Asymptotic lower bound for cardinality of weighted spherical designs

Dmitry Gorbachev

Tula State University, Russia

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3 Estimates for weighted designs

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Definition of Chebyshev-type design

• Let $S^d \in \mathbb{R}^{d+1}$ be a unit sphere.

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Definition of Chebyshev-type design

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- $X = \{x_{\nu}\}_{\nu=1}^{N} \subset S^{d}$.

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Definition of Chebyshev-type design

- Let $S^d \in \mathbb{R}^{d+1}$ be a unit sphere.
- $X = \{x_{\nu}\}_{\nu=1}^{N} \subset S^{d}$.
- If the following quadrature formula

$$\frac{1}{\max(S^d)} \int_{S^d} f(x) \, dx = \frac{1}{N} \sum_{\nu=1}^N f(x_{\nu})$$

holds for all algebraic polynomials $f(x_1, \ldots, x_{d+1})$ of degree at most τ then X is called spherical τ -design.

Main problem

For d and τ to construct a spherical design with minimal number of points. This minimal number of points is denoted by $N(d, \tau)$.

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Examples for d = 2:

- N(2,0) = 1 (point);
- N(2,1) = 2 (two poles);
- *N*(2,2) = 4 (tetrahedron);
- *N*(2,3) = 6 (octahedron);
- *N*(2,5) = 12 (icosahedron);
- N(2,7) = 24? (improved snub cube, open problem).

Lower bounds of $N(d, \tau)$

Delsarte, Goethals and Seidel (1977) proved LP-bound for $N(d, \tau)$ and obtained the well-known tight bound

$$\mathsf{N}(d, au) \geq egin{pmatrix} d+\left[rac{ au+1}{2}
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Thus for fixed d and $\tau \to \infty$ we have the following asymptotic result

$$N(d, \tau) \geq C_{DGS}(d) \, \tau^d \left(1 + o(1)\right).$$

LP-bound of $N(d, \tau)$

Let
$$\pi_k(t) = \frac{P_k^{(d/2-1,d/2-1)}(t)}{P_k^{(d/2-1,d/2-1)}(1)}$$
 be Gegenbauer polynomials;

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Theorem:

$$N(d, \tau) \geq B(d, \tau), \quad ext{where } B(d, \tau) = \max_{f} f(1).$$

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Examples:
$$B(2,3) = 6$$
, $B(2,5) = 12$, $B(2,7) = ? (\approx 21)$.

From the odd bound (Boyvalenkov and Nikova, 1994) or SDP-bound (G., 2010) we have $N(2,7) \ge 22$. On the other hand, improved snub cube implies that $N(2,7) \le 24$.

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Proof of LP-bound

• Let
$$I = \sum_{\mu,\nu=1}^{N} f(x_{\mu}x_{\nu}) = \sum_{\substack{\mu=\nu\\Nf(1)}} + \sum_{\substack{\mu\neq\nu\\\geq 0}} \ge Nf(1).$$

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• Thus $Nf(1) \leq I \leq N^2 \Rightarrow N \geq f(1).$

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DGS approach

• Let
$$f = \sum_{k=0}^{\tau} f_k \pi_k$$
 be a polynomial $(f_k = 0 \text{ for } k \ge \tau + 1).$

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$$f = \sum_{k=0}^{r} f_k \pi_k$$
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- Usage Lukach theorem on the representation of nonnegative polynomials (e.g. $f = a^2 + (1 t^2)b^2$ for $\tau = 2s$) and Schwarz inequality.
- Usage Gauss-Markov quadrature formulae (alternative approach), e.g.

$$\underbrace{\underbrace{f_{0}}_{=1}}_{=1} = \frac{\int_{-1}^{1} f(t)(1-t^{2})^{d/2-1} dt}{\int_{-1}^{1} (1-t^{2})^{d/2-1} dt} = \gamma_{0}f(1) + \underbrace{\sum_{i=1}^{s} \gamma_{i}f(r_{i})}_{\geq 0} \ge \gamma_{0}f(1).$$

Yudin lower bound

• Using an LP-bound, Yudin (1997) obtained the following inequality

$$N(d, au) \geq rac{\int_{-1}^{1} (1-t^2)^{d/2-1} \, dt}{\int_{t_ au}^{1} (1-t^2)^{d/2-1} \, dt} = C_Y(d) \, au^d \, (1+o(1)) \, .$$

Here t_{τ} is the last zero of the Jacobi polynomial $P_{\tau}^{(d/2,d/2)}$.

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- Yudin approach: construction of an admissible convolution function $f_Y = \sum_{k=0}^{\infty} f_{Yk} \pi_k$ with small support on [-1, 1].
- Conjecture: extremal function for B(d, τ) is a polynomial of degree τ (tight case) or greater than τ (general case).

Existent bounds for Chebychev-type designs

• Seymour and Zaslavsky (1984) proved that spherical designs exist for all parameters d and τ .

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• Bondarenko, Radchenko and Viazovska (2010) proved the bound

$$N(d, \tau) \leq C_d \tau^d$$

(Korevaar and Meyers conjecture).

Conjecture for the Chebychev-type designs

From DGS and BRV we have estimate $N(d, \tau) \asymp C(d)\tau^d$.

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Let C(d) be a "mystery" limit constant.

Comparison of DGS and Y asymptotic results

• DGS:
$$C(d) \geq \frac{2^{1-d}}{\Gamma(d+1)}$$

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• Y: $C(d) \ge \frac{\Gamma(d/2+1)\Gamma(d/2)}{\Gamma(d)} \left(\frac{2}{q_{d/2}}\right)^d$
where $q_{d/2}$ is first zero of Bessel function $J_{d/2}$.

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• Examples:

1. For
$$d = 2$$
: $C_Y(2) = 0.2724... > C_{DGS}(2) = 0.25.$

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$$d = 2$$
: $C_Y(2) = 0.2724... > C_{DGS}(2) = 0.25$.
Conjecture: $C(2) \le 0.5$.

2. For fixed τ and $d \rightarrow \infty$:

$$rac{C_{DGS}(d)}{C_Y(d)}\sim rac{(e/4)^d}{\pi d}\lesssim (0.68)^d\ll 1.$$

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LP-bound of sphere packing density

Let Δ_d be the sphere packing density of \mathbb{R}^d and let

$$A(d) = \frac{\operatorname{vol}(B^d)}{2^d} \min_f f(0).$$

Here f is a radial positive defined function with mean value $\hat{f}(0) = 1$, $f|_{\mathbb{R}^d \setminus B^d} \leq 0$, B^d is the unit ball.

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Yudin (1989) constructed good admissible function for A(d) problem.

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LP-bounds of Δ_d

Cohn and Elkies (2001), G. and Filippov (2004):

d	Central density of known packing	Upper bound for $\frac{A(d)}{\operatorname{vol}(B^d)}$
2	0.28868	0.28868
8	0.0625	0.06250
24	1	1.00000
36	4.4394	258.54994
72	68719476736	31734457390376

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Main result

Theorem (G.):

We obtain the following new lower bound for

$$B(d, au) \geq \underbrace{rac{1}{2^d \Gamma(d) A(d)}}_{C_{new}(d)} au^d \left(1 + o(1)
ight)$$

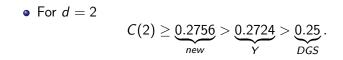
where A(d) is LP-bound of sphere packing density Δ_d .

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Examples

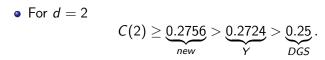
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Examples



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Examples



• Comparison of Y and new asymptotic results:

d	Lower bound for $\frac{C_{new}(d)}{C_Y(d)}$
2	1.011
8	4.591
24	20.841
36	40.684
72	274.38

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Idea of proof

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• Let f be an admissible smooth function for A(d) problem.

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- Define an even function $g(t) = f(\tau \arccos t), t \in [0, 1].$
- Using Baratella–Gatteschi (1988) Hilb-type expansion for Jacobi polynomials.
- For DGS LP-bound the function g/g_0 is admissible and the following conditions hold:

$$\begin{split} g(1) &= \max\left(S^{d-1}\right)\widehat{f}(0)\left(1+o(1)\right);\\ g_0 &= \frac{(2\pi/\tau)^d f(0)}{\int_{-1}^1 (1-t^2)^{d/2-1} \, dt} \left(1+o(1)\right);\\ g_k &= \tau^{-d-\varepsilon}\left(f(k/\tau)+\varepsilon_k\right), \quad \sum |\varepsilon_k| = o(1), \quad k \geq \tau+1. \end{split}$$

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• This implies the statement of the theorem.

Estimates for weighted spherical designs

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• X is a weighted spherical design if the following quadrature formula

$$\frac{1}{\mathrm{mes}\,(S^d)}\int_{S^d}f(x)\,dx=\sum_{\nu=1}^N\lambda_\nu f(x_\nu)$$

holds for positive weights λ_{ν} .

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- Upper existent bound follows from product of one-dimensional Gauss quadratures, e.g. elementary fact $N(2, \tau) \lesssim 0.5\tau^2$ (conjecture: $N(2, \tau) \approx 0.33\tau^2$).

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- Thus main theorem also holds for the weighted case.

Some interenet resources

- Neil J.A. Sloane: Home Page;
- D. Potts: Home page;
- Snub cube;
- Sphere packing.

Thank you for your attention!

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