Dunkl harmonic analysis and sharp Jackson inequalities in L_2-space with power weight

4th Workshop on Fourier Analysis and Related Fields
Alfre'd Re'nyi Institute of Mathematics, Budapest, Hungary
26–30 August 2013

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Introduction

Let $\alpha_1, \dots, \alpha_m \in \mathbb{R}^d$ be nonzero vectors, $k_1, \dots, k_m > 0$, $v(x) = \prod_{j=1}^m |\langle \alpha_j, x\rangle|^{k_j}$ power weight, $L_{2,v}(\mathbb{R}^d)$ the space of

Lebesgue measurable complex functions with finite norm

$$\|f\|_{2,v} = \left(\int_{\mathbb{R}^d} |f(x)|^2 v(x)dx\right)^{1/2} < \infty.$$

Our goal is to prove the sharp inequality between the best approximation of function from weighted space by entire functions of exponential type and its modulus of continuity. It is known as the Jackson inequality. To determine the modulus of continuity and to prove sharp Jackson inequality we need a rich harmonic analysis. In general case, it is difficult to hope to construct such analysis. For what weights we can do it? Here we are helped the root systems and associated reflection groups.

1. Root systems and reflection groups

Let $O(d)$ denote the group of all orthogonal maps of $\mathbb{R}^d$. For a nonzero vector $\alpha \in \mathbb{R}^d$ define the reflection $\sigma_\alpha \in O(d)$ by

$$\sigma_\alpha(x) = x - (2\langle x, \alpha\rangle/|\alpha|_2^2)\alpha, \quad x \in \mathbb{R}^d,$$

where $\langle x, \alpha\rangle = \sum_{j=1}^{d} x_j\alpha_j$, the inner product, and $|\alpha|_2^2 = \langle \alpha, \alpha\rangle$.

A finite set $R \subset \mathbb{R}^d \setminus \{0\}$ is called a root system, if $\forall \alpha \in R$

$$1)\ \sigma_\alpha(R) = R, \quad 2)\ R \cap \mathbb{R}\alpha = \{\pm\alpha\}.$$

We can define positive subsystem of root system $R_+ = \{\alpha \in R : \langle \alpha, \alpha_0\rangle > 0,\ \alpha_0 \in \mathbb{R}^d\}$, so that $R = R_+ \sqcup (-R_+)$. The subgroup $G(R) \subset O(d)$ which is generated by the reflections $\{\sigma_\alpha : \alpha \in R\}$ is called the reflection group associated with R.

2. Weight functions

Let function $k\colon R \to \mathbb{R}_+$ be invariant with respect to reflection group ($k(\alpha) = k(g\alpha)\ \forall\alpha \in R\ \forall g \in G(R)$) and let v_k denote the power weight on $\mathbb{R}^d$ defined by

$$v_k(x) = \prod\nolimits_{\alpha\in R_+} |\langle\alpha, x\rangle|^{2k(\alpha)}.$$

Example. Let $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ be standard basis in $\mathbb{R}^d$. The set $R = \{\pm e_i\}_{i=1}^d$ is a root system and $R_+ = \{e_i\}_{i=1}^d$. Reflection group is the set of diagonal matrices with ± 1 on main diagonal. Invariant function $k(\pm e_j) = \lambda_j + 1/2$, $\lambda_j \geq -1/2$ can have d different values. The power weight v_k has the form

$$v_k(x) = \prod_{j=1}^{d} |x_j|^{2\lambda_j+1}$$

(product of absolute values of coordinates in nonnegative degrees).

3. Harmonic analysis in в $L_{2,k}(\mathbb{R}^d)$-space

Let $v_k(x) = \prod\limits_{\alpha\in R_+} |\langle\alpha, x\rangle|^{2k(\alpha)}$ be power weight,

$c_k = \int\limits_{\mathbb{R}^d} e^{-|x|^2/2} v_k(x)dx$ Macdonald–Mehta–Selberg normalizing constant, $d\mu_k(x) = c_k^{-1} v_k(x)dx$, $L_{2,k}(\mathbb{R}^d)$ Hilbert space with finite norm and inner product

$$\|f\|_{2,k} = \left(\int\limits_{\mathbb{R}^d} |f(x)|^2 d\mu_k(x)\right)^{1/2}, \quad (f, g)_k = \int\limits_{\mathbb{R}^d} f(x)\overline{g(x)}d\mu_k(x).$$

Harmonic analysis in space $L_{2,k}(\mathbb{R}^d)$ was constructed by C.F. Dunkl [1–4] with the help of differential-difference and integral Dunkl operators. A great contribution to the development of this theory introduced M. Rösler, de Jeu, K. Trimeche, Y. Xu and others.

Differential-difference Dunkl operators have form

$$D_j f(x) = \frac{\partial f(x)}{\partial x_j} + \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, e_j\rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad j = 1, \dots, d.$$

For $y \in \mathbb{R}^d$ differential-difference system with initial condition

$$D_j f(x) = i y_j f(x), \quad f(0) = 1$$

has unique solution $e_k(x, y)$, which extend to entire function in $\mathbb{C}^d \times \mathbb{C}^d$. Generalized exponential $e_k(x, y)$ has properties similarly properties of usual exponential $e^{i\langle x,y\rangle}$.

Proposal 1 (C.F. Dunkl [3], M. Rösler [5]). If $g \in G(R)$, $\lambda \in \mathbb{C}$, $x, y \in \mathbb{R}^d$, $z \in \mathbb{C}^d$, $\operatorname{Im} z = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_d)$, then

$$e_k(x,y) = e_k(y,x), \quad e_k(0,y) = 1, \quad e_k(\lambda x, y) = e_k(x, \lambda y),$$
$$\overline{e_k(x,y)} = e_k(-x,y), \quad e_k(gx, gy) = e_k(x,y),$$
$$|e_k(x,y)| \le 1, \quad |e_k(z,y)| \le e^{|y||\operatorname{Im} z|}.$$

Generalized exponents are eigenvalues of Laplace-Dunkl operator

$$\Delta_k f(x) = \sum_{j=1}^{d} D_j^2 f(x): \quad -\Delta_k e_k(x,y) = |y|_2^2 e_k(x,y).$$

M. Rösler [5] got integral representation of generalized exponent

$$e_k(x,y) = \int\limits_{\mathbb{R}^d} e^{i\langle \xi, y\rangle} d\mu_k^x(\xi),$$

where μ_k^x is Borel probability measure with compact support in convex hull of orbit of x: $C(x) = \operatorname{co}\{gx, g \in G(R)\}$.

Dunkl integral transforms are defined with the help of generalized exponent

$$\widehat{f}(y)=\int_{\mathbb{R}^d} f(x)\overline{e_k(x,y)}d\mu_k(x),\quad \overset{\vee}{f}(x)=\int_{\mathbb{R}^d} f(y)e_k(x,y)d\mu_k(y).$$

Proposal 2 (C.F. Dunkl [4], M.F.E. de Jeu [6]). Dunkl integral transform realize isometric isomorphism of $L_{2,k}(\mathbb{R}^d)$-spaces and for them are fulfilled Parseval equality:

$$\widehat{f}: L_{2,k}(\mathbb{R}^d)\to L_{2,k}(\mathbb{R}^d),\quad (\widehat{f})^{-1}=\overset{\vee}{f},$$

$$\|f\|_{2,k}=\|\widehat{f}\|_{2,k},\quad (f,g)_k=(\widehat{f},\widehat{g})_k.$$

4. Best approximation

Let V be convex centrally symmetric compact body invariant with respect to reflection group, $\sigma > 0$. Let us define 3 classes of entire functions:

$$E^d_{2,k}(\sigma V) = \{f \in L_{2,k}(\mathbb{R}^d) \bigcap C_b(\mathbb{R}^d) : \operatorname{supp} \widehat{f} \subset \sigma V\},$$

$$W^d_{2,k}(\sigma V) = \{f\text{–entire in } \mathbb{C}^d : |f(z)| \le c_f e^{\sigma |z|_{V^*}},\ f \in L_{2,k}(\mathbb{R}^d)\},$$

$$\widetilde{W}^d_{2,k}(\sigma V) = \{f\text{–entire in } \mathbb{C}^d : |f(z)| \le c_f e^{\sigma |\operatorname{Im} z|_{V^*}},\ f \in L_{2,k}(\mathbb{R}^d)\}.$$

Here V^* is polar of V, $|z|_{V^*}$ norm in $\mathbb{R}^d$, defined with the help of V^*.

For $f \in E^d_{2,k}(\sigma V)$ and $z \in \mathbb{C}^d$ $\quad f(z) = \int\limits_{\sigma V} \widehat{f}(y) e_k(z, y)\, d\mu_k(y)$,

therefore

$$E^d_{2,k}(\sigma V) \subseteq \widetilde{W}^d_{2,k}(\sigma V) \subseteq W^d_{2,k}(\sigma V).$$

If $k(\alpha)\equiv 0$, then $E^d_{2,k}(\sigma V)=\widetilde{W}^d_{2,k}(\sigma V)=W^d_{2,k}(\sigma V)$ [7,8].
The first equality is known as Paley—Viener theorem. In weighted case it is proved by de Jeu for Euclidean ball [9].
He proved Paley—Viener theorem for arbitrary V in the case when function $k(\alpha)$ has only integer values [9].

Theorem 1[10]. $W^d_{2,k}(\sigma V)=\widetilde{W}^d_{2,k}(\sigma V)$.

Let us define the value of best approximation in $L_{2,k}(\mathbb{R}^d)$-space:

$$E(\sigma V, f)_{2,k}=\inf\{\|f-g\|_{2,k}:\ g\in E^d_{2,k}(\sigma V)\}.$$

As in the case of unit weight

$$E^2(\sigma V, f)_{2,k}=\int_{|y|_V\geq\sigma}|\widehat{f}(y)|^2d\mu_k(y).$$

5. Modulus of continuity

Let $M=\{\mu_s\}_{s\in\mathbb{Z}}$, $\sum_{s\in\mathbb{Z}}\mu_s=0$, $\sum_{s\in\mathbb{Z}}|\mu_s|<\infty$ be complex sequence,

$$\nu_s=\sum_{l\in\mathbb{Z}}\mu_{l+s}\overline{\mu}_l\quad \Big(\nu_0=\sum_{l\in\mathbb{Z}}|\mu_l|^2,\quad \sum_{s\in\mathbb{Z}}\nu_s=0,\quad \sum_{s\in\mathbb{Z}}|\nu_s|<\infty\Big),$$

$$\varphi(t,y)=\sum_{s\in\mathbb{Z}}\nu_s e_k(st,y),\quad t,y\in\mathbb{R}^d.$$

Using integral representation for generalized exponential we get

$$\varphi(t,y)=\int_{\mathbb{R}^d}\sum_{s\in\mathbb{Z}}\nu_s e^{is\langle\xi,y\rangle}\,d\mu_t^k(\xi)=\int_{\mathbb{R}^d}\Big|\sum_{s\in\mathbb{Z}}\mu_s e^{is\langle\xi,y\rangle}\Big|^2\,d\mu_t^k(\xi)\geq 0.$$

We have too

$$\varphi\,(t,y)\in C_b(\mathbb{R}^d\times\mathbb{R}^d),\ \varphi\,(t,y)=\varphi\,(y,t)\,,\ \varphi\,(0,y)=0.$$

It allowed us to define modulus of continuity for arbitrary convex centrally symmetric compact body U invariant with respect to reflection group:

$$\omega_M(\tau U, f)_{2,k} = \sup_{t\in\tau U}\left(\int_{\mathbb{R}^d}\varphi(t,y)|\widehat{f}(y)|^2 d\mu_k(y)\right)^{1/2}, \quad \tau > 0.$$

We have $\lim_{\tau\to 0+0}\omega_M(\tau U, f)_{2,k} = 0$. If $v_k(x) = 1$ is the unit weight,

$$\Delta_t^M f(x) = \sum_{s\in\mathbb{Z}}\mu_s f(x+st)$$

infinitely-difference operator, then

$$\omega_M(\tau U, f)_2 = \sup_{t\in\tau U}\|\Delta_t^M f(x)\|_2.$$

6. Inequality and Jackson constant

Jackson constant for the pair of bodies V, U

$$D_M(\sigma V, \tau U)_{2,k} = \sup \left\{ \frac{E(\sigma V, f)_{2,k}}{\omega_M(\tau U, f)_{2,k}} : f \in L_{2,k}(\mathbb{R}^d) \right\}$$

is the least constant in Jackson inequality

$$E(\sigma V, f)_{2,k} \leq D\omega_M(\tau U, f)_{2,k}.$$

We have

$$D_M(\sigma V, \tau U)_{2,k} = D_M(V, \sigma\tau U)_{2,k}$$

and can assume further that $\sigma = 1$.

Theorem 2. Jackson constant $D_M(V,\tau U)_{2,k}$ is continue as function of $\tau>0$.

Theorem 3. For all $\tau>0$

$$D_M(V,\tau U)_{2,k}\geq 1/\sqrt{\nu_0}.$$

Theorem 4. There is constant $\gamma=\gamma(k,M,d,V,U)>0$ such, that for every $f\in L_{2,k}(\mathbb{R}^d)$

$$E(V,f)_{2,k}\leqslant\frac{1}{\sqrt{\nu_0}}\omega_M(\gamma U,f)_{2,k}\,.$$

For unit weight theorem 4 was proved by S.N. Vasilyev [11].

Least value of γ in theorem 4 we call as optimal argument. Denote it $\tau_{k,M}(V,U)$:

$$\tau_{k,M}(V,U):=\inf\{\tau>0:D_M(V,\tau U)_{2,k}=1/\sqrt{\nu_0}\}.$$

Optimal argument depend on power weight v_k, sequence M and on geometry of bodies V,U.

7. Logan type extremal problem

Let for real function f and body V

$$\lambda(f, V) = \sup\{|x|_V : f(x) > 0\}$$

be radius of minimal ball in V norm, outside of them function f is nonpositive,

$$F_M(t, f) = -\sum_{s\neq 0} \nu_s f(st),$$

$K_M(U, V)$ be class of entire functions f for which $f \in L_{1,k}(\mathbb{R}^d)$, $\operatorname{supp} \widehat{f} \subset U$, $\widehat{f}(x) \geq 0$, $f(0) > 0$, $\lambda(F_M(f), V) < \infty$.

Logan type problem. To find the value

$$\Lambda_{k,M}(U, V) = \inf\{\lambda(F_M(f), V) : f \in K_M(U, V)\}.$$

B.F. Logan [12] posed and solved this problem for unit weight in one dimensional case and $F_M(t, f) = f(t)$.

Theorem 5. $\tau_{k,M}(V,U)=\Lambda_{k,M}(U,V)$.

Let $r\in\mathbb{N}$, $M_r=\left\{(-1)^s\binom{r}{s}\right\}_{s\in\mathbb{Z}}$. Note that $\mu_s\neq 0$ only for $s=0,1,\ldots,r$. Sequence M_r define the modulus of continuity of order r:

$$\omega_r\left(\tau U,f\right)_{2,k}=\omega_{M_r}\left(\tau U,f\right)_{2,k}.$$

In the case of unit weight

$$\omega_{M_r}\left(\tau U,f\right)_{2,k}=\sup_{t\in\tau U}\left\|\sum_{s=0}^{r}(-1)^s\binom{r}{s}f(x+st)\right\|_2=\omega_r\left(\tau U,f\right)_{2,k}.$$

For unit weight and sequence M_r theorem 5 was proved by E.E. Berdyscheva [13] ($r=1$) and D.V. Gorbachev, S.A. Strankovskiy [14]($r>1$). For general weight and sequence M_1 theorem 4 was proved by A.V. Ivanov [15].

From theorems 4 and 5 it follows that the class $K_M(U,V)$ is not empty for any sequence M and bodies U, V.

8. Optimal argument for sequence M_1

The modulus of continuity $\omega_1(\tau U, f)_{2,k}$, defined by means of sequence $M_1 = \{\dots, 0, 1, -1, 0, \dots\}$, can be written with the help of generalized translation operator T^t and self function f:

$$\omega_1(\tau U, f)_{2,k} = \sup_{t\in\tau U}\left(\int_{\mathbb{R}^d} T_y^t |f(y) - f(x)|^2|_{y=x} d\mu_k(x)\right)^{1/2}.$$

In the space $L_{2,k}(\mathbb{R}^d)$ generalized translation operator

$$T^t f(x) = \int_{\mathbb{R}^d} e_k(t, y)\widehat{f}(y) e_k(x, y) d\mu_k(x)$$

was defined by M. Rösler [16]. For every $t \in \mathbb{R}^d$ $\quad \|T^t\| = 1$.

Firstly we consider the case of unit weight ($k(\alpha)\equiv 0$). Let $J_\lambda(x)$ be Bessel function of order $\lambda\geq -1/2$, q_λ its least positive zero,

$$j_\lambda(x)=2^\lambda\Gamma(\lambda+1)\frac{J_\lambda(x)}{x^\lambda}$$

normalized Bessel function, $\lambda_1(U)$ least eigenvalue of eigenvalue problem for Laplace operator

$$-\Delta v=\lambda v,\quad v|_{\partial U}=0,\quad v\in L_2(U),$$

$$B_p^d=\left\{x\in\mathbb{R}^d:|x|_p=\left(\sum_{j=1}^d|x|^p\right)^{1/p}\leq 1\right\},\ 1\leq p<\infty,$$

$$B_\infty^d=\left\{x\in\mathbb{R}^d:|x|_\infty=\max_j|x_j|\leq 1\right\},\ p=\infty$$

balls in l_p^d-norm.

In 1981 V.A. Yudin [17] proved Jackson inequality, which gives common upper estimation

$$\tau_{0,M_1}(B_2^d, U) = \Lambda_{0,M_1}(U, B_2^d) \leq 2\lambda_1^{1/2}(U).$$

His method allows us to construct a good entire functions in the Logan problem for sequence M_1.

Conjecture 1.

$$\tau_{0,M_1}(B_2^d, U) = \Lambda_{0,M_1}(U, B_2^d) = 2\lambda_1^{1/2}(U).$$

This conjecture was verified in two cases for cube B_∞^d and ball B_2^d. In 1999 E.E. Berdyscheva [13] proved that

$$\tau_{0,M_1}(B_p^d, B_\infty^d) = \Lambda_{0,M_1}(B_\infty^d, B_p^d) = \pi d^{1/p}, \quad 1 \leq p \leq 2.$$

Extremal function in Logan problem do not depend from p. In 2000 D.V. Gorbachev [18] proved that

$$\tau_{0,M_1}(B_2^d, B_2^d) = \Lambda_{0,M_1}(B_2^d, B_2^d) = 2q_{d/2-1}.$$

Now we consider weighted case. Sharp results are obtained also in two cases, when body U is Euclidean ball or parallelepiped.

Theorem 6[15]. If $\lambda_k = d/2 - 1 + \sum\limits_{\alpha\in R_+} k(\alpha)$, then

$$\tau_{k,M_1}(B_2^d, B_2^d) = \Lambda_{k,M_1}(B_2^d, B_2^d) = 2q_{\lambda_k}. \tag{1}$$

Extremal function in Logan problem is radial function

$$f^1(x) = \frac{j_{\lambda_k}^2(|x|_2/2)}{1-(|x|_2/2q_{\lambda_k})^2}. \tag{2}$$

Consequence 1. For every $f \in L_{2,k}(\mathbb{R}^d)$, $\sigma > 0$

$$E(\sigma B_2^d, f)_{2,k} \le \frac{1}{\sqrt{2}}\omega_{M_1}\left(\frac{2q_{\lambda_k}}{\sigma}B_2^d, f\right)_{2,k}.$$

For construction of extremal function (2) in Logan problem (1) it is used Yudin method [17].
For low estimation in Logan problem (1) it is used averaging on unit sphere $S^{d-1} = \{x \in \mathbb{R}^d : |x|_2 = 1\}$, Y. Xu formula [19]

$$\int_{S^{d-1}} e_k(x,y)v_k(y)d\sigma(y) = j_{\lambda_k}(|x|_2) \int_{S^{d-1}} v_k(y)d\sigma(y)$$

and quadrature formula of Frappier–Oliver–Grozev–Rahman [20, 21]

$$\int_0^{\infty} f(x)|x|^{2\lambda+1}dx = \sum_{k=1}^{\infty} r_\lambda(k) f\left(\frac{2q_{\lambda,k}}{a}\right),$$

where f be even entire function of exponential type $a > 0$, integrable with weight $|x|^{2\lambda+1}$ $\lambda > -1/2$, $r_\lambda(k) > 0$, $\{q_{\lambda,k}\}$ positive zeros of $j_\lambda(x)$. D.V. Gorbachev [18] began the first to apply such quadrature formula in extremal problems.

Let $\lambda = (\lambda_1, \dots, \lambda_d)$, $\lambda_j \geq -1/2$, $v_k(x) = \prod_{j=1}^{d} |x_j|^{2\lambda_j+1}$,

$a = (a_1, \dots, a_d)$, $a_j > 0$, $\Pi_a = \prod_{j=1}^{d} [-a_j, a_j]$ be parallelepiped,

$b_{\lambda,a} = (2q_{\lambda_1}/a_1, \dots, 2q_{\lambda_d}/a_d)$.

Theorem 7[22]. For $1 \leq p \leq 2$

$$\tau_{k,M_1}(B_p^d, \Pi_a) = \Lambda_{k,M_1}(\Pi_a, B_p^d) = |b_{\lambda,a}|_p. \tag{3}$$

Extremal function in Logan problem depend from p and has form

$$f_p^2(x) = \Big(|\beta_{\lambda,a}|_p^p - \sum_{j=1}^{d} (a_j/2q_j)^{2-p} x_j^2\Big) \prod_{j=1}^{d} \frac{j_{\lambda_j}^2(a_j x_j/2)}{\left(1 - \left(a_j x_j/2q_{\lambda_j}\right)^2\right)^2}. \tag{4}$$

For $1 \leq p < 2$ this theorem is a new and when $v_k(x) \equiv 1$.

CONJECTURE 2. If $p > 2$, then

$$\tau_{k,M_1}(B_p^d, \Pi_a) = \Lambda_{k,M_1}(\Pi_a, B_p^d) = |b_{\lambda,a}|_2.$$

For construction of extremal function (4) in Logan problem (3) it is used some modification of Yudin method.

For low estimation in Logan problem (3) it is used multidimensional variant of Frappier–Oliver–Grozev–Rahman quadrature formula [23]

$$\int_{\mathbb{R}_+^d} f(x) \prod_{j=1}^{d} |x_j|^{2\lambda_j+1} dx = \sum_{k\in\mathbb{N}^d} \prod_{j=1}^{d} r_{\lambda_j}(k_j) f\left(\frac{2q_{\lambda_1,k_1}}{a_1}, \ldots, \frac{2q_{\lambda_d,k_d}}{a_d}\right),$$

where $f \in L_{1,\lambda}(\mathbb{R}^d)$ be even entire function of exponential type $a_j > 0$ on every variable.

Consequence 2. For every $f \in L_{2,k}(\mathbb{R}^d)$, $\sigma > 0$, $1 \le p \le 2$

$$E(\sigma B_p^d, f)_{2,k} \le \frac{1}{\sqrt{2}} \omega_{M_1} \left(\frac{|b_{\lambda,a}|_p}{\sigma} \Pi_a, f \right)_{2,k}.$$

We believe that in weighted case the conjecture 1 is true too with the replacement of the Laplace operator on Laplace-Dunkl operator.
CONJECTURE 3. For general power weight v_k

$$\tau_{k,M_1}(B_2^d, U) = \Lambda_{k,M_1}(U, B_2^d) = 2\lambda_1^{1/2}(U),$$

where $\lambda_1(U)$ is the least eigenvalue of eigenvalue problem for Laplace-Dunkl operator

$$-\Delta_k v = \lambda v, \quad v|_{\partial U} = 0, \quad v \in L_{2,k}(U).$$

9. Optimal argument for sequence M_r

Optimal argument for sequence M_r is calculated only in one case.

Теорема 8. If $\lambda_k = d/2 - 1 + \sum_{\alpha\in R_+} k(\alpha) = 1/2$, then for all $r \in \mathbb{N}$

$$\tau_{k,M_r}(B_2^d, B_2^d) = \Lambda_{k,M_r}(B_2^d, B_2^d) = 2q_{1/2} = 2\pi. \tag{5}$$

If $\lambda_k \neq 1/2$, then

$$\tau_{k,M_2}(B_2^d, B_2^d) > \tau_{k,M_1}(B_2^d, B_2^d) = 2q_{\lambda_k}. \tag{6}$$

Always

$$\tau_{k,M_r}(B_2^d, B_2^d) \leq 4q_{\lambda_k}. \tag{7}$$

Equality (5) is possible only in dimensions $1, 2, 3$. Consider the case of unit weight. Equality (5) for $d = 3$ and $r = 2$ was proved by D.V. Gorbachev and S.A. Strankovskiy [14]. Inequality (6) for $d = 1$ was proved by V.V. Arestov and A.G. Babenko [24]. Inequality (7) for $d = 1$ was proved by V.Yu. Popov [25].

Extremal function in Logan problem (5) is radial function

$$\frac{1-\cos(|x|_2)}{|x|_2^2\,((2\pi)^2-|x|_2^2)}.$$

For low estimation in Logan problem (5) we reduce problem to radial functions and prove quadrature formula

$$\int_0^\infty F_r(x)x^2dx=\sum_{n=1}^\infty c_r(n)F_r\,(2\pi n)\,,$$

where $c_r(n)>0,\ c_r(n)\asymp n^2$,

$$F_r(x)=\sum_{k=1}^r(-1)^{k-1}\binom{2r}{r-k}f(kx),$$

f be even entire function of exponential type 1, integrable with weight x^2.

Why the case of number $\lambda_k = 1/2$ is exceptional? It is connected with arithmetical properties of Bessel function zeros. Let

$$0 < q_{\alpha,1} < q_{\alpha,2} < \dots < q_{\alpha,s} < \dots$$

be all positive zeros of $j_\alpha(x)$. For $\alpha = 1/2$ $q_{\alpha,s} = \pi s$ and for all $r \in \mathbb{N}$

$$\{rq_{\alpha,s}\}_{s=1}^{\infty} \subseteq \{q_{\alpha,s}\}_{s=1}^{\infty}.$$

For $\alpha \neq 1/2$ and sufficiently large $N = N(\alpha)$

$$\{q_{\alpha,s}\}_{s=N}^{\infty} \bigcap \{2q_{\alpha,s}\}_{s=N}^{\infty} = \varnothing.$$

CONJECTURE 4. For all $\alpha \neq 1/2$

$$\{q_{\alpha,s}\}_{s=1}^{\infty} \bigcap \{2q_{\alpha,s}\}_{s=1}^{\infty} = \varnothing.$$

CONJECTURE 5. For all $\alpha \neq \pm 1/2$ and all $s \neq l$

$$\frac{q_{\alpha,s}}{q_{\alpha,l}} \notin \mathbb{Q}.$$

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