Asymptotic lower bound for cardinality of weighted spherical designs

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Definition of Chebyshev-type design

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- $X = \{x_\nu\}_{\nu=1}^N \subset S^d$.
- If the following quadrature formula

$$\frac{1}{\operatorname{mes}(S^d)} \int_{S^d} f(x)\,dx = \frac{1}{N} \sum_{\nu=1}^{N} f(x_\nu)$$

holds for all algebraic polynomials $f(x_1, \ldots, x_{d+1})$ of degree at most τ then X is called spherical τ-design.

Main problem

For d and τ to construct a spherical design with minimal number of points. This minimal number of points is denoted by $N(d, \tau)$.

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Examples for $d = 2$:

- $N(2, 0) = 1$ (point);
- $N(2, 1) = 2$ (two poles);
- $N(2, 2) = 4$ (tetrahedron);
- $N(2, 3) = 6$ (octahedron);
- $N(2, 5) = 12$ (icosahedron);
- $N(2, 7) = 24$? (improved snub cube, open problem).

Lower bounds of $N(d,\tau)$

Delsarte, Goethals and Seidel (1977) proved LP-bound for $N(d,\tau)$ and obtained the well-known tight bound

$$N(d,\tau) \geq \binom{d + \left[\frac{\tau+1}{2}\right] - 1}{d} + \binom{d + \left[\frac{\tau}{2}\right]}{d}.$$

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Thus for fixed d and $\tau \to \infty$ we have the following asymptotic result

$$N(d,\tau) \geq C_{DGS}(d)\,\tau^{d}\,(1 + o(1))\,.$$

LP-bound of $N(d,\tau)$

Let $\pi_k(t) = \dfrac{P_k^{(d/2-1,d/2-1)}(t)}{P_k^{(d/2-1,d/2-1)}(1)}$ be Gegenbauer polynomials;

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Theorem:

$$N(d,\tau) \geq B(d,\tau), \quad \text{where } B(d,\tau) = \max_f f(1).$$

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Examples: $B(2,3) = 6$, $B(2,5) = 12$, $B(2,7) = ?$ (≈ 21).

From the odd bound (Boyvalenkov and Nikova, 1994) or SDP-bound (G., 2010) we have $N(2,7) \ge 22$. On the other hand, improved snub cube implies that $N(2,7) \le 24$.

Proof of LP-bound

- Let $I = \sum_{\mu,\nu=1}^{N} f(x_\mu x_\nu) = \underbrace{\sum_{\mu=\nu}}_{Nf(1)} + \underbrace{\sum_{\mu\neq\nu}}_{\geq 0} \geq Nf(1).$

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$$=\underbrace{f_0}_{=1}\underbrace{I_0}_{=N^2}+\sum_{k=1}^{\tau} f_k \underbrace{I_k}_{=0}+\sum_{k=\tau+1}^{\infty}\underbrace{f_k}_{\leq 0}\underbrace{I_k}_{\geq 0}\leq N^2.$$

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- Thus $Nf(1) \leq I \leq N^2 \Rightarrow N \geq f(1).$

DGS approach

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- Usage Gauss–Markov quadrature formulae (alternative approach), e.g.

$$\underbrace{f_0}_{=1} = \frac{\int_{-1}^{1} f(t)(1-t^2)^{d/2-1}\,dt}{\int_{-1}^{1}(1-t^2)^{d/2-1}\,dt}$$

$$= \gamma_0 f(1) + \underbrace{\sum_{i=1}^{s} \gamma_i f(r_i)}_{\geq 0} \geq \gamma_0 f(1).$$

Yudin lower bound

- Using an LP-bound, Yudin (1997) obtained the following inequality

$$N(d,\tau) \geq \frac{\int_{-1}^{1}(1-t^2)^{d/2-1}\,dt}{\int_{t_\tau}^{1}(1-t^2)^{d/2-1}\,dt} = C_Y(d)\,\tau^d\,(1+o(1))\,.$$

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- Yudin approach: construction of an admissible convolution function $f_Y = \sum_{k=0}^{\infty} f_{Yk}\pi_k$ with small support on $[-1,1]$.
- Conjecture: extremal function for $B(d,\tau)$ is a polynomial of degree τ (tight case) or greater than τ (general case).

Existent bounds for Chebychev-type designs

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- Bondarenko, Radchenko and Viazovska (2010) proved the bound

$$N(d,\tau) \le C_d \tau^d$$

(Korevaar and Meyers conjecture).

Conjecture for the Chebychev-type designs

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Let $C(d)$ be a “mystery” limit constant.

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  1. For $d = 2$: $C_Y(2) = 0.2724\ldots > C_{DGS}(2) = 0.25$.

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  1. For $d = 2$: $C_Y(2) = 0.2724\ldots > C_{DGS}(2) = 0.25$.

  Conjecture: $C(2) \leq 0.5$.

  2. For fixed τ and $d \to \infty$:

$$\frac{C_{DGS}(d)}{C_Y(d)} \sim \frac{(e/4)^d}{\pi d} \lesssim (0.68)^d \ll 1.$$

LP-bound of sphere packing density

Let Δ_d be the sphere packing density of $\mathbb{R}^d$ and let

$$A(d) = \frac{\mathrm{vol}\,(B^d)}{2^d} \min_f f(0).$$

Here f is a radial positive defined function with mean value $\widehat{f}(0) = 1$, $f|_{\mathbb{R}^d \setminus B^d} \leq 0$, B^d is the unit ball.

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Yudin (1989) constructed good admissible function for $A(d)$ problem.

LP-bounds of Δ_d

Cohn and Elkies (2001), G. and Filippov (2004):

d	Central density of known packing	Upper bound for $\frac{A(d)}{\mathrm{vol}\,(B^d)}$
2	0.28868	0.28868
8	0.0625	0.06250
24	1	1.00000
36	4.4394	258.54994
72	68719476736	31734457390376

Main result

Theorem (G.):

We obtain the following new lower bound for

$$B(d,\tau) \geq \underbrace{\frac{1}{2^d \Gamma(d) A(d)}}_{C_{new}(d)} \tau^d (1 + o(1))$$

where $A(d)$ is LP-bound of sphere packing density Δ_d.

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$$C(2) \geq \underbrace{0.2756}_{new} > \underbrace{0.2724}_{Y} > \underbrace{0.25}_{DGS}.$$

- Comparison of Y and new asymptotic results:

d	Lower bound for $\dfrac{C_{new}(d)}{C_Y(d)}$
2	1.011
8	4.591
24	20.841
36	40.684
72	274.38

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- For DGS LP-bound the function g/g_0 is admissible and the following conditions hold:

$$g(1) = \operatorname{mes}(S^{d-1})\,\widehat{f}(0)\,(1 + o(1));$$

$$g_0 = \frac{(2\pi/\tau)^d f(0)}{\int_{-1}^{1}(1 - t^2)^{d/2-1}\,dt}\,(1 + o(1));$$

$$g_k = \tau^{-d-\varepsilon}\,(f(k/\tau) + \varepsilon_k)\,, \quad \sum |\varepsilon_k| = o(1), \quad k \geq \tau + 1.$$

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- This implies the statement of the theorem.

Estimates for weighted spherical designs

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- X is a weighted spherical design if the following quadrature formula

$$\frac{1}{\operatorname{mes}(S^d)} \int_{S^d} f(x)\, dx = \sum_{\nu=1}^{N} \lambda_\nu f(x_\nu)$$

holds for positive weights λ_ν.

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- DGS LP-bound is true for weighted designs.

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- Thus main theorem also holds for the weighted case.

Some interenet resources

- Neil J.A. Sloane: Home Page;
- D. Potts: Home page;
- Snub cube;
- Sphere packing.

Thank you for your attention!