Local chromatic number
of graphs and digraphs

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a survey

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Definition
Erdős, Füredi, Hajnal, Komjáth, Rödl, Seress 1986

\[ \psi(G) := \min_c \max_v |\{c(u) : \{u,v\} \in E(G)\}| + 1 \]

The minimization is over all proper colorings \( c \).

\( \psi(G) \) is the minimum number of colors that must appear in the most colorful closed neighborhood of a vertex in any proper coloring.

Obviously: \( \psi(G) \leq \chi(G) \).

Thm. (EFHKRS 1986): \( \forall k, \exists G : \psi(G) = 3, \chi(G) > k \).
revisiting local chromatic number
Körner, Pilotto, Simonyi, 2005

+ directed version

related to “Sperner capacity” in information theory (KPS) + an information transmission problem (Shanmugam-Dimakis-Langberg 2013)

+ study of $\psi$ and $\psi_d$ as a graph parameters, e.g.:

**Thm. (KPS 2005):** $\chi^*(G) = \psi^*(G) \leq \psi(G)$

$\chi^*$ = fractional chromatic number
Thm. (KPS 2005): $\chi^*(G) \leq \psi(G) \leq \chi(G)$

$\chi^*(G) = \chi(G) \implies \psi(G)$ is determined

examples for $\chi^*(G) \ll \chi(G)$

- Kneser graph
- Schrijver graphs
- (generalized) Mycielski graphs
- shift graphs
- ????

$\chi(G)$ is found through “topological methods”
topological method
started by Lovász, 1978

graph $G \rightarrow$ complex $B(G)$
(topological space)
$\chi(G) \geq k \iff \text{connectivity / dimension / etc of } B(G)$

often using the Borsuk-Ulam theorem
Thm. (Lusternik-Schnirelmann, 1934 ≈ B-U)
$S^{n-2}$ covered by open antipodal-free sets $\Rightarrow \geq n$ sets needed
Thm. (Ky Fan, 1952)
$S^{n-2}$ covered by open antipodal-free sets $\Rightarrow \exists x, x'$
antipodal $x$ covered by $\geq \lceil n/2 \rceil$, $x'$ by $\geq \lfloor n/2 \rfloor$ sets.
Zig-zag Thm. (Simonyi, T., 2006)
If $\chi(G) \geq k$ for “topological reasons”, then $\forall$ proper coloring of $G$ contains a rainbow-colored $K_{[k/2],[k/2]}$

Corollary: $\psi(G) \geq \lceil k/2 \rceil + 1$
Corollary: For Kneser, Scrijver, gen. Mycielski graphs:
$\psi(G) \geq \lceil \chi(G)/2 \rceil + 1$

Surprise: For large enough Schrijver and gen. Mycielski graphs:
$\psi(G) \leq \lfloor \chi(G)/2 \rfloor + 2$

Leaves a gap of 1 if $\chi$ is even.

Thm. (Simonyi, T., Vrećica, 2009): For Kneser, Schrijver and gen. Mycielski graphs:
$\psi(G) \geq \lfloor \chi(G)/2 \rfloor + 2$
Open

\[ \psi = \chi \] for all Kneser graphs?

Or \( \psi \approx \chi/2 \) for some Kneser graphs?
Thm. (KPS 2005): $\chi^*(G) \leq \psi(G) \leq \chi(G)$

$\chi^*(G) = \chi(G) \quad \Rightarrow \quad \psi(G) \text{ is determined}$

Examples for $\chi^*(G) \ll \chi(G)$:

- Kneser graph
- Schrijver graphs
- (generalized) Mycielski graphs

$\chi(G)$ is found through “topological methods”

- shift graphs — topological methods fail
- ????
shift graphs $H_m$

$V(H_m) = \{(i,j) \mid 1 \leq i < j \leq m\}$

$E(H_m) = \{(i,j),(j,k) \mid 1 \leq i < j < k \leq m\}$

**Thm. (Simonyi, T., 2011)**

$\chi(H_m) - \psi(H_m) \leq 1$

$\chi(H_m) = \psi(H_m)$ \quad \text{if} \quad 2^k + 2^{k-1} < m \leq 2^{k+1}$

**Proof:** Bollobás type inequalities

Is $\chi(H_m) = \psi(H_m)$ for all $m$?

(I’m sure.)
Definitions

Erdős, Füredi, Hajnal, Komjáth, Rödl, Seress 1986

$$\psi(G) := \min_c \max_v |\{c(u) : \{u,v\} \in E(G)\}| + 1$$

The minimization is over all proper colorings $$c$$.

$$\psi(G)$$ is the minimum number of colors that must appear in the most colorful closed neighborhood of a vertex in any proper coloring.

Körner, Pilotto, Simonyi, 2005

$$\vec{G}$$ is a directed graph

$$\psi_d(\vec{G}) := \min_c \max_v |\{c(u) : (v,u) \in E(\vec{G})\}| + 1$$

The minimization is over all proper colorings $$c$$ of the underlying graph $$G$$.

$$\psi_d(\vec{G})$$ is the minimum number of colors that must appear in the most colorful closed outneighborhood of a vertex in any proper coloring of $$G$$.

Obviously: $$\psi_d(\vec{G}) \leq \psi(G)$$.
Thm. (Simonyi, T., 2011)

If $\chi(G) \geq k$ for “topological reasons”,
then $\psi_d(\overrightarrow{G}) \geq \lceil k/4 \rceil + 1$.

Tight for large enough Schrijver and gen. Mycielski graphs.
local chromatic number of $\vec{G}$ versus $G$

$\psi_d(\vec{G}) \leq \psi(G)$

if both direction appears in $\vec{G}$ of every edge of $G$

Can = achieved always with an orientation (= no edge in both directions)?

Thm. (Simonyi, T., Zsbán, 2013)

$\exists G$ with $\psi(G) = 4$, $\forall$ orientation $\psi_d(\vec{G}) = 3$

How far can $\psi_d(\vec{G})$ and $\psi(G)$ be? Very far.

$\psi(H_m) \approx \log m$, but for the “natural” orientation of $\psi_d(H_m) = 2$

Thm. (KPS 2005): $\chi^*(G) = \psi^*(G) \leq \psi(G)$

$\chi^*(H_m) < 4$. How far can $\psi_d(G)$ and $\chi^*(G)$ be? Not far.

Thm. (STZs, 2013)

$\sup \chi^*(G)/\psi_d(\vec{G}) = \sup \chi^*(G)/\psi^*(\vec{G}) = e$

independently Shanmugam-Dimakis-Langberg with a worse constant

OPEN: can the gap be arbitrarily large?
graphs on surfaces

Mohar, Simonyi, T., 2013: Generalization of Youngs, Archdeacon-Hutchinson-Nakamoo-Negami-Ota, Mohar-Seymour results from chromatic to local chromatic number.

+ graph on genus 5 surface with $\psi \neq \chi$.

OPEN: planar graph with $\psi \neq \chi$?

need $\chi = 4, \chi^* = 3$