In which domains can one do Fourier Analysis?

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Standard Fourier Analysis

$L^2([0,1])$ has an orthogonal basis of complex exponentials

$$e_n(x) = e^{2\pi i n \cdot x}, \quad n \in \mathbb{Z}.$$

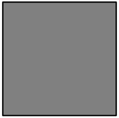
Frequencies:

$$\cdots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad \cdots$$

Functions $f \in L^2([0,1])$ can be written

$$f(x) = \sum_{n \in \mathbb{Z}} f_n \, e_n(x).$$

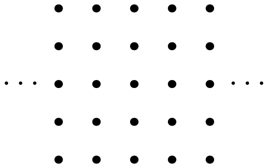
In higher dimension as well



$L^2([0,1]^2)$ has an orthogonal basis of complex exponentials

$$e_{(m,n)}(x,y) = e^{2\pi i (m,n)\cdot(x,y)}, \quad m, n \in \mathbb{Z}.$$

Frequencies: the lattice $\mathbb{Z}^2$



Functions $f \in L^2([0,1]^2)$ can be written

$$f(x,y) = \sum_{(m,n)\in\mathbb{Z}^2} f_{(m,n)}\, e_{(m,n)}(x,y).$$

Unusual Fourier Analysis

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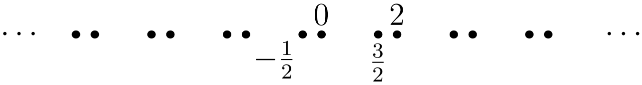
$L^2([0, \frac{1}{2}] \cup [1, \frac{3}{2}])$ also has an orthogonal basis of complex exponentials

$$e^{2\pi i(2n)\cdot x}, \quad e^{2\pi i(2n-\frac{1}{2})\cdot x}, \quad n \in \mathbb{Z}.$$

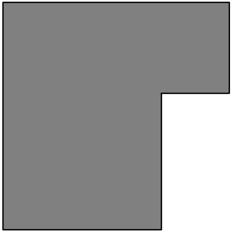
Frequencies:

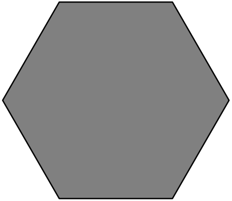


Functions $f \in L^2([0, \frac{1}{2}] \cup [1, \frac{3}{2}])$ can be written

$$f(x) = \sum_{n \in \mathbb{Z}} f_n \, e^{2\pi i(2n)\cdot x} + f'_n \, e^{2\pi i(2n-\frac{1}{2})\cdot x}.$$

Unusual Fourier Analysis in higher dimension





These also have an orthogonal basis of exponentials (lattice frequencies).

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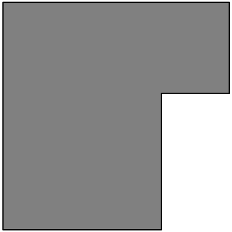

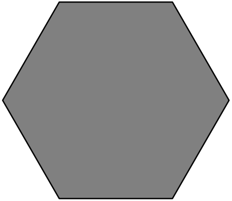

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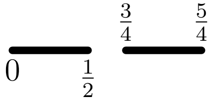
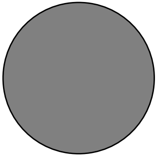
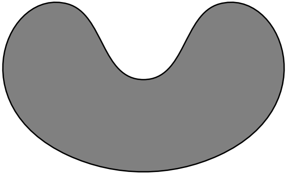
But NOT these:





$[0, \frac{1}{2}] \cup [\frac{3}{4}, \frac{5}{4}]$ and the disk have no orthogonal basis of exponentials.

Which domains are spectral



Domain $\Omega \subseteq \mathbb{R}^d$ is *spectral* if it has an orthogonal basis of exponentials

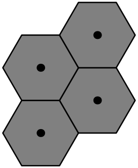
$$e^{2\pi i \lambda \cdot x}, \quad \lambda \in \Lambda.$$

The set of *frequencies* $\Lambda \subseteq \mathbb{R}^d$ is a *spectrum* of Ω.

A spectral set may have many different spectra.

The Fuglede Conjecture (1974)

"Ω is spectral $\Longleftrightarrow$ it can tile space by translations"



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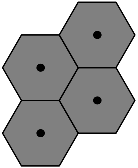

Ω tiles when translated at the locations T if

$$\sum_{t \in T} \mathbf{1}_\Omega(x-t) = 1, \quad \text{for a.e. } x.$$

Its T translates cover $\mathbb{R}^d$ exactly (except for measure 0).

Orhogonality and completeness via the Fourier Transform

Our Fourier Transform: $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x)\, dx$.

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- So $\lambda \perp \mu$ (i.e. $e^{2\pi i \lambda \cdot x} \perp e^{2\pi i \mu \cdot x}$) $\iff \quad \lambda - \mu \in Z\left(\widehat{\mathbf{1}_\Omega}\right)$=zeros of $\widehat{\mathbf{1}_\Omega}$.

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- Fuglede's Conjecture in geometric language makes more sense:

$$\Omega \text{ tiles at level } 1 \iff \left|\widehat{\mathbf{1}_\Omega}\right|^2 \text{ tiles at level } |\Omega|^2.$$

Tiling in Fourier space

Define the measure $\delta_T = \sum_{t \in T} \delta_t$ (unit point masses at $t \in T$).

- $\sum_{t \in T} f(x - t) = \text{const. a.e.} \iff f * \delta_T = \text{const.}$

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- *Poisson Summation Formula*:

$$\widehat{\delta_T} = \frac{1}{|\det A|}\delta_{T^*}$$

  implies

$$f * \delta_T = \text{const.} \iff \widehat{f} \equiv 0 \text{ on } T^* \setminus 0.$$

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- Having T^* as spectrum

$$
\begin{aligned}
&\iff \left|\widehat{\mathbf{1}_\Omega}\right|^2 \text{ tiles with } T^* \\
&\iff T \setminus \{0\} \subseteq (\Omega - \Omega)^c \\
&\iff (T - T) \cap (\Omega - \Omega) = \{0\} \\
&\iff \Omega + T \text{ is a packing (i.e. no overlaps)} \\
&\iff \Omega + T \text{ is a tiling (by volume-density matching \& periodicity).}
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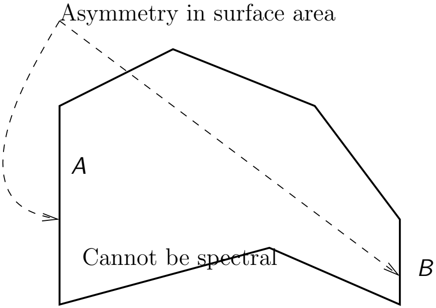
- Conjecture true for convex bodies in $\mathbb{R}^2$ (Iosevich, Katz and Tao, 2003).

  $\Longrightarrow$ only parellelograms and symmetric hexagons are spectral among planar convex sets.

Fuglede Conjecture: Positive results for general domains

For each normal direction of a spectral polytope the same area measure looks forward and backward.
(K. and Papadimitrakis, 2002)

Same is obviously true for polytopes that are tiles.



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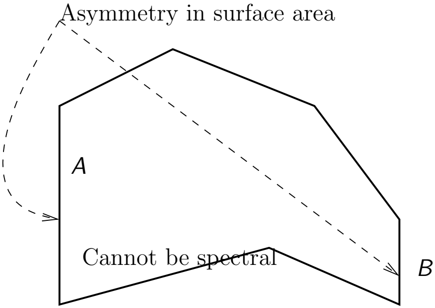

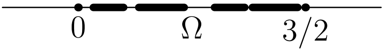

If $\Omega \subseteq (0, \frac{3}{2} - \epsilon)$ and $|\Omega| = 1$

$\Longrightarrow$ conjecture true for Ω (K. and Łaba, 2001).

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- First construct counterexamples in finite groups:
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- In the group $\mathbb{Z}_2^n$ orthogonal exponentials (characters) on

  $$\Omega = \{e_1, e_2, \ldots, e_n\}, \quad (\{e_j\} \text{ a "standard basis"}),$$

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- For example:

  A 12×12 Hadamard matrix gives a spectral set of size 12 in $\mathbb{Z}_2^{12}$.

  Not a tile for divisibility reasons.

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- Conjecture still open in both directions for $d = 1, 2$.
- May be true for all convex bodies.

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  Harmonic Analysis proof.

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Question: Are one-dimensional spectra periodic?

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- If Ω is a finite union of intervals and Λ is a spectrum of Ω
  $\Longrightarrow \Lambda$ is periodic (Bose and Madan, 2010 and K. 2011).
- If Ω is any bounded set and Λ is a spectrum of Ω
  $\Longrightarrow \Omega$ is periodic (Iosevich and K., 2011).

We describe the proof of the last result in some detail.

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$$\Lambda \cap [\lambda, \lambda + w], \quad \lambda \in \Lambda,$$
are finitely many.
- We view $\Lambda = \{\ldots, \lambda_{-1}, \lambda_0 = 0, \lambda_1, \ldots\}$ as a *double sequence of symbols*
$(\lambda_{j+1} - \lambda_j)_{j\in\mathbb{Z}}$.
- We identify Λ with an element of $\Sigma^{\mathbb{Z}}$,
Σ a finite alphabet of the Λ-gaps allowed.

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  Similarly by left half-lines.
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- Compactness (diagonal argument) $\Longrightarrow$

if X is determined by left half-lines and by right half-lines then X is determined by a finite window size.

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$$x^n_{-n} = y^n_{-n}, x^n_{-n+1} = y^n_{-n+1}, \ldots, x^n_{-1} = y^n_{-1}, \text{ but } x^n_0 \neq y^n_0.$$

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- Passing to subsequence: $\exists x, y \in X : x^n \to x, y^n \to y$.
- Then $x_n = y_n$ for $n < 0$, $x_0 \neq y_0$ (contradiction).

Pigeonhole principle $\Longrightarrow$

If X is determined by windows of size w then each $x \in X$ is periodic of period $\leq |\Sigma|^w$.

Symbolic sequences with a spectral gap

- Identify Λ with the double sequence $(\lambda_{n+1} - \lambda_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$.
- $\sum_{\lambda\in\Lambda} \left|\widehat{\mathbf{1}_\Omega}\right|^2 (x-\lambda) \equiv 1 \Longrightarrow \operatorname{supp}\widehat{\delta_\Lambda} \subseteq \{0\} \cup \{\mathbf{1}_\Omega * \mathbf{1}_{-\Omega} = 0\}$

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- For some $a > 0$ we have a *spectral gap*: $\operatorname{supp} \widehat{\delta_\Lambda} \cap (0, a) = \emptyset$ (*).
- Let $X \subseteq \Sigma^{\mathbb{Z}}$ be all Λ satisfying (*). X is
  (i) shift-invariant (obvious) and
  (ii) closed.
- *Proof.* Let $X \ni \Lambda^n \to \Lambda$ and $\phi \in C^\infty(0, a)$. Then $\widehat{\delta_{\Lambda^n}}(\phi) = 0$.

$$\widehat{\delta_\Lambda}(\phi) = \delta_\Lambda(\widehat{\phi}) = \sum_{\lambda\in\Lambda} \widehat{\phi}(\lambda) = \lim_n \sum_{\lambda\in\Lambda^n} \widehat{\phi}(\lambda)$$
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Determined by half-lines

- Suppose $\Lambda^1, \Lambda^2 \in X$ (with $\Lambda^1_0 = \Lambda^2_0 = 0$) are identical to the left of 0:

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- Define $\mu = \delta_{\Lambda^1} - \delta_{\Lambda^2}$. Then
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- Take $\psi \in C^\infty(-\frac{a}{10}, \frac{a}{10})$ with $\widehat{\psi} > 0$, define bounded measure $\nu = \widehat{\psi}\mu$. Then $\operatorname{supp} \widehat{\nu} = \operatorname{supp} (\psi * \widehat{\mu})$ and

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- By the F. & M. Riesz Theorem (Uncertainty Principle): $\widehat{\nu}$ vanishes on set of 0 measure. This contradicts the spectral gap of ν, so $\nu \equiv 0$ and $\mu \equiv 0$ and $\Lambda^1 = \Lambda^2$.

Periodicity. Conclusion of the argument.

- X is determined by half-lines
  $\Longrightarrow X$ is determined by some finite window size w.
- Therefore any $\Lambda \in X$ is periodic with period $\leq |\Sigma|^w$.