Gallai, colourings and critical graphs – a 50 year anniversary

Talk by Bjarne Toft

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at the Erdős Centennial Conference, Budapest, July 2013
We all have a favorite paper -
I have two (both exactly 50 years old):
Or rather: I have three favorites!

- Gallai's beautiful theory of alternating paths.
- The Gallai-Edmonds decomposition theorem.
- AN INTRIGUING FOOTNOTE:
- With the present methods I have succeeded in getting factorization theorems for general graphs besides $\sigma=1$ only for $\sigma=2$. I shall discuss these results on another occasion.
Gallai’s papers are mostly in German

• Gallai’s work is not as well-known as it should be, due to the fact that most of the papers are available in German only.

• It might be an idea to make translations into English available, either in bookform or on the internet (but it is a demanding and non-trivial task to create such translations).

• The papers deserve to be studied in their original form – Gallai is an outstanding writer with a very careful style and a lot of interesting details.
Let $\chi(G)$ denote the chromatic number of the graph $G$ (without loops and multiple edges), and $\varphi(x)$ the degree of the vertex $x$ of $G$. $G$ is said to be $k$-critical (shortly critical) if $\chi(G) = k$ and if $\chi(G') < \chi(G)$ is valid for each proper subgraph $G'$ of $G$. If $G$ be a $k$-critical graph, then $\varphi(x) \geq k - 1$ holds for each vertex $x$ of $G$ (see [1]), and if $\varphi(x) = k - 1$ or $\varphi(x) > k - 1$ we call $x$ a secondary or primary vertex of $G$, respectively. There exist critical graphs containing secondary vertices only. The complete graphs and odd circuits (circuits containing an odd number of edges) are such graphs and from the theorem of Brooks it follows that no other graphs have the same property. We have found, that in every critical graph the subgraph spanned by the secondary vertices is of a very simple structure. There holds the following

**Theorem 1.** Let $G$ be a $k$-critical graph and $G'$ the subgraph of $G$ spanned by its secondary vertices. Then the lobes of $G'$ are complete $j$-graphs $(0 \leq j \leq k)$ and odd circuits.

(The lobes (see [8], [9]) of a graph $G$ are the maximal connected subgraphs of $G$ with respect to the property that every pair of edges is contained in a circuit of $G$. The isolated vertices are also lobes. A complete $j$-graph is a complete graph with $j$ vertices ($j \geq 0$), the complete 0-graph is the void graph.)

Ignoring the complete $k$-graphs the property expressed in Theorem 1 together with a trivial additional condition is sufficient for the characterization of the subgraphs $G'$ in the case $k \geq 4$. This is asserted in the following

**Theorem 2.** Let $k \geq 4$ and let $G'$ be a graph whose lobes are complete $j$-graphs $(0 \leq j \leq k - 1)$ and odd circuits, and the degrees of whose vertices are $\leq k - 1$. Then there exists a $k$-critical graph $G$ such that the subgraph spanned by the secondary vertices of $G$ is isomorphic to $G'$.

In Theorem 2 there is also contained the statement that for $k \geq 4$ there are $k$-critical graphs without secondary vertices. In the proof of this statement the most important step is the determination of a $4$-critical graph without secondary vertices. (With the aid of such a graph it is easy to construct $k$-critical graphs without secondary vertices.)
THEORY OF GRAPHS
AND ITS APPLICATIONS

Proceedings of the Symposium
held in Smolenice
in June 1963

A
Publishing House
of the Czechoslovak
Academy of Sciences
and LONDON

B
Academic Press
NEW YORK

Smolenice June 1963
Critical graphs I

- The blocks in the minor subgraphs are complete and/or odd cycles (the minor graph is a forest of Gallai trees)
- This is best possible (by construction)
- If G is k-critical \((k \geq 4)\) on n vertices \((n>k)\) then
  \[2e \geq (k - 1)n + \frac{(k - 3)}{(k^2 - 3)}n\]
- Gallai’s Conjecture:
  \[2e \geq \frac{[(k^2 - k - 2)n - k(k - 3)]}{(k - 1)}\] and this is sharp for \(n=1 \mod (k-1)\)
4-critical graphs with empty minor graph

\[ \begin{array}{ccc}
\text{max} & \text{min} & |V(G')| \\
G \text{ k-chromatic} & G' \text{ odd cycle} & G' \subseteq G
\end{array} \]

\[ \begin{array}{c}
1 & 2 & 12 \rightarrow +1 \\
2 & 1 & 21 \rightarrow -1
\end{array} \]

\[ \Sigma_{c \text{ always}} = 0 \]

\[ \Sigma \neq 0 \]
Critical graphs II

• A k-critical graph with $\leq 2k-2$ vertices has disconnected complement
• The proof uses Gallai’s theory of alternating paths from the 1950 paper
• Other proofs by Molloy 1999 and Stehlik 2003
• The right minimum number of edges for all $n \leq 2k-1$
Balatonfüred and Budapest, Aug.-Dec. 1969. Ivan Tafteberg Jakobsen and I were in Hungary as part of a PhD-program

- **Erdős** was present in Budapest the whole fall
- **Rényi and Turán** came to the Institute regularly
- **Vera Sós** conducted a combinatorics course at the University in English
- **Hajós** was at the University
- **Gallai** spent one full day at the Institute each week, working in the Library and meeting people, maybe going out for lunch
- And many young people around!

- Ivan and I had a weekly meeting with Gallai for about one hour
- He showed us ideas and problems
- He was very kind, very open, but extremely modest about his own achievements
Gabriel Andrew Dirac

CRITICAL GRAPHS FIRST
DEFINED IN G.A. DIRAC’s
PhD-thesis 1951

Gabriel Andrew Dirac (1925-1984)

HUNGARIA (formerly New York) near intersection of Rakoczi ut & Korut (ut = street). Water go there.
BELVAROSI KAVEHAZ (= cafe) near Pest end of new Elisabeth bridge.
GERBEAUD in Vorosmarty Square at the end of Vaci utca (= street) very good and pleasant cafe, but not for breakfast.
GELLERT hotel cafe.

There are also pleasant and historic cafes at the intersection of Andrassy ut & Körut.
Critical \( k \)-chromatic graphs (on \( n \) vertices) with just one Major vertex

- Min max \(|C| \leq c \log n\)
- BEST POSSIBLE –
- Alon, Krivelevich, Seymour 2000
- Shapira&Thomas 2011
- Max min \(|C| \sim c \log n\)
  - Erdős 1959 and 1962
- Max min \(|\text{odd } C| \) ??
Critical 4-chromatic graphs with long shortest odd cycles

- $\text{Max min } |\text{odd } C| \geq c \sqrt{n}$
- BEST POSSIBLE

In 1973, P. Erdős conjectured that for each $k \geq 2$, there exists a constant $c_k$ so that if $G$ is a graph on $n$ vertices and $G$ has no odd cycle with length less than $c_k n^{1/k}$, then the chromatic number of $G$ is at most $k+1$. Constructions due to Lovász and Schriver show that $c_k$, if it exists, must be at least $1$. In this paper we settle Erdős' conjecture in the affirmative. We actually prove a stronger result which provides an upper bound on the chromatic number of a graph in which we have a bound on the chromatic number of subgraphs with small diameter.
Planar 4-critical graphs without vertices of degree 3 (Koester 1984).
4-critical graphs with all vertices of high degree (max min $\delta(G)$)

- Simonovits and Toft 1971
- $\text{Max min } d(G) \geq c \sqrt[3]{|\mathcal{V}(G)|}$
- BEST POSSIBLE ??
Sperner’s Simplex Lemma
(as told to us by Gallai)

\[ \text{3-colouring:} \]

\[ \sum_c = 0 \quad \text{always} \]

\[ \sum_c \neq 0 \]

\[ \text{THE RED GRAPH IS 4-CHR.} \]
A generalization:

1972 Budapest

1974 Prague
A maximum independent set of vertices in a bipartite graph

• has the same number of elements as a minimum set of edges covering all vertices (we assume that $\delta \geq 1$)
• Equivalently: the complement of a bipartite graph is perfect
• Due to König and Gallai 1932
• Published by Gallai in 1958 and 1959
• Gallai told me that he proved this in 1932 as an answer to a question posed in a lecture by König, but when Gallai told him, König said that he already knew. So Gallai did not count it as his own result.
MAX MIN THEOREMS

MAXIMUM-MINIMUM SÄTZE ÜBER GRAPHEN

Von
T. GALLAI (Budapest)
(Vorgelegt von O. HAJOS)

Einleitung

Im folgenden behandeln wir in vereinfachter Weise die Ergebnisse einer kürzlich in ungarischer Sprache erschienenen Arbeit [9], [10]. Wir beweisen mehrere, dem Mengerschen „a-Kettenzatz“ ([14], S. 222) ähnliche „Maximum-Minimum“ Sätze. Eigenvärs verallgemeinerte (in matrizentheoretischer Formulierung) den auf paare Graphen bezüglichen Spezialfall des Mengerschen Satzes in solcher Weise, daß er die Kanten der Graphen mit Zahlen bewertete und statt des Maximums der Kantenzahlen gewisser Kantenmengen das Maximum der Wertsommen der betrachteten Kantenmengen nahm ([3], S. 17, 1). Den allgemeinen Mengerschen Satz kann man nicht in der gleichen Richtung ausdehnen. Es gelingt aber, einen der Eigenvärs Verallgemeinerung ähnlichen Satz dadurch zu finden, daß man statt ungerichteter Graphen gerichtete, statt Wege gerichtete Kreise nimmt. In gleicher Weise kann man aus dem „max-flow min-cut“ Satz [1], [5], [6], [7], der den Mengerschen Satz als Spezialfall enthält, mehrere der Eigenvärs Verallgemeinerung entsprechende Sätze herleiten (Sätze (2.1.6), (3.1.4), (3.2.3), (3.2.6)). Wir werden jedem dieser Sätze je einen „dualen“ Satz zur Seite stellen (Sätze (2.1.7), (3.1.5), (3.2.4), (3.2.7)). Durch Anwendung der erhaltenen Sätze auf besondere Graphen bzw. Bewertungen gelangen wir zu weiteren Maximum-Minimum Sätzen (Sätze (4.2.3), (4.2.5), (4.2.7), (4.2.9), (4.3.1), (4.3.3), (4.3.5), (4.4.12), (4.4.13)), die sich auf Wege bzw. auf Kanten beziehen, und die als Spezialfall des „max-flow min-cut“ Satz, den erwähnten Eigenvärschen Satz und einen von DILWORTH stammenden, auf halbeordnete Mengen bezüglichen Satz ([2], S. 161, 1, 1) enthalten.

Es ist bemerkenswert, daß in den Sätzen die Gangzeitdertigkeit der Bewertung die Gangzeitdertigkeit der anderen auftretenden Zahlenwerte nach sich zieht. Wir werden unsere Sätze erst unter Berücksichtigung der Gangzeitdertigkeit ableiten und nur nachträglich zeigen, daß sie auch mit nicht ganzzahligen Bewertungen in Kraft bleiben (Abschnitt 2.5).


Gallai, T.

Maximum-minimum Sätze und verallgemeinerte Faktoren von Graphen. (German. Russian summary)


This paper makes an important advance in the theory of the factorization of graphs. To each vertex $X$ of a finite graph $\Gamma$ let there be assigned two non-zero integers $\kappa(X)$ and $\kappa'(X)$. Let $Q$ denote a set of paths in $\Gamma$, each passing through no vertex more than once except for a possible return to the initial point. (A re-entering path is considered to have two coincident ends.) Such a system $Q$ is called admissible if no two members of $Q$ have a common edge, the paths of $Q$ have in all at most $\kappa(X)$ ends at $X$, and these paths pass through $X$, other than as an endpoint at most $\kappa'(X)$ times. The author obtains a formula for the maximum number of admissible paths for given functions $\kappa$ and $\kappa'$.

He finds that his main result includes Menger’s theorem and the factorization theorems given by Petersen, Baeble, Tutte, Belck and Ore.

Reviewed by W. T. Tutte

Gallai, T.

Über extreme Punkt- und Kantenmengen. (German)


Given a finite graph $G$ without isolated vertices, a set $P$ of vertices of $G$ is called independent if no two elements of $P$ are adjacent. Independence of a set of edges is defined similarly. $P$ spans $G$ if every edge of $G$ is incident with some $p \in P$; similarly for sets of edges. Let $p_{\text{max}}$, $p_{\text{min}}$ be the maximum [minimum] number of independent [spanning] vertices of $G$. $k_{\text{max}}$ and $k_{\text{min}}$ the analogous numbers for edges. Then the following “duality theorem” is proved: $p_{\text{max}} + k_{\text{min}} = p_{\text{min}} + k_{\text{max}}$ is the number of vertices of $G$. The result is extended to graphs on the non-negative integers.

Reviewed by G. Sabidussi

(c) Copyright American Mathematical Society 1962, 2013
Scribbles by Gallai, summer 1972
Other Gallai classes of graphs

$n = 3(k-1) + 1$
$h_i + h_{i+1} \leq k-1$
$h_2 + h_1 \leq k-1$
4-critical graphs with many edges/high min degree
Critical k-chromatic graphs with precisely two Major vertices

- IF there are precisely two major vertices and they are independent
- THEN the minor graph is disconnected
- Gallai’s Conjecture: the number of conn. components in the minor graph is at least the number of components in the major graph

- PROVED by Stiebitz in 1982
- USED by Krivelevich in 1997
Tibor Gallai (1912 – 1992)
Perhaps graph theory owes more to the contact of mankind with mankind, than to the contact of mankind with nature.

Vielleicht noch mehr als der Berührung der Menschheit mit der Natur verdankt die Graphentheorie der Berührung der Menschen untereinander.
König in Göttingen 1904/05

...
Sós, Erdös, Sved and Gallai
(from *N is a number* by George Csicsery)

ANONYMUS IN THE CITY PARK
(Városliget)
Lieber Kollege Dr. Toft,


Hoffentlich ist in Odense die graphentheoretische Literatur Ihnen zugänglich und haben Sie genugend viel Zeit weitere schöne Ergebnisse zu erreichen.

Mit vielen herzlichen Grüßen

Ihr

Tibor Gallai
Another letter 1977

To
Professor
Kalman,
Eidgenössische
Technische
Hochschule
Zürich

Another...
Dear Teft,

Many thanks for your interesting letter. I was in Germany and France and just returned and will be here for another 3-6 weeks. I have lunch with Gallai today and Heidts tomorrow and in a few days will want to go to see Heidt's talk.

On Feb. 7 there will be a small meeting in Cambridge in honor of Lovett. I will be in. I would love it if we could meet then. The year old son is still staying at the Gallai and visited only here that I left my son at home and I had to borrow one from Gallai.

Let N be a graph with n vertices and no triangle. Assume that every vertex has degree > 1. How large can the maximal degree of G be? Fix the degree of the vertex le a(x) = a(x) = ... Can you determine?

max a(0) + a(1) + ... + a(x), you will of course recognize that I came to these problems by looking at your nice paper. It does not guarantee of course that the problems are well posed.

Let G be a graph of a vertices and e edges. Assume that it is the subgraph of a maximal four

Many thanks for sending us your very interesting booklet. I am just having supper with Gallai. I had no time or place to study it carefully.

I think the answer to your question on p. 2.3 is negative if G(m) has no triangle then its chromatic number is 0(m^3). The reason is that By a theorem of Ajtai, Komlo and Szemeredi a G(m) which has no triangle always has an independent set of size 1/2 log m. They strengthened the slightly weaker result of Groso and Joch.

(1963). Now the greedy algorithm leaves at every step you take the largest independent set and color the whole graph with

Best regards to you and your family, colleagues, we miss

E.P.

[Signature]

[Address: 40, Rue de la 11, Brouxel 1040]
Two Gallai conjectures

Gallai made the following conjecture: Let $G(m)$ be a $k$-chromatic edge critical graph of $m$ vertices. Is it true that the number of $K_{k-1}$ contained in our graph is $\leq m$ equality only if $m = k$ and $G(m)$ is a $k$-clique. His second conjecture states that the number of $(k-1)$ chromatic critical subgraphs of our $G(m)$ is $\geq m$. There conjecture are open even for $k=4$.

Gallai came to the conjecture from a conjecture of Reed: Let $G(m)$ be a $K_k$ chromatic critical and not a $K_k$, then it contains a critical $(k-1)$ chromatic graph which is not $K_{k-1}$. Yedidia and I stated in one of our papers the following conjecture: Let $G(m)$ be a graph of $m$ vertices, assume that every subgraph of $G(m)$ contains an independent set of size $\frac{m}k$. Then the
Thank you!

Erdős, Keszthely 1973, conducting a problemsession on the boat on Lake Balaton during the conference excursion