Cumulants on the Wiener space and Quantitative Central Limit Theorems

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The model for a quantitative CLT

Let X_j a sequence of independent identically distributed random variables such that $\mathbb{E}(X_1) = 0$, $\mathbb{E}(X_1^2) = 1$ and $\mathbb{E}(|X_1|^3)$ is finite. If

$$V_n := \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}},$$

then V_n converges in distribution to $N \sim \mathcal{N}(0,1)$.
Moreover

Theorem Berry-Esseen 1942. *There exists some universal constant C such that, for all real z,*

$$|P(V_n < z) - P(N < z)| \leq \frac{C\mathbb{E}(|X_1|^3)}{\sqrt{n}}.$$

Kolmogorov Distance $d_{\mathrm{Kol}}(X, Y) := \sup_z |P(X < z) - P(Y < z)|$.

$$d_{\mathrm{Kol}}(V_n, N) \leq \frac{C\mathbb{E}(|X_1|^3)}{\sqrt{n}}.$$

The setting of Breuer Major Theorem.

Let $(X(j))_{j\in\mathbb{Z}}$ a sequence of Centered Gaussian random variables such that $\mathbb{E}(X(j)X(j+k)) = \rho(k)$ (Stationary Centered Gaussian Time series). We are interested in proving that

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} F(X(k)) \xrightarrow{d} \mathcal{N}(0,\sigma^2)$$

for some functionals F. In particular polynomials, and specifically Hermite polynomials

$$H_q(t) = (-1)^q e^{\frac{t^2}{2}} \frac{d^q}{dt^q}\left(e^{-\frac{t^2}{2}}\right).$$

When it is possible, give the speed of convergence.

Theorem Breuer-Major 1983. *Let $\rho \in \ell^q(\mathbb{Z})$. Then TFAE*

$$\text{(i)} \qquad \mathrm{Var}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_q(X(k))\right) \longrightarrow \sigma_q^2,$$

$$\text{(ii)} \qquad \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_q(X(k)) \xrightarrow{d} \mathcal{N}(0, \sigma_q^2),$$

with

$$\sigma_q^2 = q! \sum_{k \in \mathbb{Z}} \rho(k)^q.$$

Problem. Have quantitative versions.

Numerous partial results, in particular Nourdin, Peccati, Podolskij (2010) and Biermé, B., Léon (2010).

Wiener chaos and Fourth Moment Approach

$$L^2(X) := L^2(\Omega, \mathcal{A}, P) = \bigoplus \mathcal{H}_q$$

where $\bigoplus \mathcal{H}_q$ is the qth *Wiener chaos*.

$\mathcal{H}_q$ generated by the $H_q(\sum_{\text{finite}} a_j X(j))$. In particular $H_q(X(k))$ belongs to the Wiener chaos $\mathcal{H}_q$.

Particular case of quantitative CLT for F_n, when F_n belongs to $\mathcal{H}_q$.

Fourth Moment Theorem (Nualart Peccati 2005.) *Let $\{F_n : n \geq 1\}$ a sequence of random variables in $\mathcal{H}_q$ such that $\mathbb{E}[F_n^2] = 1$ for all $n \geq 1$. Then F_n converges in distribution to $N \sim \mathcal{N}(0,1)$ if and only if $\mathbb{E}[F_n^4] \to 3$.*

Fourth moment and cumulants

Moreover (Nourdin, Peccati 2009)

$$d_{\mathrm{Kol}}(F_n, N) \leq \sqrt{\mathbb{E}[F_n^4] - 3}.$$

$\mathbb{E}[N^4] = 3$. Stein's Method: $Y \sim \mathcal{N}(0,1)$ if and only if $\mathbb{E}(f'(Y)) = \mathbb{E}(Yf(Y))$, f smooth.

Let F a real-valued random variable, $\phi_F(t) = \mathbb{E}[e^{itF}]$ its characteristic function. The jth cumulant of F, denoted by $\kappa_j(F)$, is

$$\kappa_j(F) = (-i)^j \frac{d^j}{dt^j} \log \phi_F(t)|_{t=0}.$$

$\kappa_j(N) = 0$ for all $j > 2$.
When $\mathbb{E}(F) = 0$ and $\mathbb{E}(F^2) = 1$, then
$\kappa_3(F) = \mathbb{E}(F^3), \kappa_4(F) = \mathbb{E}(F^4) - 3$.

$$\mathbb{E}[F^{m+1}] - \kappa_{m+1}(F) = \sum_{s=1}^{m} \binom{m}{s-1} \kappa_s(F) \mathbb{E}[F^{m+1-s}].$$

Estimates of cumulants

Theorem (BBNP). *There exists universal constants* $c_s(q)$ *such that, for* $q \geq 2$ *and* $s > 4$*, whenever* F *is in the chaos* $\mathcal{H}_q$ *and satisfies* $\mathbb{E}(F) = 0$, $\mathbb{E}(F^2) = 1$, *then*

$$\kappa_s(F) \leq c_s(q) \, [\kappa_4(F)]^{\frac{s}{4}} \, .$$

Compare to the bounds for moments given by hypercontractivity:

$$\mathbb{E}(|F|^s) \leq (s-1)^{sq/2}.$$

All moments (and cumulants) are bounded in terms of the second one, but all cumulants (except for the second and the third one) are bounded in terms of the fourth cumulant.

The theorem was known for $q = 2$, with $c_s(q) = \frac{(s-1)!}{2 \times 3^{s/4}}$.

The speed of convergence for a smooth distance.

Recall:

$$d_{\mathrm{Kol}}(F,N) \leq \sqrt{\mathbb{E}[F^4]-3}.$$

Wasserstein distance

$$d_{\mathrm{Wass}}(X,Y) := \sup_{\|h'\|_\infty \leq 1} \left|\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]\right|.$$

"smooth" distance

$$d(X,Y) := \sup_{\|h''\|_\infty \leq 1} \left|\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]\right|.$$

Theorem (BBNP). *Whenever* F *is in the chaos* $\mathcal{H}_q$ *and satisfies* $\mathbb{E}(F)=0$, $\mathbb{E}(F^2)=1$, *then*

$$d_{\mathrm{Wass}}(F,N) \leq C \max\left\{|E[F^3]|, (E[F^4]-3)^{3/4}\right\}.$$

$$d(F,N) \leq C \max\left\{|E[F^3]|, E[F^4]-3\right\}.$$

Sharpness of the estimates.

Moreover the last result is sharp:
for F_n in the chaos $\mathcal{H}_q$ such that $\mathbb{E}(F_n) = 0$, $\mathbb{E}(F_n^2) = 1$ and converging in distribution to $N \sim \mathcal{N}(0,1)$, there exists $c > 0$ such that

$$d(F_n, N) \geq c \max\left\{|E[F_n^3]|, E[F_n^4] - 3\right\}.$$

For $q \geq 4$ even, the maximum can be obtained by each of the two terms (examples given in the Breuer Major setting).

Back to the theorem of Breuer Major.

Recall that $\{X(j), j \in \mathbb{Z}\}$ is a Gaussian time series with

$$\mathbb{E}(X(j)X(j+k)) = \rho(k) \quad \text{so that} \quad \mathbb{E}[H_q(X(j))H_q(X(j+k))] = q!\rho(k)^q.$$

$$\text{We define} \qquad F_n := \frac{H_q(X(1)) + \cdots + H_q(X(n))}{\sqrt{nv_n}},$$

with

$$\begin{aligned} v_n : &= \frac{1}{n}\mathbb{E}\left[(H_q(X(1)) + \cdots + H_q(X(n)))^2\right] \\ &= \frac{q!}{n}\sum_{k,k'=1}^{n} \rho(k-k')^q = q!\sum_{k=-n}^{n}\left(1 - \frac{|k|}{n}\right)\rho(k)^q. \end{aligned}$$

v_n tends to $q!\sum_{k \in \mathbb{Z}} \rho(k)^q = \sigma_q^2$, which is the variance of the limit law for $\sqrt{v_n}F_n$.

The third and fourth cumulants.

In the same way, computation of the cumulants of order three (for q even, otherwise 0) and four.

$$v_n^{3/2}\sqrt{n}\,\kappa_3(F_n) = \frac{1}{n}\sum_{j,k,l=1}^{n} \rho(k-l)^{q/2}\rho(k-j)^{q/2}\rho(l-j)^{q/2}$$
$$= \sum_{k,l=-n}^{n} \left(1 - \frac{\max(k,l)}{n} + \frac{min(k,l)_+}{n}\right) \rho(k-l)^{q/2}\rho(k)^{q/2}\rho(l)^{q/2}.$$

tends to $\langle \rho^{q/2} * \rho^{q/2}, \rho^{q/2}\rangle = \int_{\mathbb{T}} g^3 dt$, with g non negative, for $g(t) := \sum \rho(k)^{q/2} e^{ikt}$ under the assumption that ρ belongs to $\ell^{3q/4}(\mathbb{Z})$.

Theorem (BBNP). *For q even and $\rho \in \ell^{3q/4}(\mathbb{Z})$, for F_n defined as above and $h \in \mathcal{C}^1$ with a bounded derivative and such that $\gamma(h) := \int_{-\infty}^{+\infty} h(t)H_3(t)e^{-t^2/2}dt \neq 0$. Then*

$$\mathbb{E}(h(F_n)) - \mathbb{E}(h(N)) \sim \frac{c_q\gamma(h)\int_{\mathbb{T}} g^3 dt}{\sqrt{n}}.$$

Odd and even Wiener chaos behave differently.

Using previous work of Nourdin Peccati, one can prove that

Theorem . *For q even and $\rho \in \ell^{4/3}(\mathbb{Z})$ when $q = 2$ or $\ell^2(\mathbb{Z})$ for $q \geq 2$, for F_n defined as above, then, for $z \in \mathbb{Z}$ with $z \neq \pm 1$,*

$$P(F_n < z) - P(N < z) \sim \frac{c_q \int_{\mathbb{T}} g^3 dt \times (z^2 - 1)e^{-z^2/2}}{\sqrt{n}}.$$

In particular, one cannot expect a speed of convergence better than $n^{-1/2}$ for q even.

For q odd, one has a better convergence, in $1/n$, for the smooth distance.

Comparison between different distances?

First tool: Wiener-Itô stochastic integrals.

Here we can assume that

$$\rho(k) = \int_{\mathbb{T}} e^{ikt} d\mu(t).$$

μ, which is positive, is the spectral measure of the Gaussian time series. When it is absolutely continuous, $d\mu(t) = g(t)dt$, then we can assume that

$$X(k) = \int e^{ikt} (g(t))^{1/2} dW(t)$$

with W a complex Brownian Motion.

More generally, let $\mathfrak{H} := L^2(A, \mu)$ be a real Hilbert space and ε_k is such that $\langle \varepsilon_k, \varepsilon_l \rangle_{\mathfrak{H}} = \rho(k - l)$. The Wiener-Itô stochastic integral $X(h)$ is defined for $h \in \mathfrak{H}$ and such that

$$\mathbb{E}\left[X(h)X(g)\right] = \langle h, g \rangle_{\mathfrak{H}}.$$

The q-th fold multiple Wiener-Itô stochastic integral is defined on $\mathfrak{H}^{\odot q} = L^2{}_{\mathrm{sym}}(A^q, \mu^{\otimes q})$ by

$$I_q(h \otimes \cdots \otimes h) := H_q(X(h))$$

and is an isometry between $\mathfrak{H}^{\odot q}$ and the Wiener chaos $\mathcal{H}_q$.

Multiplication Formula for $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f)I_q(g) = \sum_{r=0}^{\min(p,q)} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g).$$

Malliavin calculus and integration by parts.

Define D, *the Malliavin derivative*, by

$$D(I_q(h \otimes \cdots \otimes h)) := I_{q-1}(\underbrace{h \otimes \cdots \otimes h}_{q-1 \text{ times}})\, h.$$

For $F = \sum_{\text{finite}} I_q(f_q)$, then $L^{-1}F := -\sum_{q \geq 1} q^{-1} I_q(f_q)$.
With these notations, integration by parts

$$\mathbb{E}(FG) := \mathbb{E}(F)\mathbb{E}(G) - \mathbb{E}(\langle DF, DL^{-1}G\rangle_{\mathfrak{H}}).$$

In particular,

$$\mathbb{E}(f(F)G) := \mathbb{E}(f(F))\mathbb{E}(G) - \mathbb{E}(f'(F)\langle DF, DL^{-1}G\rangle_{\mathfrak{H}}).$$

Stein's method and the end.

For fixed h consider f solution of $f'(t) - tf(t) = h(t) - \mathbb{E}(h(N))$ with control at infinity. Then

$$\mathbb{E}(h(N)) - \mathbb{E}(h(F)) = \mathbb{E}(Ff(F)) - \mathbb{E}(f'(F)).$$

Moreover, by using inductively integration by parts,

$$\mathbb{E}[Ff(F)] = \sum_{s=0}^{M-1} \frac{\kappa_{s+1}(F)}{s!}\mathbb{E}[f^{(s)}(F)] + \mathbb{E}[\Gamma_M(F) f^{(M)}(F)].$$

where $\Gamma_s(F)$ is inductively defined by $\Gamma_0(F) = F$ and, for every $j \geq 1$,

$$\Gamma_j(F) = \langle DF, -DL^{-1}\Gamma_{j-1}(F)\rangle_{\mathfrak{H}}.$$

We use the fact that (Nourdin, Peccati 2010) $\mathbb{E}(\Gamma_j(F)) = \frac{\kappa_{j+1}(F)}{j!}$.
The key point is the inequality, valid for $q \geq 2$ and $s \geq 3$,

$$\mathbb{E}[|\Gamma_s(F)|] \leq c_s(q) \times \left(\mathbb{E}[F^4] - 3\right)^{\frac{s+1}{4}}.$$

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