Observability of square plates on small sets

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Outline

1. Elementary results
2. Ingham–Beurling type theorems
3. Ingham–Kahane type theorems

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Internal observability of plates

We consider the small transversal vibrations of a hinged plate:

$$\begin{cases} u'' + \Delta^2 u = 0 & \text{in} \quad \mathbb{R} \times \Omega, \\ u = \Delta u = 0 & \text{on} \quad \mathbb{R} \times \Gamma, \\ u(0) = u_0, \quad u'(0) = u_1 & \text{in} \quad \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is some given bounded domain.

We observe the vibrations on some subset $S \subset \Omega$ during some time T.

Is the linear map $(u_0, u_1) \mapsto u|_{(0,T)\times S}$ one-to-one? If yes, we say that $(0, T) \times S$ is an *observability set*.

Square plates

We consider henceforth a square plate $\Omega = (0, \pi) \times (0, \pi)$. Then the solutions have the form

$$u(t, x, y) = \sum_{k,n=1}^{\infty} \left(a_{kn} e^{i(k^2+n^2)t} + b_{kn} e^{-i(k^2+n^2)t} \right) \sin kx \sin ny,$$

and the observability is equivalent to

$$u = 0 \quad \text{on} \quad (0, T) \times S \Longrightarrow a_{kn} = b_{kn} = 0 \quad \text{for all} \quad k, n.$$

An elementary result

Proposition

$(0,2\pi)\times\Omega$ *is an observability set.*

Proof.

The series

$$u(t,x,y)=\sum_{k,n=1}^{\infty}\left(a_{kn}e^{i(k^2+n^2)t}+b_{kn}e^{-i(k^2+n^2)t}\right)\sin kx\sin ny$$

is orthogonal in $L^2((0,2\pi)\times\Omega)$, so that

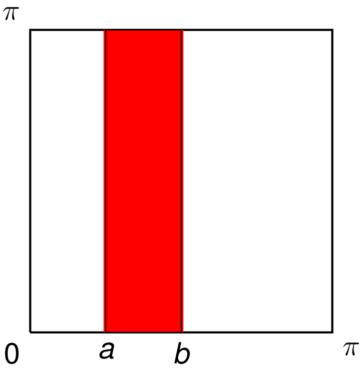
$$\int_{(0,2\pi)\times\Omega}|u(t,x,y)|^2\ dx\ dy\ dt=\frac{\pi^3}{2}\sum_{k,n=1}^{\infty}\left(|a_{kn}|^2+|b_{kn}|^2\right).$$

□

An improvement

Proposition

$(0, 2\pi) \times (a, b) \times (0, \pi)$ *is an observability set for any* $0 \leq a < b \leq \pi$.



Proof of the improvement

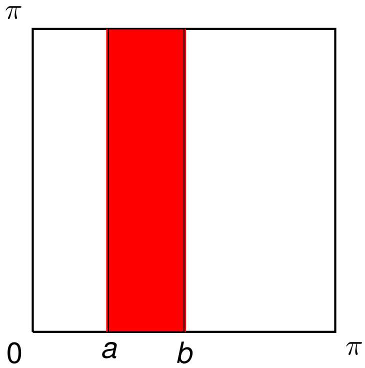
$$\begin{aligned}
&\int_{(0,2\pi)\times S} |u(t,x,y)|^2 \ dx\, dy\, dt \\
&= \frac{\pi}{2}\sum_{n=1}^{\infty}\int_a^b \int_0^{2\pi} \Big|\sum_{k=1}^{\infty}\left(a_{kn}e^{i(k^2+n^2)t} + b_{kn}e^{-i(k^2+n^2)t}\right) \\
&\qquad\qquad\qquad\qquad \times \sin kx\Big|^2 \ dt\, dx \\
&= \pi^2 \sum_{k,n=1}^{\infty}\left(|a_{kn}|^2 + |b_{kn}|^2\right)\int_a^b \sin^2 kx\, dx \\
&\asymp \sum_{k,n=1}^{\infty}\left(|a_{kn}|^2 + |b_{kn}|^2\right).
\end{aligned}$$

We have used the relations $0 < \int_a^b \sin^2 kx \to (b-a)/2 > 0$.

Haraux's theorem

Theorem

$(0, T) \times (a, b) \times (0, \pi)$ *is an observability set for any* $T > 0$ *and* $0 \leq a < b \leq \pi$.



Proof of Haraux's theorem

$$\int_{(0,T)\times S} |u(t,x,y)|^2 \ dx\ dy\ dt$$

$$= \frac{\pi}{2}\sum_{n=1}^{\infty}\int_a^b\int_0^T\Big|\sum_{k=1}^{\infty}\left(a_{kn}e^{i(k^2+n^2)t}+b_{kn}e^{-i(k^2+n^2)t}\right)$$

$$\times \sin kx\Big|^2\ dt\ dx$$

$$\asymp \sum_{k,n=1}^{\infty}\left(|a_{kn}|^2+|b_{kn}|^2\right)\int_a^b \sin^2 kx\ dx$$

$$\asymp \sum_{k,n=1}^{\infty}\left(|a_{kn}|^2+|b_{kn}|^2\right).$$

We have used a generalization of Parseval's equality:

Crucial inequality of the proof

We have used the following estimate for each fixed n and x:

$$\int_0^T \left| \sum_{k=1}^{\infty} \left(a_{kn} e^{i(k^2+n^2)t} + b_{kn} e^{-i(k^2+n^2)t} \right) \times \sin kx \right|^2 dt$$
$$\asymp \sum_{k=1}^{\infty} \left(|a_{kn}|^2 + |b_{kn}|^2 \right) \sin^2 kx$$

or equivalently

$$\int_0^T \left| \sum_{k=1}^{\infty} \left(a'_{kn} e^{i(k^2+n^2)t} + b'_{kn} e^{-i(k^2+n^2)t} \right) \right|^2 dt \asymp \sum_{k=1}^{\infty} \left(|a'_{kn}|^2 + |b'_{kn}|^2 \right).$$

They are not obvious for $T < 2\pi$.

Ingham's inequality, 1936

Theorem

If a family $\{\omega_k : k \in K\}$ of real numbers satisfies the gap condition

$$\gamma = \gamma(K) := \inf \{|\omega_k - \omega_n| \ : \ k \neq n\} > 0,$$

then we have

$$\int_{-R}^{R} \Big| \sum_{k \in K} x_k e^{i\omega_k t} \Big|^2 \, dt \asymp \sum_{k \in K} |x_k|^2$$

for every $R > \pi/\gamma$.

Remark

For $\{\omega_k\} = \mathbb{Z}$ this follows from Parseval's equality (even for $R = \pi/\gamma$).

Beurling's improvement (equivalent form), 1958

Theorem

Let $\{\omega_k : k \in K\} \subset \mathbb{R}$ *satisfy the gap condition*

$$\gamma = \gamma(K) := \inf_{k \neq n} |\omega_k - \omega_n| > 0.$$

If

$$R > \frac{\pi}{\gamma(K_1)} + \cdots + \frac{\pi}{\gamma(K_m)}$$

for some finite partition $K = K_1 \cup \cdots \cup K_m$ *of* K*, then*

$$\int_{-R}^{R} \Big| \sum_{k \in K} x_k e^{i\omega_k t} \Big|^2 \, dt \asymp \sum_{k \in K} |x_k|^2 .$$

Example

Let $\{\omega_k\} = \{\pm k^2 \ : \ k = 0, 1, \ldots\} = \{0, \pm 1, \pm 4, \pm 9, \ldots\}$. For each fixed $m = 1, 2, \ldots$, the partition

$$\{\omega_k\} = \left\{\pm k^2 \ : \ k = m, m+1, \ldots\right\} \cup \bigcup_{k=-m+1}^{m-1} \{k\}$$

satisfies

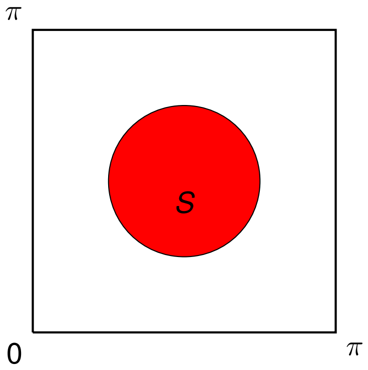
$$\frac{\pi}{\gamma(K_1)} + \cdots + \frac{\pi}{\gamma(K_{2m})} = \frac{\pi}{2m+1} + \frac{\pi}{\infty} + \cdots + \frac{\pi}{\infty} \to 0$$

as $m \to \infty$.

A theorem of Jaffard

Proposition

$(0, T) \times S$ is an observability set for any open subset S and $T > 0$.



Multidimensional Ingham type theorem

Theorem

(Baiocchi, K., Loreti) If a family $\{\omega_k : k \in K\}$ of vectors in $\mathbb{R}^N$ satisfies the gap condition

$$\gamma = \gamma(K) := \inf_{k \neq n} |\omega_k - \omega_n| > 0,$$

then we have

$$\int_{B_R} \left| \sum_{k=-\infty}^{\infty} x_k e^{i\omega_k \cdot t} \right|^2 dt \asymp \sum_{k=-\infty}^{\infty} |x_k|^2$$

for every $R > \frac{\sqrt{\mu}}{\gamma}$ where B_R denotes the open ball of radius R in $\mathbb{R}^N$ and μ denotes the first Dirichlet eigenvalue of $-\Delta$ in the unit ball B_1 of $\mathbb{R}^N$.

Comments on the theorem

- In the one-dimensional case $N = 1$ we recover Ingham's theorem: in $B_1 = (-1, 1)$ the first eigenfunction is

$$H(x) = \cos(\pi x).$$

Since

$$-H'' = \pi^2 H,$$

we have

$$\sqrt{\mu} = \pi.$$

- In case $N > 1$ the optimality of the condition $R > \frac{\sqrt{\mu}}{\gamma}$ is an open question.

Kahane type improvement

Theorem

Let $\{\omega_k : k \in K\} \subset \mathbb{R}^N$ *satisfy the gap condition*

$$\gamma = \gamma(K) := \inf_{k \neq n} |\omega_k - \omega_n| > 0.$$

If

$$R > \frac{\sqrt{\mu}}{\gamma(K_1)} + \cdots + \frac{\sqrt{\mu}}{\gamma(K_m)}$$

for some finite partition $K = K_1 \cup \cdots \cup K_m$ *of* K*, then*

$$\int_{B_R} \Big| \sum_{k=-\infty}^{\infty} x_k e^{i\omega_k \cdot t} \Big|^2 \, dt \asymp \sum_{k=-\infty}^{\infty} |x_k|^2 .$$

Sparse sets

A set $\{\omega_k : k \in K\} \subset \mathbb{R}^N$ is called *sparse* if for each $\varepsilon > 0$ there exists a finite cover $K \subset K_1 \cup \cdots \cup K_m$ of K satisfying

$$\frac{1}{\gamma(K_1)} + \cdots + \frac{1}{\gamma(K_m)} < \varepsilon.$$

Example

The set $\{\omega_k\} = \left\{k^2 \;:\; k \in \mathbb{Z}\right\} = \{0, 1, 4, 9, \ldots\} \subset \mathbb{Z}$ is sparse.

Remark

The union of finitely many sparse sets is also sparse.

Jaffard's main lemma

Corollary

If $\{\omega_k : k \in K\} \subset \mathbb{R}^N$ *is a sparse set having a uniform gap, then*

$$\int_{B_R} \left| \sum_{k=-\infty}^{\infty} x_k e^{i\omega_k \cdot t} \right|^2 dt \asymp \sum_{k=-\infty}^{\infty} |x_k|^2$$

for every $R > 0$.

Theorem

(Jaffard) The set

$$X := \left\{ (k_1, k_2, k_1^2 + k_2^2) : k_1, k_2 \in \mathbb{Z} \right\} = \left\{ (k, |k|^2) : k \in \mathbb{Z}^2 \right\} \subset \mathbb{Z}^3$$

is sparse.

Proof of Jaffard's theorem I

Lemma

If $k \in \mathbb{Z}^2$ is a non-zero vector and I a bounded interval, then

$$X(k, I) := \left\{(m, |m|^2) \ : \ m \in \mathbb{Z}^2, m \cdot k \in I\right\}$$

is sparse.

Proof.

We may assume by a finite partition that $|I| < 1$. Fix an arbitrary $(m, |m|^2) \in Z(k, I)$. If $(m + n, |m + n|^2)$ is another element, then $n \perp k$ because $n \cdot k \in \mathbb{Z}$ and $|n \cdot k| < 1$. Hence

$$X(k, I) = \left\{(m + in', \left|m + in'\right|^2) \ : \ i \in \mathbb{Z}\right\}$$

for some n', and we may argue as for $\left\{k^2 \ : \ k \in \mathbb{Z}\right\}$. □

Proof of Jaffard's theorem II

Fix a large number $c > 1$. Set

$$K := \left\{ k \in \mathbb{Z}^2 \ : \ 0 < |k| < c \right\}$$

and

$$X(c) := \left\{ (m, |m|^2) \in X \ : \ |m \cdot k| \geq c^2 \quad \text{for all} \quad k \in K \right\}.$$

Observe that

$$X \setminus X(c) = \bigcup_{k \in K} X(k, (-c^2, c^2))$$

is sparse. The theorem will follow from the relation

$$\lim_{c \to \infty} \gamma(X(c)) = \infty.$$

Proof of Jaffard's theorem III

Lemma

If $c \geq 1$, $K := \{k \in \mathbb{Z}^2 \ : \ 0 < |k| < c\}$ *and*

$$X(c) := \left\{(m, |m|^2) \in X \ : \ |m \cdot k| \geq c^2 \quad \text{for all} \quad k \in K\right\},$$

then $\gamma(X(c)) > c$.

Proof.

If $(m, |m|^2)$ and $(m + k, |m + k|^2)$ are two distinct elements, then either $|k| \geq c$, or

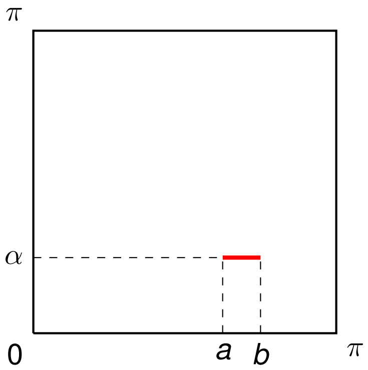
$$\left| |m + k|^2 - |m|^2 \right| \geq 2\,|m \cdot k| - |k|^2 \geq 2c^2 - c^2 = c^2 \geq c.$$

□

Observability by segments

Theorem

(K.–Loreti, 2013) $(0, T) \times S$ *is an observability set for any small horizontal segment* $S = (a, b) \times \{\alpha\}$ *and* $T > 0$*, where* $0 < \alpha/\pi < 1$ *is irrational.*



Main tool

We use the following improvement of Jaffard's estimates:

Theorem

(Tenenbaum–Tucsnak, 2009) The set

$$X := \left\{(k_1, k_1^2 + k_2^2) : k_1, k_2 \in \mathbb{Z}\right\} \subset \mathbb{Z}^2$$

is sparse.

Remark

It would be interesting to find an elementary proof.

Observability by segments. Proof if α/π is irrational.

We have

$$\int_0^T \int_a^b |u(t,x,\alpha)|^2 \ dx\ dt$$

$$= \int_0^T \int_a^b \Big| \sum_{k_1,k_2=1}^{\infty} \left(a_k e^{i|k|^2 t} + b_k e^{-i|k|^2 t} \right) \sin k_1 x \sin k_2 \alpha \Big|^2 \ dt\ dx$$

$$= \frac{1}{4} \int_0^T \int_a^b \Big| \sum_{k_1,k_2=1}^{\infty} \left(a_k e^{i|k|^2 t} + b_k e^{-i|k|^2 t} \right)$$

$$\times \left(e^{ik_1 x} - e^{-ik_1 x} \right) \sin k_2 \alpha \Big|^2 \ dt\ dx$$

$$\asymp \sum_{k_1,k_2=1}^{\infty} \left(|a_k|^2 + |b_k|^2 \right) \sin^2 k_2 \alpha.$$

We conclude by observing that $\sin^2 k_2 \alpha > 0$ for all k_2.

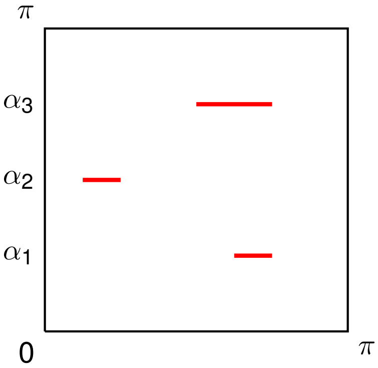
When α/π is a quadratic irrational number

If α/π is a quadratic irrational number, then we have a better estimate:

$$\begin{aligned}
&\int_0^T \int_a^b |u(t,x,\alpha)|^2 \ dx\ dt \\
&\asymp \sum_{k_1,k_2=1}^{\infty} \left(|a_k|^2 + |b_k|^2\right) \sin^2 k_2\alpha \\
&\ge 4 \sum_{k_1,k_2=1}^{\infty} \left(|a_k|^2 + |b_k|^2\right) \operatorname{dist}\left(k_2\alpha/\pi, \mathbb{Z}\right)^2 \\
&\ge c \sum_{k_1,k_2=1}^{\infty} \left(|a_k|^2 + |b_k|^2\right) k_2^{-2} \\
&\ge c \sum_{k_1,k_2=1}^{\infty} \left(|a_k|^2 + |b_k|^2\right) |k|^{-2} .
\end{aligned}$$

Observation on several segments I

We observe simultaneously on several segments $(a_j, b_j) \times \{\alpha_j\}$, $j = 1, \ldots, M$.



Observation on several segments II

Assume that $\alpha_1,\dots,\alpha_M$, π are linearly independent over the field of rational numbers, and that they belong to a real algebraic field of degree $M+1$. Then we have

$$\sum_{j=1}^{M}\int_0^T\int_{a_j}^{b_j}\left|u(t,x,\alpha_j)\right|^2\,dx\,dt \asymp \sum_{k_1,k_2=1}^{\infty}\left(|a_k|^2+|b_k|^2\right)\sum_{j=1}^{M}\sin^2 k_2\alpha_j$$

$$\ge 4\sum_{k_1,k_2=1}^{\infty}\left(|a_k|^2+|b_k|^2\right)\sum_{j=1}^{M}\operatorname{dist}\left(k_2\alpha_j/\pi,\mathbb{Z}\right)^2$$

$$\ge c\sum_{k_1,k_2=1}^{\infty}\left(|a_k|^2+|b_k|^2\right)k_2^{-2/M}\ge c\sum_{k_1,k_2=1}^{\infty}\left(|a_k|^2+|b_k|^2\right)|k|^{-2/M}.$$

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