{\infty} x_n^n$ has the property that $Q|_H$ is a polynomial for every finitely generated subgroup H of G.

Definition

$p : G \to \mathbb{C}$ is a local polynomial if $f|_H$ is a polynomial for every finitely generated subgroup H of G.

Theorem

For every $f : G \to \mathbb{C}$,
f is a polynomial $\Longrightarrow$ f is a generalized polynomial $\Longrightarrow$ f is a local polynomial.

If G is finitely generated, then
f is a polynomial $\Longleftrightarrow$ f is a generalized polynomial $\Longleftrightarrow$ f is a local polynomial.

Local exponential polynomial:

$Q(x) = \sum_{n=1}^{\infty} x_n^n$ has the property that $Q|_H$ is a polynomial for every finitely generated subgroup H of G.

Definition

$p : G \to \mathbb{C}$ is a local polynomial if $f|_H$ is a polynomial for every finitely generated subgroup H of G.

Theorem

For every $f : G \to \mathbb{C}$,
f is a polynomial $\implies$ f is a generalized polynomial $\implies$ f is a local polynomial.

If G is finitely generated, then
f is a polynomial $\iff$ f is a generalized polynomial $\iff$ f is a local polynomial.

Local exponential polynomial: $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a local polynomial and m_i is an exponential $(i = 1, \ldots, n)$.

Definition

Local spectral synthesis holds on G if every variety on G is spanned by local exp. polynomials.

Definition

Local spectral synthesis holds on G if every variety on G is spanned by local exp. polynomials.

Definition

Local spectral synthesis holds on G if every variety on G is spanned by local exp. polynomials.

Theorem (G. Székelyhidi, M.L. 2004)

Spectral analysis holds on G; that is, every variety $V \neq \{0\}$ contains an exponential, if and only if $r_0(G) < 2^\omega$.

Definition

Local spectral synthesis holds on G if every variety on G is spanned by local exp. polynomials.

Theorem (G. Székelyhidi, M.L. 2004)

Spectral analysis holds on G; that is, every variety $V \neq \{0\}$ contains an exponential, if and only if $r_0(G) < 2^\omega$.

Definition

Local spectral synthesis holds on G if every variety on G is spanned by local exp. polynomials.

Theorem (G. Székelyhidi, M.L. 2004)

Spectral analysis holds on G; that is, every variety $V \neq \{0\}$ contains an exponential, if and only if $r_0(G) < 2^\omega$.

Lemma

If V is a variety, $f \in V$, $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a local polynomial and m_i is an exponential $(i = 1, \ldots, n)$, then $p_i \cdot m_i \in V$ and $m_i \in V$ for every $i = 1, \ldots, n$.

Definition

Local spectral synthesis holds on G if every variety on G is spanned by local exp. polynomials.

Theorem (G. Székelyhidi, M.L. 2004)

Spectral analysis holds on G; that is, every variety $V \neq \{0\}$ contains an exponential, if and only if $r_0(G) < 2^\omega$.

Lemma

If V is a variety, $f \in V$, $f = \sum_{i=1}^n p_i \cdot m_i$, where p_i is a local polynomial and m_i is an exponential $(i = 1, \ldots, n)$, then $p_i \cdot m_i \in V$ and $m_i \in V$ for every $i = 1, \ldots, n$.

Definition

Local spectral synthesis holds on G if every variety on G is spanned by local exp. polynomials.

Theorem (G. Székelyhidi, M.L. 2004)

Spectral analysis holds on G; that is, every variety $V \neq \{0\}$ contains an exponential, if and only if $r_0(G) < 2^\omega$.

Lemma

If V is a variety, $f \in V$, $f = \sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a local polynomial and m_i is an exponential $(i = 1, \ldots, n)$, then $p_i \cdot m_i \in V$ and $m_i \in V$ for every $i = 1, \ldots, n$.

Corollary

If $r_0(G) \geq 2^\omega$ (e.g. if G is torsion free and $|G| \geq 2^\omega$), then local spectral synthesis fails on G.

Theorem (ML 2012)

There exists a cardinal $\aleph_1 \leq \kappa \leq 2^\omega$ such that, for every Abelian group, local spectral synthesis holds on G if and only if $r_0(G) < \kappa$.

Theorem (ML 2012)

There exists a cardinal $\aleph_1 \le \kappa \le 2^\omega$ *such that, for every Abelian group, local spectral synthesis holds on* G *if and only if* $r_0(G) < \kappa$.

Theorem (ML 2012)

There exists a cardinal $\aleph_1 \leq \kappa \leq 2^\omega$ such that, for every Abelian group, local spectral synthesis holds on G if and only if $r_0(G) < \kappa$.

Lemma

If local spectral synthesis holds on G, then the same is true for every subgroup of G and for every homomorphic image of G.

Theorem (ML 2012)

There exists a cardinal $\aleph_1 \leq \kappa \leq 2^\omega$ *such that, for every Abelian group, local spectral synthesis holds on* G *if and only if* $r_0(G) < \kappa$.

Lemma

If local spectral synthesis holds on G*, then the same is true for every subgroup of* G *and for every homomorphic image of* G.

Theorem (ML 2012)

There exists a cardinal $\aleph_1 \leq \kappa \leq 2^\omega$ *such that, for every Abelian group, local spectral synthesis holds on* G *if and only if* $r_0(G) < \kappa$.

Lemma

If local spectral synthesis holds on G*, then the same is true for every subgroup of* G *and for every homomorphic image of* G.

Lemma

If local spectral synthesis holds on G*, and if* T *is torsion, then local spectral synthesis holds on* $G \times T$.

Theorem (ML 2012)

There exists a cardinal $\aleph_1 \leq \kappa \leq 2^\omega$ *such that, for every Abelian group, local spectral synthesis holds on* G *if and only if* $r_0(G) < \kappa$.

Lemma

If local spectral synthesis holds on G*, then the same is true for every subgroup of* G *and for every homomorphic image of* G.

Lemma

If local spectral synthesis holds on G*, and if* T *is torsion, then local spectral synthesis holds on* $G \times T$.

Theorem (ML 2012)

There exists a cardinal $\aleph_1 \leq \kappa \leq 2^\omega$ such that, for every Abelian group, local spectral synthesis holds on G if and only if $r_0(G) < \kappa$.

Lemma

If local spectral synthesis holds on G, then the same is true for every subgroup of G and for every homomorphic image of G.

Lemma

If local spectral synthesis holds on G, and if T is torsion, then local spectral synthesis holds on $G \times T$.

Let F_λ denote the free Abelian group of rank λ. Let κ denote the smallest cardinal such that local spectral synthesis does not hold on F_κ. Then $\kappa \leq 2^\omega$, and local spectral synthesis holds on G if and only if $r_0(G) < \kappa$.

Theorem (ML 2012)

There exists a cardinal $\aleph_1 \leq \kappa \leq 2^\omega$ *such that, for every Abelian group, local spectral synthesis holds on* G *if and only if* $r_0(G) < \kappa$.

Lemma

If local spectral synthesis holds on G*, then the same is true for every subgroup of* G *and for every homomorphic image of* G.

Lemma

If local spectral synthesis holds on G*, and if* T *is torsion, then local spectral synthesis holds on* $G \times T$.

Let F_λ denote the free Abelian group of rank λ. Let κ denote the smallest cardinal such that local spectral synthesis does not hold on F_κ. Then $\kappa \leq 2^\omega$, and local spectral synthesis holds on G if and only if $r_0(G) < \kappa$.

We need: $\kappa \geq \aleph_1$. That is, we need:

Theorem

Local spectral synthesis holds on F_ω.

Theorem

Local spectral synthesis holds on F_ω.

Theorem

Local spectral synthesis holds on F_ω.

Lefranc's approach for $G = \mathbb{Z}^n$: duality between varieties on $\mathbb{Z}^n$ and ideals of measures $\mu = \sum c_{i_1 \ldots i_n} \cdot \delta_{i_1 \ldots i_n}$.

Theorem

Local spectral synthesis holds on F_ω.

Lefranc's approach for $G = \mathbb{Z}^n$: duality between varieties on $\mathbb{Z}^n$ and ideals of measures $\mu = \sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_n}$.

$V^\perp = \{\mu : \int p \, d\mu = 0 \text{ for every } p \in V\}$.

Theorem

Local spectral synthesis holds on F_ω.

Lefranc's approach for $G = \mathbb{Z}^n$: duality between varieties on $\mathbb{Z}^n$ and ideals of measures $\mu = \sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_n}$.

$V^\perp = \{\mu : \int p \, d\mu = 0$ for every $p \in V\}$.

$p \in V \iff \int p \, d\mu = 0$ for every $\mu \in V^\perp$.

Theorem

Local spectral synthesis holds on F_ω.

Lefranc's approach for $G = \mathbb{Z}^n$: duality between varieties on $\mathbb{Z}^n$ and ideals of measures $\mu = \sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_n}$.

$V^\perp = \{\mu : \int p \, d\mu = 0$ for every $p \in V\}$.

$p \in V \iff \int p \, d\mu = 0$ for every $\mu \in V^\perp$.

$\sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_k} \mapsto \sum c_{i_1 \dots i_n} \cdot x_1^{i_1} \dots x_n^{i_n}$ maps $V^\perp$ into an ideal of $\mathbb{C}[x_1, \dots, x_n]$.

Theorem

Local spectral synthesis holds on F_ω.

Lefranc's approach for $G = \mathbb{Z}^n$: duality between varieties on $\mathbb{Z}^n$ and ideals of measures $\mu = \sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_n}$.

$V^\perp = \{\mu : \int p \, d\mu = 0 \text{ for every } p \in V\}$.

$p \in V \iff \int p \, d\mu = 0 \text{ for every } \mu \in V^\perp$.

$\sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_k} \mapsto \sum c_{i_1 \dots i_n} \cdot x_1^{i_1} \dots x_n^{i_n}$ maps $V^\perp$ into an ideal of $\mathbb{C}[x_1, \dots, x_n]$.

Differential operators on $\mathbb{C}[x_1, \dots, x_n] \leftrightarrow$ exponential polynomials on $\mathbb{Z}^n$.

Theorem

Local spectral synthesis holds on F_ω.

Lefranc's approach for $G = \mathbb{Z}^n$: duality between varieties on $\mathbb{Z}^n$ and ideals of measures $\mu = \sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_n}$.

$V^\perp = \{\mu : \int p \, d\mu = 0 \text{ for every } p \in V\}$.

$p \in V \iff \int p \, d\mu = 0$ for every $\mu \in V^\perp$.

$\sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_k} \mapsto \sum c_{i_1 \dots i_n} \cdot x_1^{i_1} \dots x_n^{i_n}$ maps $V^\perp$ into an ideal of $\mathbb{C}[x_1, \dots, x_n]$.

Differential operators on $\mathbb{C}[x_1, \dots, x_n] \leftrightarrow$ exponential polynomials on $\mathbb{Z}^n$.

Equivalent reformulation of the spectral synthesis on $\mathbb{Z}^n$:

Theorem

Local spectral synthesis holds on F_ω.

Lefranc's approach for $G = \mathbb{Z}^n$: duality between varieties on $\mathbb{Z}^n$ and ideals of measures $\mu = \sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_n}$.

$V^\perp = \{\mu : \int p \, d\mu = 0$ for every $p \in V\}$.

$p \in V \iff \int p \, d\mu = 0$ for every $\mu \in V^\perp$.

$\sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_k} \mapsto \sum c_{i_1 \dots i_n} \cdot x_1^{i_1} \dots x_n^{i_n}$ maps $V^\perp$ into an ideal of $\mathbb{C}[x_1, \dots, x_n]$.

Differential operators on $\mathbb{C}[x_1, \dots, x_n] \leftrightarrow$ exponential polynomials on $\mathbb{Z}^n$.

Equivalent reformulation of the spectral synthesis on $\mathbb{Z}^n$:

If $p \in \mathbb{C}[x_1, \dots, x_n] \setminus I$, then there is a root $c = (c_1, \dots, c_n)$ of I and there is a differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$.

Theorem

Local spectral synthesis holds on F_ω.

Lefranc's approach for $G = \mathbb{Z}^n$: duality between varieties on $\mathbb{Z}^n$ and ideals of measures $\mu = \sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_n}$.

$V^\perp = \{\mu : \int p \, d\mu = 0$ for every $p \in V\}$.

$p \in V \iff \int p \, d\mu = 0$ for every $\mu \in V^\perp$.

$\sum c_{i_1 \dots i_n} \cdot \delta_{i_1 \dots i_k} \mapsto \sum c_{i_1 \dots i_n} \cdot x_1^{i_1} \dots x_n^{i_n}$ maps $V^\perp$ into an ideal of $\mathbb{C}[x_1, \dots, x_n]$.

Differential operators on $\mathbb{C}[x_1, \dots, x_n] \leftrightarrow$ exponential polynomials on $\mathbb{Z}^n$.

Equivalent reformulation of the spectral synthesis on $\mathbb{Z}^n$:

If $p \in \mathbb{C}[x_1, \dots, x_n] \setminus I$, then there is a root $c = (c_1, \dots, c_n)$ of I and there is a differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$. (This is Krull's theorem.)

Reformulation of the spectral synthesis on F_ω :

Reformulation of the spectral synthesis on F_ω :

For $i = (i_1, i_2, \ldots) \in F_\omega$ $(i_n = 0 \ (n > m))$

$$D_i = \frac{\partial^{i_1+\ldots+i_m}}{(\partial x_1)^{i_1} \cdots (\partial x_m)^{i_m}}$$

Reformulation of the spectral synthesis on F_ω :
For $i = (i_1, i_2, \ldots) \in F_\omega$ $(i_n = 0\ (n > m))$

$$D_i = \frac{\partial^{i_1 + \ldots + i_m}}{(\partial x_1)^{i_1} \cdots (\partial x_m)^{i_m}}$$

Example. $R = \mathbb{C}[x_1, x_2, \ldots]$, $J = (x_1^2, x_1 - x_2, x_1 - x_3, x_1 - x_4, \ldots)$.

Reformulation of the spectral synthesis on F_ω :
For $i = (i_1, i_2, \ldots) \in F_\omega \ (i_n = 0 \ (n > m))$

$$D_i = \frac{\partial^{i_1+\ldots+i_m}}{(\partial x_1)^{i_1} \cdots (\partial x_m)^{i_m}}$$

Example. $R = \mathbb{C}[x_1, x_2, \ldots]$, $J = (x_1^2, x_1 - x_2, x_1 - x_3, x_1 - x_4, \ldots)$.

$x_1 \notin J$.

Reformulation of the spectral synthesis on F_ω :
For $i = (i_1, i_2, \ldots) \in F_\omega$ $(i_n = 0\ (n > m))$

$$D_i = \frac{\partial^{i_1+\ldots+i_m}}{(\partial x_1)^{i_1} \cdots (\partial x_m)^{i_m}}$$

Example. $R = \mathbb{C}[x_1, x_2, \ldots]$, $J = (x_1^2, x_1 - x_2, x_1 - x_3, x_1 - x_4, \ldots)$.

$x_1 \notin J$. The only root of J is $(0, 0, \ldots)$.

Reformulation of the spectral synthesis on F_ω :
For $i = (i_1, i_2, \ldots) \in F_\omega$ $(i_n = 0\ (n > m))$

$$D_i = \frac{\partial^{i_1+\ldots+i_m}}{(\partial x_1)^{i_1} \cdots (\partial x_m)^{i_m}}$$

Example. $R = \mathbb{C}[x_1, x_2, \ldots]$, $J = (x_1^2, x_1 - x_2, x_1 - x_3, x_1 - x_4, \ldots)$.

$x_1 \notin J$. The only root of J is $(0, 0, \ldots)$.

There is no differential operator D s.t. $Df(0) = 0$ for every $f \in J$, and $Dx_1(0) \neq 0$.

Reformulation of the spectral synthesis on F_ω :

For $i = (i_1, i_2, \ldots) \in F_\omega$ $(i_n = 0$ $(n > m))$

$$D_i = \frac{\partial^{i_1 + \ldots + i_m}}{(\partial x_1)^{i_1} \cdots (\partial x_m)^{i_m}}$$

Example. $R = \mathbb{C}[x_1, x_2, \ldots]$, $J = (x_1^2, x_1 - x_2, x_1 - x_3, x_1 - x_4, \ldots)$.

$x_1 \notin J$. The only root of J is $(0, 0, \ldots)$.

There is no differential operator D s.t. $Df(0) = 0$ for every $f \in J$, and $Dx_1(0) \neq 0$.

$$D = \sum_{i=1}^{\infty} \frac{\partial}{\partial x_i}$$

Reformulation of the spectral synthesis on F_ω :
For $i = (i_1, i_2, \ldots) \in F_\omega$ $(i_n = 0\ (n > m))$

$$D_i = \frac{\partial^{i_1+\ldots+i_m}}{(\partial x_1)^{i_1} \cdots (\partial x_m)^{i_m}}$$

Example. $R = \mathbb{C}[x_1, x_2, \ldots]$, $J = (x_1^2, x_1 - x_2, x_1 - x_3, x_1 - x_4, \ldots)$.

$x_1 \notin J$. The only root of J is $(0, 0, \ldots)$.

There is no differential operator D s.t. $Df(0) = 0$ for every $f \in J$, and $Dx_1(0) \neq 0$.

$$D = \sum_{i=1}^{\infty} \frac{\partial}{\partial x_i}$$

$Df(0) = 0$ for every $f \in J$, and $Dx_1(0) \neq 0$!

Reformulation of the spectral synthesis on F_ω :

For $i = (i_1, i_2, \ldots) \in F_\omega$ $(i_n = 0\ (n > m))$

$$D_i = \frac{\partial^{i_1+\ldots+i_m}}{(\partial x_1)^{i_1} \cdots (\partial x_m)^{i_m}}$$

Example. $R = \mathbb{C}[x_1, x_2, \ldots]$, $J = (x_1^2, x_1 - x_2, x_1 - x_3, x_1 - x_4, \ldots)$.

$x_1 \notin J$. The only root of J is $(0, 0, \ldots)$.

There is no differential operator D s.t. $Df(0) = 0$ for every $f \in J$, and $Dx_1(0) \neq 0$.

$$D = \sum_{i=1}^{\infty} \frac{\partial}{\partial x_i}$$

$Df(0) = 0$ for every $f \in J$, and $Dx_1(0) \neq 0$!

Definition

Generalized differential operator: $\sum_{i \in F_\omega} a_i D_i$, where, for every m, the set $\{i \in F_\omega : a_i \neq 0 \text{ and } i_n > 0\ \forall\ n > m\}$ is finite.

Theorem

Let I be an ideal of $\mathbb{C}[x_1, x_2, \ldots]$, and let $p \in \mathbb{C}[x_1, x_2, \ldots] \setminus I$. Then there is a root $c = (c_1, c_2, \ldots)$ of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$.

Theorem

Let I be an ideal of $\mathbb{C}[x_1, x_2, \dots]$, and let $p \in \mathbb{C}[x_1, x_2, \dots] \setminus I$. Then there is a root $c = (c_1, c_2, \dots)$ of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$.

Theorem

Let I be an ideal of $\mathbb{C}[x_1, x_2, \ldots]$, and let $p \in \mathbb{C}[x_1, x_2, \ldots] \setminus I$. Then there is a root $c = (c_1, c_2, \ldots)$ of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$.

The theorem is true for every uncountable and algebraically closed field in place of $\mathbb{C}$.

Theorem

Let I be an ideal of $\mathbb{C}[x_1, x_2, \ldots]$, and let $p \in \mathbb{C}[x_1, x_2, \ldots] \setminus I$. Then there is a root $c = (c_1, c_2, \ldots)$ of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$.

The theorem is true for every uncountable and algebraically closed field in place of $\mathbb{C}$.

Question

Let X be an infinite set of indeterminates, and let Ω be an algebraically closed field with $|X| < |\Omega|$. Is it true that for every ideal I of the polynomial ring $\Omega[X]$ and for every polynomial $p \in \Omega[X]$ there is a root c of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$?

Theorem

Let I be an ideal of $\mathbb{C}[x_1, x_2, \ldots]$, and let $p \in \mathbb{C}[x_1, x_2, \ldots] \setminus I$. Then there is a root $c = (c_1, c_2, \ldots)$ of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$.

The theorem is true for every uncountable and algebraically closed field in place of $\mathbb{C}$.

Question

Let X be an infinite set of indeterminates, and let Ω be an algebraically closed field with $|X| < |\Omega|$. Is it true that for every ideal I of the polynomial ring $\Omega[X]$ and for every polynomial $p \in \Omega[X]$ there is a root c of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$?

Theorem

Let I be an ideal of $\mathbb{C}[x_1, x_2, \dots]$, and let $p \in \mathbb{C}[x_1, x_2, \dots] \setminus I$. Then there is a root $c = (c_1, c_2, \dots)$ of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$.

The theorem is true for every uncountable and algebraically closed field in place of $\mathbb{C}$.

Question

Let X be an infinite set of indeterminates, and let Ω be an algebraically closed field with $|X| < |\Omega|$. Is it true that for every ideal I of the polynomial ring $\Omega[X]$ and for every polynomial $p \in \Omega[X]$ there is a root c of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$?

Question

What is the "real" value of κ?

Theorem

Let I be an ideal of $\mathbb{C}[x_1, x_2, \ldots]$, and let $p \in \mathbb{C}[x_1, x_2, \ldots] \setminus I$. Then there is a root $c = (c_1, c_2, \ldots)$ of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$.

The theorem is true for every uncountable and algebraically closed field in place of $\mathbb{C}$.

Question

Let X be an infinite set of indeterminates, and let Ω be an algebraically closed field with $|X| < |\Omega|$. Is it true that for every ideal I of the polynomial ring $\Omega[X]$ and for every polynomial $p \in \Omega[X]$ there is a root c of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$?

Question

What is the "real" value of κ?

Theorem

Let I be an ideal of $\mathbb{C}[x_1, x_2, \ldots]$, and let $p \in \mathbb{C}[x_1, x_2, \ldots] \setminus I$. Then there is a root $c = (c_1, c_2, \ldots)$ of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$.

The theorem is true for every uncountable and algebraically closed field in place of $\mathbb{C}$.

Question

Let X be an infinite set of indeterminates, and let Ω be an algebraically closed field with $|X| < |\Omega|$. Is it true that for every ideal I of the polynomial ring $\Omega[X]$ and for every polynomial $p \in \Omega[X]$ there is a root c of I and there is a generalized differential operator D such that $Df(c) = 0$ for every $f \in I$, and $Dp(c) \neq 0$?

Question

What is the "real" value of κ? Is it true that $\kappa = \aleph_1$ independently of the vaue of 2^ω?

Lemma

If $r_0(G) < \infty$, *then every local polynomial on* G *is a polynomial.*

Lemma

If $r_0(G) < \infty$, then every local polynomial on G is a polynomial.

Lemma

If $r_0(G) < \infty$, *then every local polynomial on* G *is a polynomial.*

Corollary

Spectral synthesis holds on every countable Abelian group with $r_0(G) < \infty$.

Lemma

If $r_0(G) < \infty$*, then every local polynomial on* G *is a polynomial.*

Corollary

Spectral synthesis holds on every countable Abelian group with $r_0(G) < \infty$*.*

Lemma

If $r_0(G) < \infty$*, then every local polynomial on* G *is a polynomial.*

Corollary

Spectral synthesis holds on every countable Abelian group with $r_0(G) < \infty$*. In particular, spectral synthesis holds on* $\mathbb{Q}^n \times T$ *if* $n \in \mathbb{N}$ *and* T *is torsion.*

Lemma

If $r_0(G) < \infty$, then every local polynomial on G is a polynomial.

Corollary

Spectral synthesis holds on every countable Abelian group with $r_0(G) < \infty$. In particular, spectral synthesis holds on $\mathbb{Q}^n \times T$ if $n \in \mathbb{N}$ and T is torsion. Spectral synthesis holds on every G with $r_0(G) < \infty$.

$$\sum_{i=1}^{n} a_i f(x + b_i y) = 0 \qquad (f : \mathbb{C} \to \mathbb{C},\ a_i, b_i \in \mathbb{C}). \tag{1}$$

$$\sum_{i=1}^{n} a_i f(x + b_i y) = 0 \qquad (f : \mathbb{C} \to \mathbb{C},\ a_i, b_i \in \mathbb{C}). \tag{1}$$

$\mathbb{Q}(b_1)\ldots(b_n) = K, \qquad K^* = K \setminus \{0\}$ is a countable group under multiplication.

$$\sum_{i=1}^{n} a_i f(x + b_i y) = 0 \qquad (f : \mathbb{C} \to \mathbb{C},\ a_i, b_i \in \mathbb{C}). \tag{1}$$

$\mathbb{Q}(b_1)\dots(b_n) = K, \qquad K^* = K \setminus \{0\}$ is a countable group under multiplication.

K^* is not finitely generated; $r_0(K^*) = \infty$ for every K. Thus spectral synthesis does not hold on K^*.

$$\sum_{i=1}^{n} a_i f(x + b_i y) = 0 \qquad (f : \mathbb{C} \to \mathbb{C},\ a_i, b_i \in \mathbb{C}). \tag{1}$$

$\mathbb{Q}(b_1)\dots(b_n) = K, \qquad K^* = K \setminus \{0\}$ is a countable group under multiplication.

K^* is not finitely generated; $r_0(K^*) = \infty$ for every K. Thus spectral synthesis does not hold on K^*.

Since K^* is countable, local spectral synthesis holds on K^*.

$$\sum_{i=1}^{n} a_i f(x + b_i y) = 0 \qquad (f : \mathbb{C} \to \mathbb{C},\ a_i, b_i \in \mathbb{C}). \tag{1}$$

$\mathbb{Q}(b_1)\dots(b_n) = K, \qquad K^* = K \setminus \{0\}$ is a countable group under multiplication.

K^* is not finitely generated; $r_0(K^*) = \infty$ for every K. Thus spectral synthesis does not hold on K^*.

Since K^* is countable, local spectral synthesis holds on K^*.

The additive solutions of (1) restricted to K^* constitute a variety V_a.

$$\sum_{i=1}^{n} a_i f(x + b_i y) = 0 \qquad (f : \mathbb{C} \to \mathbb{C},\ a_i, b_i \in \mathbb{C}). \tag{1}$$

$\mathbb{Q}(b_1)\dots(b_n) = K, \qquad K^* = K \setminus \{0\}$ is a countable group under multiplication.

K^* is not finitely generated; $r_0(K^*) = \infty$ for every K. Thus spectral synthesis does not hold on K^*.

Since K^* is countable, local spectral synthesis holds on K^*.

The additive solutions of (1) restricted to K^* constitute a variety V_a.

Theorem (G. Kiss, ML 2012)

In V_a every local exponential polynomial is an exponential polynomial.

$$\sum_{i=1}^{n} a_i f(x + b_i y) = 0 \qquad (f : \mathbb{C} \to \mathbb{C},\ a_i, b_i \in \mathbb{C}). \tag{1}$$

$\mathbb{Q}(b_1)\dots(b_n) = K$, $\qquad K^* = K \setminus \{0\}$ is a countable group under multiplication.

K^* is not finitely generated; $r_0(K^*) = \infty$ for every K. Thus spectral synthesis does not hold on K^*.

Since K^* is countable, local spectral synthesis holds on K^*.

The additive solutions of (1) restricted to K^* constitute a variety V_a.

Theorem (G. Kiss, ML 2012)

In V_a every local exponential polynomial is an exponential polynomial.

$$\sum_{i=1}^{n} a_i f(x + b_i y) = 0 \qquad (f : \mathbb{C} \to \mathbb{C},\ a_i, b_i \in \mathbb{C}). \tag{1}$$

$\mathbb{Q}(b_1)\dots(b_n) = K, \qquad K^* = K \setminus \{0\}$ is a countable group under multiplication.

K^* is not finitely generated; $r_0(K^*) = \infty$ for every K. Thus spectral synthesis does not hold on K^*.

Since K^* is countable, local spectral synthesis holds on K^*.

The additive solutions of (1) restricted to K^* constitute a variety V_a.

Theorem (G. Kiss, ML 2012)

In V_a every local exponential polynomial is an exponential polynomial. Consequently, spectral synthesis holds on V_a.

$$\sum_{i=1}^{n} a_i f(x+b_i y) = 0 \qquad (f:\mathbb{C}\to\mathbb{C},\ a_i, b_i \in \mathbb{C}). \tag{1}$$

$\mathbb{Q}(b_1)\dots(b_n) = K, \qquad K^* = K \setminus \{0\}$ is a countable group under multiplication.

K^* is not finitely generated; $r_0(K^*) = \infty$ for every K. Thus spectral synthesis does not hold on K^*.

Since K^* is countable, local spectral synthesis holds on K^*.

The additive solutions of (1) restricted to K^* constitute a variety V_a.

Theorem (G. Kiss, ML 2012)

In V_a every local exponential polynomial is an exponential polynomial. Consequently, spectral synthesis holds on V_a.

The exponential polynomial solutions of (1) are easy to desribe. This gives a complete description of the set of additive solutions of (1).