Heisenberg Uniqueness pairs

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Bordeaux

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Joint work with K. Kellay

Heisenberg Uniqueness Pairs

- μ : finite measure on $\mathbb{R}^2$ $\widehat{\mu}(x,y) = \int_{\mathbb{R}^2} e^{i(sx+ty)}\,\mathrm{d}\mu(s,t)$.
- Γ : finite union of disjoint curves
- $\mathcal{M}(\Gamma)$: measures supported on Γ
- $\mathcal{AC}(\Gamma)$: $\mu \in \mathcal{M}(\Gamma)$ absolutely continuous w.r.t arc length.
- $\Lambda \subset \mathbb{R}^2$: set of lines

Definition

(Γ,Λ) Heisenberg Uniqueness Pair (HUP) if $\mu \in \mathcal{AC}(\Gamma)$, $\widehat{\mu} = 0$ on $\Lambda \Rightarrow \mu = 0$.

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- N. Lev *Uniqueness theorems for Fourier transforms.* Bull. Sci. Math., 135 (2011), 134-140.
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HUP 2

Inv 1 Fix $(s_0, t_0), (x_0, y_0) \in \mathbb{R}^2$. Then (Γ, Λ) is a HUP if and only if $(\Gamma - (s_0, t_0), \Lambda - (x_0, y_0))$ is a HUP.

Inv 2 Fix T a linear invertible transformation $\mathbb{R}^2 \to \mathbb{R}^2$ and denote by T^* its adjoint. Then (Γ, Λ) is a HUP if and only if $(T^{-1}(\Gamma), T^*(\Lambda))$ is a HUP.

If $\Gamma = \{s, \gamma(s), s \in I\}$ (Γ, λ) HUP $\Leftrightarrow f \in L^1(I)$ s.t.

$$\forall (x, y) \in \Lambda \quad \int_I f(s) e^{-i(sx + \gamma(s)y)} \, ds = 0 \Rightarrow f = 0$$

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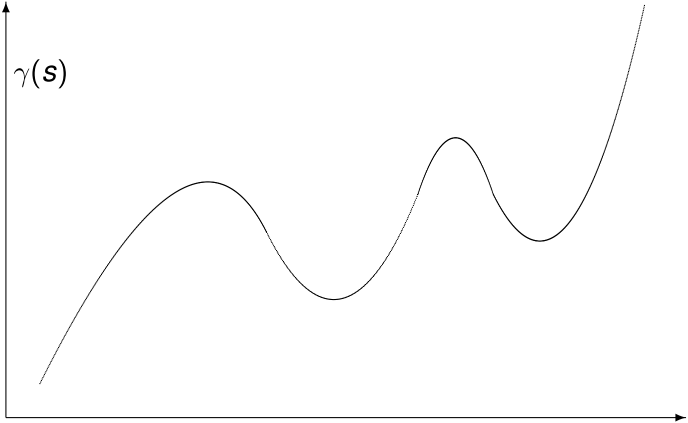
Basic lemma



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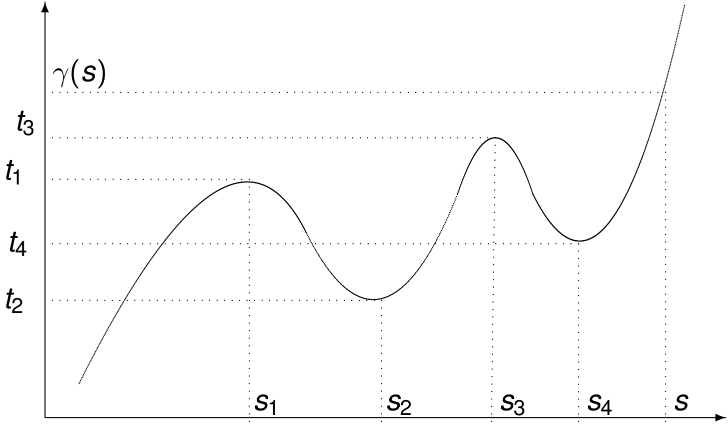

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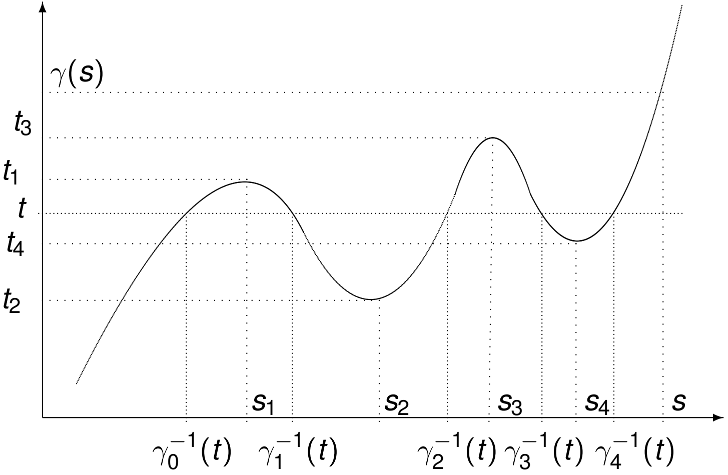

$\gamma(s)$
t_3
t_1
t
t_4
t_2
s_1
s_2
s_3
s_4
s
$\gamma_0^{-1}(t)$
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$$= \sum_{k=0}^{m}\int_{t_k}^{t_{k+1}} \frac{f\left(\gamma_k^{-1}(t)\right)}{\gamma'\left(\gamma_k^{-1}(t)\right)}e^{itx}\,\mathrm{d}t$$

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&= \int_{\min t_k}^{\max t_k} \sum_{k=0}^{m} \mathbf{1}_{[t_k,t_{k+1}]}(t) \frac{f\left(\gamma_k^{-1}(t)\right)}{\gamma'\left(\gamma_k^{-1}(t)\right)} e^{itx}\,\mathrm{d}t.
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Thus $(\Gamma, \{x = 0\})$ HUP

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\end{aligned}$$

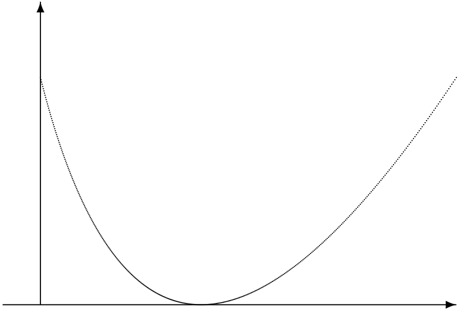
Thus $(\Gamma, \{x=0\})$ HUP $\Leftrightarrow \displaystyle\sum_{k=0}^{m} \mathbf{1}_{[t_k, t_{k+1}]}(t) \frac{f\left(\gamma_k^{-1}(t)\right)}{\gamma'\left(\gamma_k^{-1}(t)\right)} = 0.$

Example 1

γ is one-to-one, then $(\Gamma, \{x = 0\})$ HUP

Example 2

γ has one local extrema.



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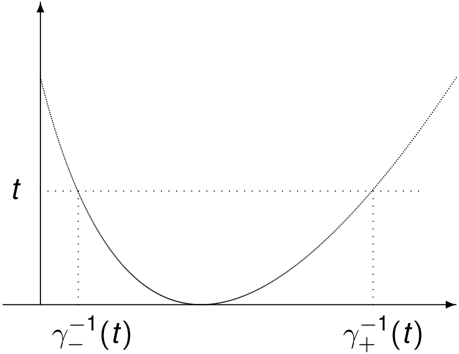

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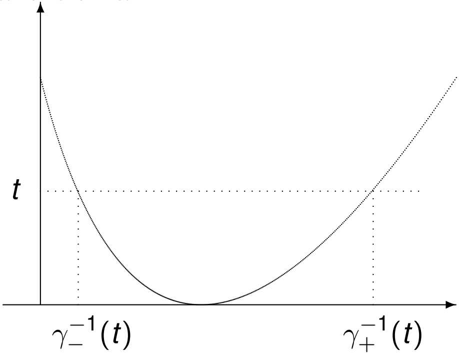

$$\frac{f(\gamma_+^{-1}(t))}{\gamma'(\gamma_+^{-1}(t))} = -\frac{f(\gamma_-^{-1}(t))}{\gamma'(\gamma_-^{-1}(t))}.$$

Lemma

$$\frac{f\left(\gamma_+^{-1}(t)\right)}{\gamma'\left(\gamma_+^{-1}(t)\right)} = -\frac{f\left(\gamma_-^{-1}(t)\right)}{\gamma'\left(\gamma_-^{-1}(t)\right)}.$$

and for $\alpha_+, \beta_+ > s_1$ and $\alpha_- = \gamma_-^{-1}\left(\gamma(\alpha_+)\right)$, $\beta_- = \gamma_-^{-1}\left(\gamma(\beta_+)\right)$, we have

$$\int_{\alpha_-}^{\beta_-} |f(s)|\, ds = -\int_{\alpha_+}^{\beta_+} |f(s)|\, ds$$

Theorem : two lines

γ smooth, $a > b$, $\psi(t) = \gamma(t) + at$ and $\chi(t) = \gamma(t) + bt$

– ψ (resp. χ) have a unique local minimum in s_1 (resp s_2)

– $\psi(t), \chi(t) \to +\infty$ when $t \to \pm\infty$.

$f \in L^1(\mathbb{R})$ s.t.

$$\int_{\mathbb{R}} f(s)e^{i\psi(s)x}\,ds = \int_{\mathbb{R}} f(s)e^{i\chi(s)x}\,ds = 0$$

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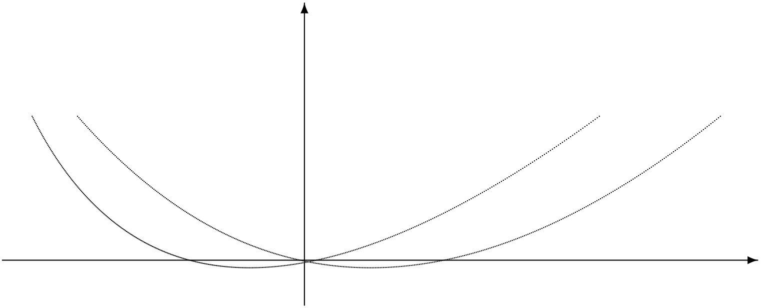

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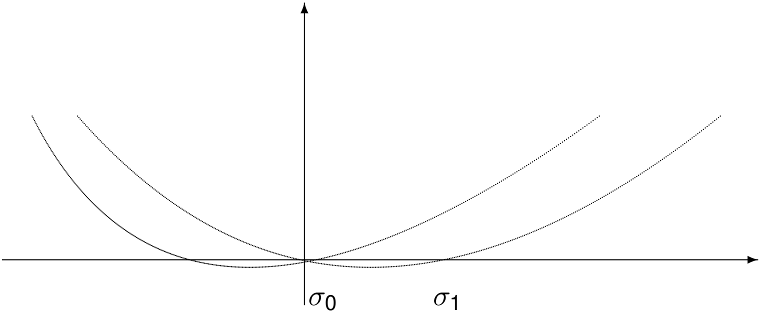

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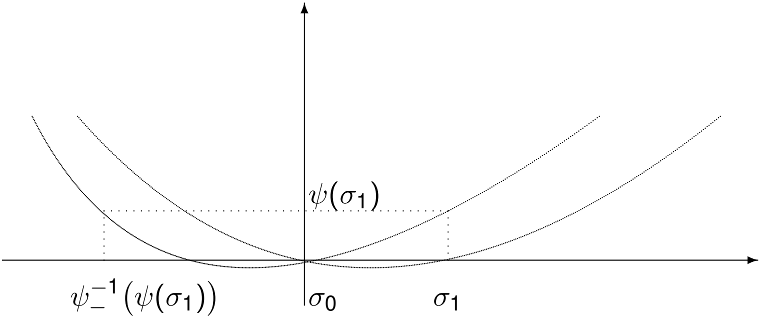

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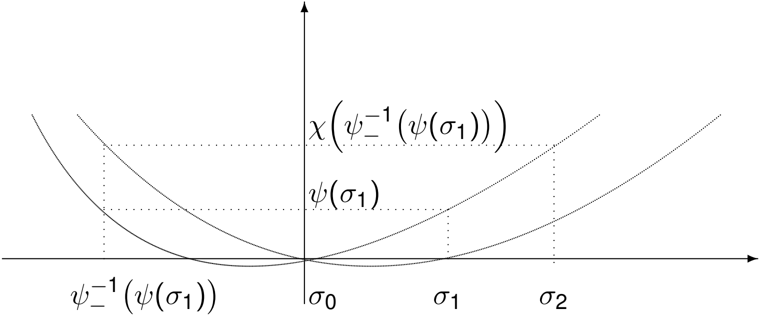

$\sigma_k \to +\infty$.

Proof 2

If $f \neq 0$, $\int_{\sigma_k}^{\sigma_{k+1}} |f(s)|\,ds \neq 0$

$$\int_{\sigma_{k+1}}^{\sigma_{k+2}} |f(s)|\,ds = \int_{\sigma_k}^{\sigma_{k+1}} |f(s)|\,ds.$$

$$\int_0^{+\infty} |f(s)|\,ds = \sum_{k\geq 0} \int_{\sigma_k}^{\sigma_{k+1}} |f(s)|\,ds = +\infty.$$

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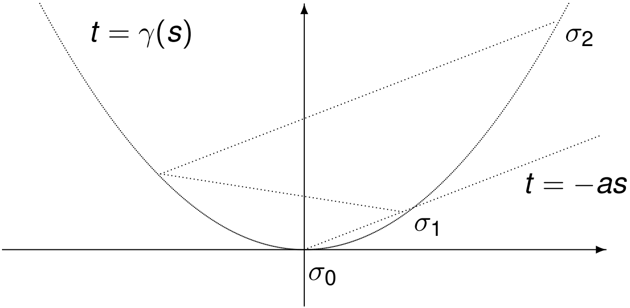
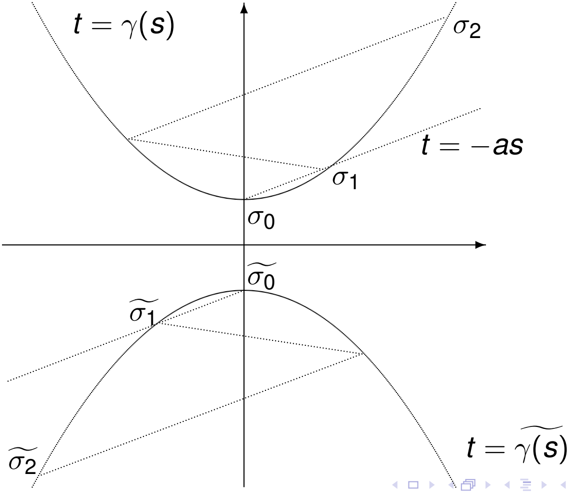
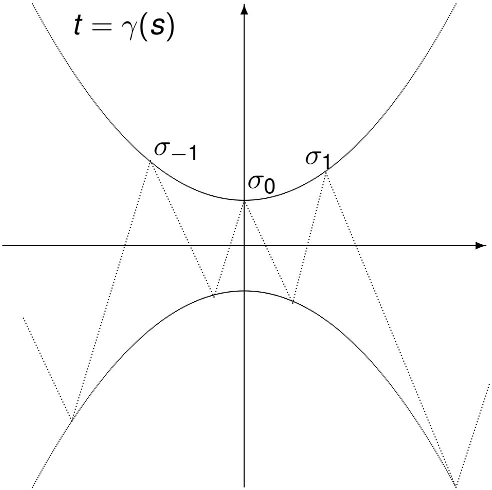
A better picture



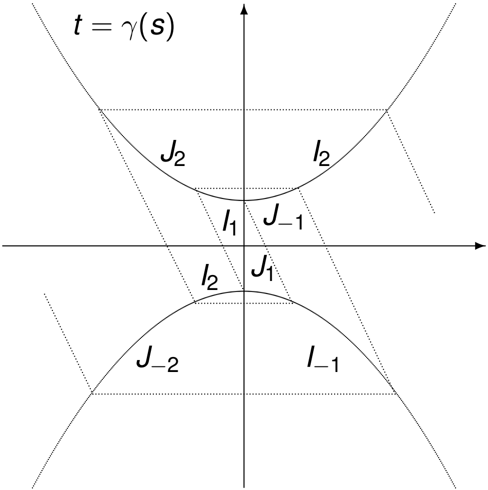
Hyperbola, case 1



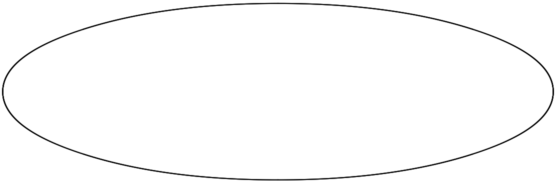
Hyperbola, case 2



Hyperbola, case 3



Closed curve



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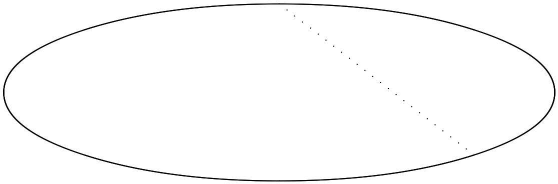

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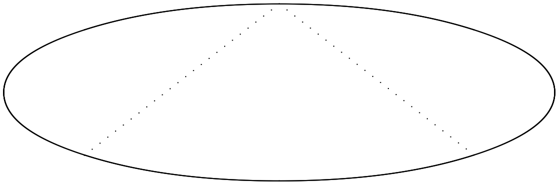

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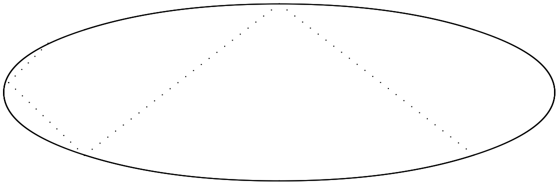

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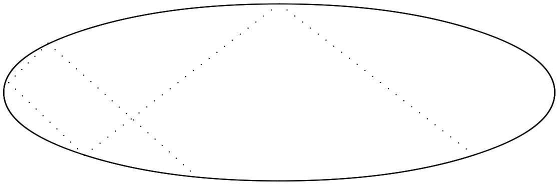

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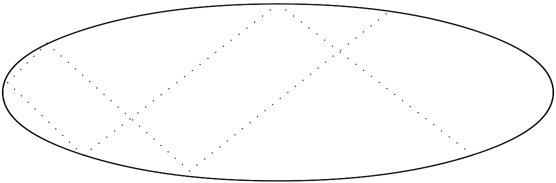
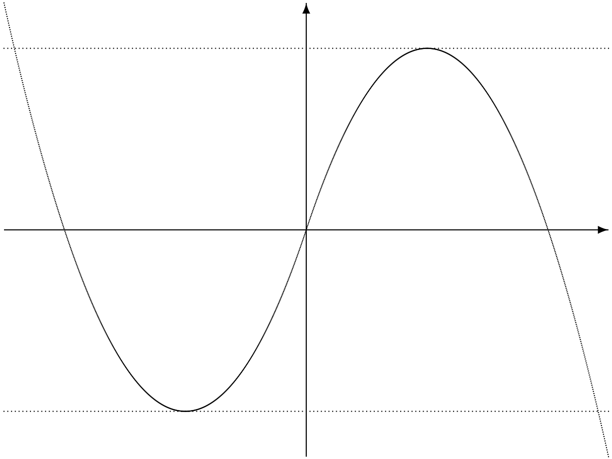


One local max, one local min



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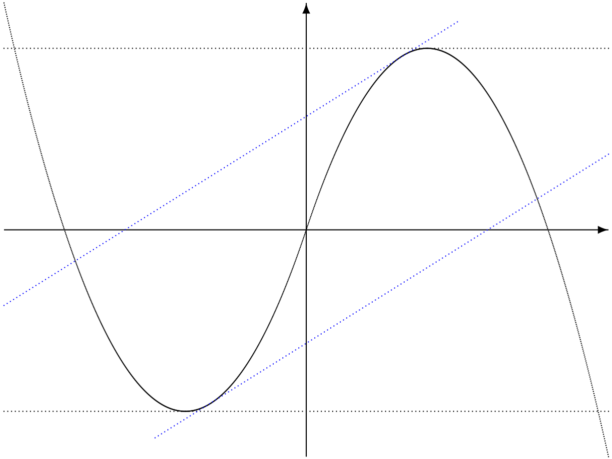

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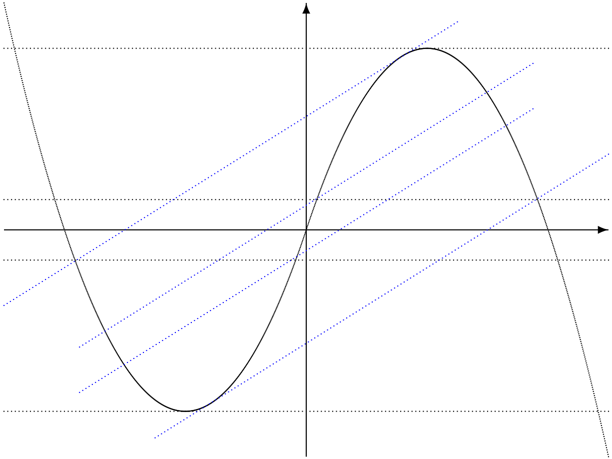

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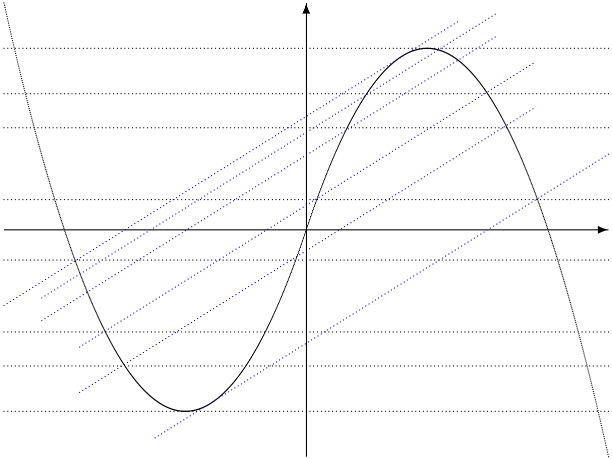

That's all !

Thank you for your attention !