Convergence of trigonometric series in $L^p (0 < p < 1)$

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Motivation

What necessary and/or sufficient conditions can be given for the L^p-convergence $(0 < p \leq 1)$ of a single

$$f_1(x) \sim \sum_{k \in \mathbb{Z}} c_k e^{ikx}, \quad x \in \mathbb{T} = [-\pi, \pi)$$

or a double

$$f_2(x, y) \sim \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} c_{k\ell} e^{i(kx+\ell y)}, \quad (x, y) \in \mathbb{T}^2$$

Fourier series with the use of the coefficients?

L^1-convergence of a single Fourier series

Theorem (A. S. Belov, 2008)

Assume $f_1 \in L^1(\mathbb{T}^2)$ *and*

$$\|s_n(f_1) - f_1\| \to 0 \quad as \quad n \to \infty$$

i.e.

$$\frac{1}{2\pi}\int_{\mathbb{T}} \Big| \sum_{|k|\le n} c_k e^{ikx} - f_1(x) \Big| dx \to 0 \quad as \quad n \to \infty.$$

Then

$$\sum_{k=[n/2]}^{2n} \frac{|c_k| + |c_{-k}|}{(|k-n|+1)} \to 0 \quad as \quad n \to \infty.$$

A necessary condition is

$$\sum_{k=[n/2]}^{2n} \frac{|c_k| + |c_{-k}|}{(|k-n|+1)} \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently,

$$(c_n + c_{-n}) \ln n \to 0 \quad \text{as} \quad n \to \infty$$

is a necessary and sufficient condition for the L^1-convergence under some additional conditions such as the coefficients are decreasing or general monotone.

L^1-convergence of a double Fourier series

Theorem (F. Móricz, 2010)

Assume $f_2 \in L^1(\mathbb{T}^2)$ *and*

$$\|s_{mn}(f_2) - f_2\| \to 0 \quad \textit{as } m, n \to \infty \textit{ independently,}$$

$$\textit{i.e.} \quad \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} \Big| \sum_{|k| \le m} \sum_{|\ell| \le n} c_{k\ell} e^{i(kx+\ell y)} - f_2(x,y) \Big| dxdy \to 0.$$

Then

$$\sum_{k=[m/2]}^{2m} \sum_{\ell=[n/2]}^{2n} \frac{|c_{k\ell}| + |c_{-k,\ell}| + |c_{k,-\ell}| + |c_{-k,-\ell}|}{(|k-m|+1)(|\ell-n|+1)} \to 0 \textit{ as } m, n \to \infty.$$

Although for decreasing or general monotone coefficients

$$\sum_{k=[m/2]}^{2m} \sum_{\ell=[n/2]}^{2n} \frac{|c_{k\ell}|+|c_{-k,\ell}|+|c_{k,-\ell}|+|c_{-k,-\ell}|}{(|k-m|+1)(|\ell-n|+1)} \to 0 \text{ as } m,n \to \infty$$

implies

$$(c_{mn}+c_{-m,n}+c_{m,-n}+c_{-m,-n}) \ln m \ln n \to 0 \quad \text{as} \quad m,n \to \infty,$$

this does not seem to be a sufficient condition for the L^1-convergence. However, there are some recent calculations to resolve this issue by giving a modified condition.

L^p-convergence $(0 < p < 1)$

For $f_1 \in L^p(\mathbb{T})$, $0 < p < 1$, the L^p-metric is defined by

$$\|f_1\|_{L^p} = \|f_1\|_p = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f_1(x)|^p dx\right)^{1/p}.$$

Similarly, for $f_2 \in L^p(\mathbb{T}^2)$, $0 < p < 1$, the L^p-metric is defined by

$$\|f_2\|_{L^p} = \|f_2\|_p = \left(\frac{1}{4\pi^2}\iint_{\mathbb{T}^2} |f_2(x,y)|^p dxdy\right)^{1/p}.$$

$\|\cdot\|_p$ is not a norm, it does not satisfies the triangle inequality, but it is known as a quasi-norm.

L^p-convergence of a single Fourier series

Theorem (P. Kórus, X. Krasniqi, F. Móricz)

Assume that $f \in L^p(\mathbb{T})$, $0 < p < 1$, *and*

$$\|s_n(f) - f\|_p \to 0 \quad as \quad n \to \infty.$$

Then

$$\left(\sum_{k=[n/2]}^{2n} \frac{|c_k|^p + |c_{-k}|^p}{(|k-n|+1)^{2-p}} \right)^{1/p} \to 0 \quad as \quad n \to \infty.$$

L^p-convergence of a double Fourier series

Theorem (P. Kórus, X. Krasniqi, F. Móricz)

Assume $f \in L^p(\mathbb{T}^2)$, $0 < p < 1$, *and*

$$\|s_{mn}(f) - f\|_p \to 0 \quad \text{as} \;\; m, n \to \infty$$

independently of one another. Then

$$\left(\sum_{k=[m/2]}^{2m} \sum_{\ell=[n/2]}^{2n} \frac{|c_{k\ell}|^p + |c_{-k,\ell}|^p + |c_{k,-\ell}|^p + |c_{-k,-\ell}|^p}{((|k-m|+1)(|\ell-n|+1))^{2-p}} \right)^{1/p} \to 0$$

$$\text{as} \;\; m, n \to \infty.$$

Elements of proofs

In case of L^1-convergence, Hardy's inequality

Lemma

If

$$\varphi(z) = \sum_{k=0}^{\infty} c_k z^k, \ |z| < 1 \ \text{and} \ \varphi \in H^1,$$

then

$$\frac{1}{\pi} \sum_{k=0}^{\infty} \frac{|c_k|}{k+1} \leq \|\varphi\|_1.$$

and the Bernstein–Zygmund inequalities were needed.

Lemma

Let $T_n(x)$ be a trigonometrical polynomial,

$$T_n(x) = \sum_{|k|\le n} c_k e^{ikx}$$

and $\widetilde{T}_n(x)$ is its conjugate,

$$\widetilde{T}_n(x) = \sum_{|k|\le n} (-i \operatorname{sign} k) c_k e^{ikx}.$$

Then

$$\max\left\{\|T_n'\|_1, \|\widetilde{T}_n'\|_1\right\} \le n\|T_n\|_1.$$

In case of L^p-convergence,

Lemma (Hardy, Littlewood, 1927)

If

$$\varphi(z) = \sum_{k=0}^{\infty} c_k z^k, \; |z| < 1 \; \text{ and } \; \varphi \in H^p, \, 0 < p < 1,$$

then

$$\sum_{k=0}^{\infty} \frac{|c_k|^p}{(k+1)^{2-p}} \leq K_p \|\varphi\|_p^p.$$

Lemma (V. Arestov, 1981)

Let $T_n(x)$ be a trigonometrical polynomial of order n and $0 < p < 1$. Then the inequality

$$\|T_n^{(r)}\|_p \leq n^r \|T_n\|_p$$

holds true.

and

Lemma (K. Runovski, H.-J. Schmeisser, 2004)

Let $T_n(x)$ be a trigonometrical polynomial of order n and $r > 0$. Then the inequality

$$\|\widetilde{T}_n^{(r)}\|_p \le K_{p,r}\, n^r \|T_n\|_p$$

holds true if and only if $p > 1/(r+1)$.

are needed.

For single Fourier series

Lemma

For every $n \in \mathbb{N}$ *and* $0 < p < 1$, *we have*

$$\min\left\{\left\|\sum_{k=0}^{n} c_k e^{ikx}\right\|_p^p, \left\|\sum_{k=0}^{n} c_k e^{-ikx}\right\|_p^p\right\}$$

$$\geq K_p \max\left\{\sum_{k=0}^{n} \frac{|c_k|^p}{(k+1)^{2-p}}, \sum_{k=0}^{n} \frac{|c_k|^p}{(n-k+1)^{2-p}}\right\}.$$

Lemma

For $0 \le n < \nu$ and $0 < p < 1$, we have

$$\left\| \sum_{k=n+1}^{\nu} k^r c_k e^{ikx} \right\|_p \ge K_p \max \left\{ \left(\sum_{k=n+1}^{\nu} (k-n)^{p-2} \left| k^r c_k \right|^p \right)^{1/p}, \left(\sum_{k=n+1}^{\nu} (\nu - k + 1)^{p-2} \left| k^r c_k \right|^p \right)^{1/p} \right\}$$

$$\left\| \sum_{k=n+1}^{\nu} (-k)^r c_{-k} e^{-ikx} \right\|_p \ge K_p \max \left\{ \left(\sum_{k=n+1}^{\nu} (k-n)^{p-2} \left| k^r c_{-k} \right|^p \right)^{1/p}, \left(\sum_{k=n+1}^{\nu} (\nu - k + 1)^{p-2} \left| k^r c_{-k} \right|^p \right)^{1/p} \right\}.$$

Lemma

For all $-1 \leq n < \nu$, $r = 1, 2, \ldots,$ *and* $1/(r+1) < p < 1$, *we have*

$$\max\left\{\left\|\sum_{k=n+1}^{\nu} k^r c_k e^{ikx}\right\|_p, \left\|\sum_{k=n+1}^{\nu} (-k)^r c_{-k} e^{-ikx}\right\|_p\right. \\ \leq K_{p,r}\nu^r \left\|s_\nu - s_n\right\|_p.$$

For double Fourier series, by more calculation, we have analogous (more complicated) lemmas using the Hardy–Littlewood, Arestov and Runovski–Schmeisser theorems mentioned before (more times).

Lemma

$$\min\left\{\left\|\sum_{k=0}^{m}\sum_{\ell=0}^{n} c_{k\ell}e^{i(kx+\ell y)}\right\|_p, \left\|\sum_{k=0}^{m}\sum_{\ell=0}^{n} c_{k\ell}e^{i(-kx+\ell y)}\right\|_p,\right.$$
$$\left.\left\|\sum_{k=0}^{m}\sum_{\ell=0}^{n} c_{k\ell}e^{i(kx-\ell y)}\right\|_p, \left\|\sum_{k=0}^{m}\sum_{\ell=0}^{n} c_{k\ell}e^{i(-kx-\ell y)}\right\|_p\right\}$$
$$\geq K_p \max\left\{\left(\sum_{k=0}^{m}\sum_{\ell=0}^{n}[(k+1)(\ell+1)]^{p-2}\left|c_{k\ell}\right|^p\right)^{1/p},\right.$$
$$\left(\sum_{k=0}^{m}\sum_{\ell=0}^{n}[(m-k+1)(\ell+1)]^{p-2}\left|c_{k\ell}\right|^p\right)^{1/p},$$
$$\left(\sum_{k=0}^{m}\sum_{\ell=0}^{n}[(k+1)(n-\ell+1)]^{p-2}\left|c_{k\ell}\right|^p\right)^{1/p},$$
$$\left.\left(\sum_{k=0}^{m}\sum_{\ell=0}^{n}[(m-k+1)(n-\ell+1)]^{p-2}\left|c_{k\ell}\right|^p\right)^{1/p}\right\}.$$

Lemma

$$\left\| \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} k^{r_1} \ell^{r_2} c_{k\ell} e^{i(kx+\ell y)} \right\|_p$$

$$\geq K_{p,r_1,r_2} \max \left\{ \left(\sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} [(k-m)(\ell-n)]^{p-2} \left|k^{r_1} \ell^{r_2} c_{k\ell}\right|^p \right)^{1/p}, \right.$$

$$\left(\sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} [(\mu-k+1)(\ell-n)]^{p-2} \left|k^{r_1} \ell^{r_2} c_{k\ell}\right|^p \right)^{1/p},$$

$$\left(\sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} [(k-m)(\nu-\ell+1)]^{p-2} \left|k^{r_1} \ell^{r_2} c_{k\ell}\right|^p \right)^{1/p},$$

$$\left. \left(\sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} [(\mu-k+1)(\nu-\ell+1)]^{p-2} \left|k^{r_1} \ell^{r_2} c_{k\ell}\right|^p \right)^{1/p} \right\},$$

and three other analogous inequalities hold involving $c_{-k,\ell}$, $c_{k,-\ell}$, and $c_{-k,-\ell}$.

Lemma

For all $(m, n) \in \mathbb{N}^2$ *and* $\max\{1/r_1, 1/r_2\} < p < 1$ *we have*

$$\max\Bigg\{\left\|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\sum_{|k|\le m}\sum_{|\ell|\le n} c_{k\ell}e^{i(kx+\ell y)}\right\|_p,$$
$$\left\|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\sum_{|k|\le m}\sum_{|\ell|\le n}(-i\,\mathrm{sign}\,k)c_{k\ell}e^{i(kx+\ell y)}\right\|_p,$$
$$\left\|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\sum_{|k|\le m}\sum_{|\ell|\le n}(-i\,\mathrm{sign}\,\ell)c_{k\ell}e^{i(kx+\ell y)}\right\|_p,$$
$$\left\|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\sum_{|k|\le m}\sum_{|\ell|\le n}(-i\,\mathrm{sign}\,k)(-i\,\mathrm{sign}\,\ell)c_{k\ell}e^{i(kx+\ell y)}\right\|_p\Bigg\}$$
$$\le K_{p,r_1,r_2}m^{r_1}n^{r_2}\left\|\sum_{|k|\le m}\sum_{|\ell|\le n}c_{k\ell}e^{i(kx+\ell y)}\right\|_p.$$

Lemma

For all $-1 \le m < \mu$, $-1 \le n < \nu$ *and* $\max\{1/r_1, 1/r_2\} < p < 1$ *we have*

$$\max\left\{\left\|\sum_{k=m+1}^{\mu}\sum_{\ell=n+1}^{\nu} k^{r_1}\ell^{r_2} c_{k\ell} e^{i(kx+\ell y)}\right\|_p, \left\|\sum_{k=m+1}^{\mu}\sum_{\ell=n+1}^{\nu} (-k)^{r_1}\ell^{r_2} c_{-k\ell} e^{i(-kx+\ell y)}\right\|_p,\right.$$
$$\left\|\sum_{k=m+1}^{\mu}\sum_{\ell=n+1}^{\nu} k^{r_1}(-\ell)^{r_2} c_{k,-\ell} e^{i(kx-\ell y)}\right\|_p,$$
$$\left.\left\|\sum_{k=m+1}^{\mu}\sum_{\ell=n+1}^{\nu} (-k)^{r_1}(-\ell)^{r_2} c_{-k,-\ell} e^{i(-kx-\ell y)}\right\|_p\right\}$$
$$\le K_{p,r_1,r_2}\mu^{r_1}\nu^{r_2}\left\|s_{\mu\nu} - s_{m\nu} - s_{\mu n} + s_{mn}\right\|_p.$$

Open questions

What sufficient conditions can be given for the L^p-convergence ($0 < p < 1$) of a single or a double Fourier series which meet the previous necessary conditions somehow?

Can we give other/better necessary conditions similarly to the L^1 case?