Summation of one and two-dimensional Walsh-Fourier series

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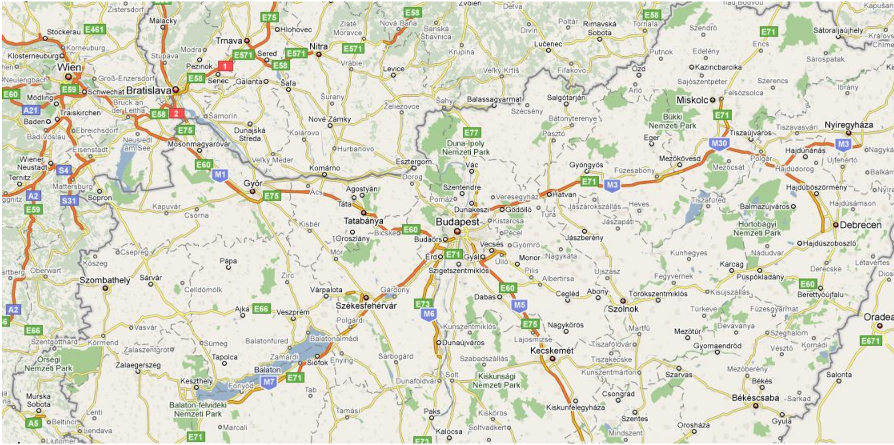

The Walsh functions

expansion with resp. the binary number system

$n = \sum_{i=0}^{\infty} n_i 2^i \in \mathbb{N}, \ x = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} \in [0,1) \quad n_i, x_i \in \{0,1\}$
$i = 0, 1, \ldots,$

If x is a dyadic rational number ($x \in \{\frac{p}{2^n} : p, n \in \mathbb{N}\}$) we choose the expansion which terminates in 0 's.

Walsh function

n-th Walsh-Paley function: $\omega_n(x) := (-1)^{\sum_{i=0}^{\infty} n_i x_i}$

Paley, *A remarkable series of orthogonal functions*, Proceedings of the London Mathematical Society (1932).

Can take $+1$ and -1 as a value.

Dirichlet and Fejér kernel functions:

Dirichlet and Fejér kernel functions

$$D_n := \sum_{k=0}^{n-1} \omega_k, \quad K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k,$$

Fourier coefficients, partial sums of Fourier series, Fejér means:

$$\hat{f}(n) := \int_0^1 f(x)\omega_n(x)dx \ (n \in \mathbb{N}),$$

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y) = \int_0^1 f(x+y)D_n(x)dx$$

$$\sigma_n f(y) := \frac{1}{n} \sum_{k=0}^{n-1} S_k f(y) = \int_0^1 f(x+y)K_n(x)dx$$

Fejér means:

trigonometric system: $(\exp(2\pi \imath kx), k \in \mathbb{Z})$ functions

Fejér Lipót, 1925 Henri Lebesgue, 1926 $\sigma_n f(x) \to f(x)$ for a.e. x.
,,Reconstruction the function"

Walsh case

Walsh-Paley system

N.J. Fine, Trans. Am. Math. Soc., 1949.
For the Walsh-Kaczmarz system

G. Gát. On (C; 1) summability of integrable functions with respect to the Walsh-Kaczmarz system. Stud. Math., 1998.

What does this Walsh-Kaczmarz system mean?

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Walsh-Kaczmarz system:

Let the Walsh-Kaczmarz functions be defined by $\kappa_0 = 1$ and for $n \geq 1, |n| = \lfloor \log_2 n \rfloor$

$$\kappa_n(x) := r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The Walsh-Paley system is $\omega := (\omega_n : n \in \mathbb{N})$ and the Walsh-Kaczmarz system is $\kappa := (\kappa_n : n \in \mathbb{N})$.

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_k : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbb{N}$ and $\kappa_0 = \omega_0$. A dyadic blockwise „rearrangement".

(application: Hadamard transform)

(C, α) means:

The (C, α) Cesàro means of the integrable function f is

$$\sigma_n^\alpha f(y) := \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f(y) = \int_0^1 f(x) K_n^\alpha (y-x) dx,$$

where $A_k^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!} \quad (\alpha \neq -k)$.

Walsh-Paley system:

Fine, 1955, Proc. Nat. Acad. Sci. U.S.A.
(C, α) means for $\alpha > 0$: $f \in L^1 \Rightarrow \sigma_n^\alpha f \to f$ a.e.

Walsh-Kaczmarz system:

Simon, 2000, Journal of Approximation Theory.

Walsh-Paley Fejér kernel function

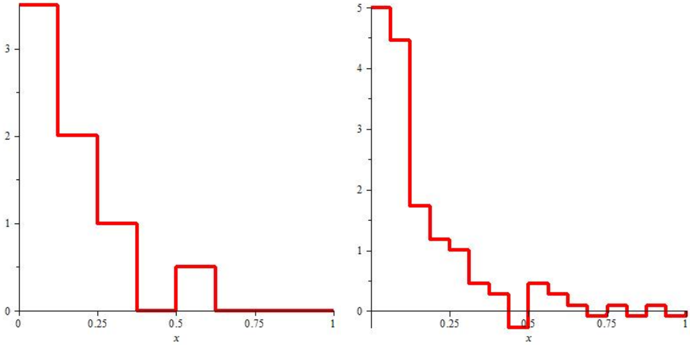

Figure: A K_8 and K_{11} Walsh-Paley-Fejér kernels

$K_n(x) \to 0$ $(n \to \infty)$ for every $x \neq 0$.

Walsh-Kaczmarz Fejér kernel function

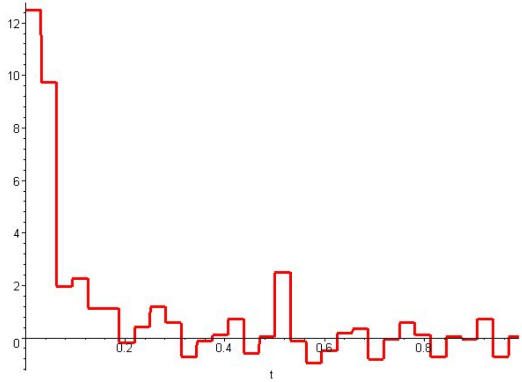

Figure: A K_{26} Walsh-Kaczmarz

$K_n(x) \to \infty$ $(n \to \infty)$ at every dyadic rational.

Introduced by N. Ja. Vilenkin in 1947
(see e.g. the book of G.H. Agaev, N.Ja. Vilenkin, G.M. Dzhafarli, and A.I. Rubinstein, 1981)
$m := (m_k, k \in \mathbb{N})$ $(\mathbb{N} := \{0, 1, \ldots\}$, seq. of integers each of them not less than 2.
Z_{m_k} the discrete cyclic group of order m_k. $\mu_k(\{j\}) = 1/m_k$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

The group operation on G_m is the coordinate-wise addition, the measure, and the topology are the product measure and topology.
If $\sup_{n\in\in\mathbb{N}} m_n < \infty$, G_m a bounded Vilenkin group.
If m is not bounded, then G_m is said to be an unbounded Vilenkin group.
If $m_j = 2$ for each j, then G_m is the Walsh group.

metrics

Generalized powers: M_n (Walsh case: $M_n = 2^n$) on the next slide

A Vilenkin group is metrizable in the following way:

$$d(x,y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric and the product topology above is the same.

generalized powers

$M_0 := 1, M_{n+1} := m_n M_n \, (n \in \mathbb{N})$

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, ..., m_i - 1\}, \ i \in \mathbb{N}),$$

Generalized Rademacher functions:

$$r_n(x) := \exp(2\pi\imath \frac{x_n}{m_n}) \quad (x \in G_m, n \in \mathbb{N}, \imath := \sqrt{-1}).$$

The n^{th} Vilenkin function:

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system.

Fourier coefficients, the partial sums of the Fourier series, the Fejér means: (in the usual way)

$$\hat{f}(n) := \int_{G_m} f\bar{\psi}_n,$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_k f.$$

Back to problems...

In 1936 Zalcwasser (Stud. Math.) asked how "rare" can be the sequence of strictly monotone increasing integers $a(n)$ such that

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n)}f \to f. \qquad (1)$$

For trigonometric system solved for continuous functions (uniform convergence) by Salem, 1955 (Am. J. Math.): if the sequence a is convex, then the condition $\sup_n n^{-1/2} \log a(n) < +\infty$ is necessary and sufficient for the uniform convergence for every continuous function.

For the time being, this issue with respect to the Walsh-Paley system has not been solved. Only, a sufficient condition is known, which is the same as in the trigonometric case (Glukhov, 1986 (Ukr. Math. J.)).

With respect to convergence almost everywhere, and integrable functions the situation is more complicated.

Belinksky, 1984 (Anal. Math.) for the trigonometric system the existence of a sequence $a(n) \sim \exp(\sqrt[3]{n})$ such that the relation (1) holds a.e. for every integrable function. In this paper Belinksky also conjectured that if the sequence a is convex, then the condition $\sup_n n^{-1/2} \log a(n) < +\infty$ is necessary and sufficient again. So, that would be the answer for the problem of Zalcwasser in this point of view (trigonometric system, a.e. convergence and L^1 functions).

We proved this is not the case for the Walsh-Paley system.

Theorem. (Gát, J. of Approx. Theory, 2010) Let $a : \mathbb{N} \to \mathbb{N}$ be a sequence with property $\frac{a(n+1)}{a(n)} \geq q > 1 \, (n \in \mathbb{N})$. Then for all $f \in L^1([0,1))$ we have the a.e. relation

$$\frac{1}{N} \sum_{n=1}^{N} S_{a(n)} f \to f.$$

Theorem. (Gát, J. of Approx. Theory, 2010) Let $a : \mathbb{N} \to \mathbb{N}$ be a convex sequence with property $a(+\infty) = +\infty$. Then for each $f \in L^1([0,1))$ we have the a.e. relation

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{S_{a(n)} f}{n} \to f.$$

Problems: Vilenkin, Walsh-Kaczmarz, trigonometric systems, other summation methods.

To investigate the Walsh-Kaczmarz case

$$E_n f(x) := 2^n \int_{I_n(x)} f, \quad I_n(x) := \{y : y_i = x_i, i = 0, \ldots, n-1\}.$$

$$E_n^A f(x) := \frac{1}{2^A} \sum_{i=0}^{2^A - 1} E_n f(x + i/2^A). \qquad (+ \text{ is mod } 1)$$

$$E^* f := \sup_{A,n,A<n} E_n^A |f|.$$

If E^* is of weak type (L^1, L^1), then it is highly likely:
If $a : \mathbb{N} \to \mathbb{N}$ be a sequence with property $\frac{a(n+1)}{a(n)} \geq q > 1 \, (n \in \mathbb{N})$, then for all $f \in L^1([0,1))$ we have the a.e. relation

$$\frac{1}{N} \sum_{n=1}^{N} S_{a(n)} f \to f.$$

Logarithmic means??

(Two) more dimensional functions:

$f : [0,1) \times [0,1) \to \mathbb{C}$, $\omega_{k,n}(x,y) := \omega_k(x)\omega_n(y)$,
Two-dimensional Fourier coefficients

$$\hat{f}(k,n) := \int_0^1 \int_0^1 f(x,y)\omega_{k,n}(x,y)dxdy,$$

Partial sums of the two-dimensional Fourier series

$$S_{M,N}f(x,y) := \sum_{k=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(k,n)\omega_{k,n}(x,y).$$

Two-dimensional Fejér means

$$\sigma_{M,N}f(x,y) := \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{n=0}^{N-1} S_{k,n}f(x,y)$$

Two-dimensional Fejér means

How can a function be reconstructed, knowing only its Walsh-Fourier coefficients?
That is, under what conditions $\sigma_{M,N} f(x,y) \to f(x,y)$

Trigonometric case: Two historical results

- In 1935 Jessen, Marcinkiewicz and Zygmund:
  $\sigma_{M,N} f \to f$ a.e., as if $\min\{m,n\} \to \infty$ for $f \in L^1 \log^+ L^1$
- In 1939 Marcinkiewicz and Zygmund:
  $\sigma_{M,N} f \to f$ a.e., as if $1/\beta \le M/N \le \beta$.

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Walsh-Paley case:

- F. Móricz, F. Schipp, and W.R. Wade, 1992 Trans Amer. Math. Soc. The unconditonal (Pringsheim sense) convergence for functions in $L\log^{+} L$, and the restricted one for L^1 functions. But, only $\sigma_{2^n,2^m} f \to f, |n-m| \leq C$.
- F. Weisz, 1996, Trans. Amer. Math. Soc.
  G. Gát, 1996, Annales Univ. Sci. Budapestiensis.
  The L^1 situation for all index pairs, not only powers of two.

Kaczmarz system

P. Simon, 2001, Monatsh Math.
Both situation (Pringsheim and restr.)

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It is also of interest to ask that it is possible to weaken the cone restriction condition with preserving the a.e. convergence for functions in L^1.

We give an answer in the negative both in the view of space and in the view of restrictness at the same time.
First result:

Gát, 2000, Proc. Amer. Math. Soc.
Let $\delta : [0,+\infty) \to [0,+\infty)$ be measurable, and $\lim_{t\to\infty} \delta(t) = 0$.
Then $\exists f \in L^1([0,1)^2)$ such as $f \in L\log^+ L\delta(L)$,
i.e. $\int_{[0,1)^2} |f(x)| \log^+(|f(x)|)\delta(|f(x)|)dx < \infty$ and
$\sigma_{n_1,n_2} f$ does not converge to f a.e. as $\min(n_2, n_2) \to \infty$.

The answer is the negative both in the view of space and in the view of restrictness at the same time

Gát, Analysis Math., 2001: Let δ as on the previous page, $w : \mathbb{N} \to [1, \infty)$ monotone increasing, $w(+\infty) = +\infty$.
$\Rightarrow$

there exists a function $f \in L^1([0,1)^2)$ such as $f \in L(\log^+ L)\delta(L)$ and

$$\lim_{\substack{\wedge n \to +\infty, \\ \vee n / \wedge n \leq w(\wedge n)}} \sigma_{n_1, n_2} f$$

does not equal with f a.e.

$(C, 1)$ divergence:

- Bounded Vilenkin case solved (Gát, AMH, 2005)
- Unbounded Vilenkin case is open. Interesting situation...
- Walsh-Kaczmarz case also open.
- What about (C, α) summability (div.)? Case $1 > \alpha > 0$ triv. Case $\alpha > 1$ open.

Fejér means of $L^1([0,1)^2)$ functions

What positive can be said for indices "not close to each other"?

Gat, G., Acta Math. Hungar., 2012 for the Walsh-Paley system

Let $a = (a_1, a_2) : \mathbb{N} \to \mathbb{N}^2$ be a sequence with property $a_j(+\infty) = +\infty$ $(j = 1, 2)$. Suppose that there exists an $\alpha > 0$ such that $a_j(n+1) \geq \alpha \sup_{k \leq n} a_j(k)$ $(j = 1, 2,\ n \in \mathbb{N})$. Then for each integrable function $f \in L^1([0,1)^2)$ we have the a.e. relation

$$\lim_{n \to \infty} \sigma_{a(n)} f = f.$$

Gát, G., J. of Math. Anal. and Appl., 2012

Same result as above for the trigonometric system.

For the Walsh-Kaczmarz system is open. Bounded Vilenkin case is to appear.

Marcinkiewicz means for $f \in L^1(I^2)$:

$$t_n f(x) := \frac{1}{n} \sum_{k=0}^{n-1} S_{k,k} f(x).$$

Marcinkiewicz (Ann Soc. Polon. Math. (1937)) for functions in the space $L \log^+ L$ a.e. relation $t_n f \to f$ with respect to the trigonometric system. The „L^1 result" for the trigonometric, Walsh-Paley, Walsh-Kaczmarz system:

- Zhizhiasvili (Izv. Akad. nauk USSR Ser Mat. (1968)) (trigonometric system),
- Weisz (J. Math. Anal. Appl. (2000)), (Walsh system)

Marcinkiewicz means

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- Nagy (J. of Approx. Theory (2006) (Walsh-Kaczmarz system)
- Gát (Georgian Math. Journ.) (2004)) (for bounded Vilenkin systems)
- unbounded Vilenking groups: nothing. Hopeful:

$$t_{M_n} f(x) = \frac{1}{M_n} \sum_{k=0}^{M_n-1} S_{k,k} f(x), \quad t_{M_n} \to f$$

Problems. Zalcwasser type: Let $(a(n))$ be a sequence of strictly monotone increasing integers. How "rare" can $(a(n))$ be with prop.

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n),a(n)}f \to f.$$

a.e. for $f \in L^1(I^2)$. Is it true that for any $\alpha(n) \nearrow +\infty$ $\exists \beta(n) \geq \alpha(n)$ such that for all $a(n) \geq \beta(n)$ we have the a.e. conv. for each integrable f.

Open for Walsh-Paley, Walsh-Kaczmarz, Vilenkin, trigonometric systems. Connected with next few slides and generalizations. - differs from 1-dim. case.

With resp. to L^1-norm conv. there are counterexamples. I.e. $\exists\, a(n) \nearrow +\infty$, for every subseq. $a(n_k)$ $\exists f \in L^1$:

$$\left\| \frac{1}{N}\sum_{k=1}^{N} S_{a(n_k),a(n_k)}f \right\|_1 \to \infty$$

Define the following Marcinkiewicz-like means: ($|n| = \lfloor \log_2 n \rfloor$), $\alpha = (\alpha_1, \alpha_2) : \mathbb{N}^2 \to \mathbb{N}^2$

$$t_n^\alpha f(x) := \frac{1}{n} \sum_{k=0}^{n-1} S_{\alpha_1(|n|,k),\alpha_2(|n|,k)} f(x).$$

$$\# \{ l \in \mathbb{N} : \alpha_j(|n|, l) = \alpha_j(|n|, k), l < n \} \leq C \quad (k < n, j = 1, 2)$$

$$\max \{ \alpha_j(|n|, k) : k < n \} \leq Cn \quad (n \in \mathbb{P}, j = 1, 2).$$

Theorem of convergence, Gát, 2012, J. of Approx. Theory.

Let α satisfy the two cond. above. Then $t_n^\alpha f \to f$ $(f \in L^1(I^2))$.

Theorem of divergence, Gát, 2012, J. of Approx. Theory.

Let $\gamma : \mathbb{N} \to \mathbb{N}$ be any function with property $\gamma(+\infty) = +\infty$. Then there exists a function α satisfying first condition and $\max\{\alpha_1(|n|, k) : k < n\} \leq Cn$, $\max\{\alpha_2(|n|, k) : k < n\} \leq Cn\gamma(n)$ and $f \in L^1(I^2)$ such that $\limsup_{n\in\mathbb{N}} |t_n^\alpha f| = +\infty$ almost everywhere.

Corollary (later we will use it)

$$\frac{1}{2^n}\sum_{k=0}^{2^n-1} S_{\alpha_1(n,k),\alpha_2(n,k)}f(x) \to f$$

a.e. for all $f \in L^1$.
$(\alpha_j(n,k) \leq C2^n)$

Marcinkiewicz-like means, triangular means, Walsh-Paley case

Open for Walsh-Kaczmarz, Vilenkin (even bounded case), trigonometric systems.

The rectangular partial sums of the 2-dimensional Walsh–Fourier series (recall)

$$S^{\square}_{M,N}f(x,y) := S_{M,N}f(x,y) := \sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\hat{f}(i,j)\omega_i(x)\omega_j(y).$$

The triangular partial sums of the 2-dimensional Walsh–Fourier series

Triangular means

$$S_k^{\triangle} f(x,y) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \hat{f}(i,j)\, w_i(x)\, w_j(y).$$

Denote the rectangular kernel

$$D_k^{\square}(x,y) := D_k(x)\, D_k(y)$$

and the triangular kernel

$$D_k^{\triangle}(x,y) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} w_i(x)\, w_j(y).$$

Marcinkiewicz-like means, triangular means, Walsh-Paley case

Triangle Fejér means:

$$t_n^{\triangle} f := \frac{1}{n} \sum_{k=1}^{n} S_k^{\triangle} f.$$

From conv. (JAT 2012) ⇒ for Walsh-Paley sys.:

$\|t_n^{\triangle} f - f\|_1 \to 0 \ (f \in L^1(I^2))$.

Theorem of convergence, For trig. sys.: Herriot, (Duke Math. J.,(1944))

$t_n^{\triangle} f \to f \ (f \in L^1(I^2))$ a.e. and in norm.

Difference: kernel problems. (no closed form Walsh)

Marcinkiewicz-like means, triangular means, Walsh-Paley case

Open problem: a.e. conv. $t_n^{\triangle} f \to f$ (Walsh-Paley)

However,

Goginava and Weisz (2012, Georgian Math. J.):
$t_{2^n}^{\triangle} f \to f$ $(f \in L^1(I^2))$ a.e.

From conv. (JAT 2012) follows corollary on slide 29 $\Rightarrow$ for Walsh-Paley sys.:

- $t_{a_n}^{\triangle} f \to f$ $(f \in L^1(I^2))$ a.e. for every (a_n) lacunar.
- Moreover, $\|t_n^{\triangle} f - f\|_1 \to 0$ $(f \in L^1(I^2))$.

Open problems: Vilenkin, Walsh-Kaczmarz system, $t_n f \to f$ a.e. problem.

Thank you for your attention!