On sum of powers of the positive integers

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I regret to say that, as far as I know, Erdős has never considered the sums $1^k + 2^k + \ldots + n^k$, which are my subject today\(^1\). Theorem 2 of my talk is, however, similar to the theorem Erdős proved in his paper *On integers of the form* $2^k + p$ *and some related problems* in 1950.

W. Bednarek asked in a letter for a characterization of pairs of positive integers $(k, m)$ such that for every positive integer $n$

$$1^k + 2^k + \ldots + n^k \mid 1^{km} + 2^{km} + \ldots + n^{km}. \quad (1)$$

\(^1\)The first sentence is not quite true, since Erdős and Moser formulated a conjecture about the Diophantine equation $1^k + 2^k + \ldots + (x - 1)^k = x^k$ (added during the conference).
The following theorem contains a partial answer with the help of Bernoulli numbers. They are denoted by \( B_n \):

\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \ldots, \quad B_{2l+1} = 0,
\]

and the Bernoulli polynomial \( \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} \) by \( B_n(x) \).
Theorem 1.  If the divisibility $(1)$ holds for every positive integer $n$, then $m$ is odd and

\[ \frac{B_{km}}{B_k} \in \mathbb{Z} \text{ for } k \text{ even}, \]
\[ mB_{km-1}/B_{k-1} \in \mathbb{Z} \text{ for } k \text{ odd } \geq 3. \]  

(2)

The condition is sufficient for $k \leq 3$, but insufficient for $k = 4$ and infinitely many $m$. 

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In fact we propose

Conjecture. For $k > 3$ the divisibility \((1)\) holds for every positive integer \(n\) only for \(m = 1\).

To support this conjecture we have

Theorem 2. For $k = 4$, \(n \equiv 58966743 \pmod{5^6 \cdot 11251^2}\) the divisibility \((1)\) holds only for \(m = 1\).

Theorem 3. For $m = n = 3$ the divisibility \((1)\) holds only for \(k \leq 3\).
Proof of Theorem 1

The proof of Theorem 1 is based on five lemmas.

**Lemma 1.** For all positive integers $k$ and $n$

$$1^k + \ldots + (n-1)^k = S_k(n) = \frac{1}{k+1}(B_{k+1}(n) - B_{k+1}).$$

This is classical.

**Lemma 2.** If $P, Q \in \mathbb{Q}[x]$ and $P(n)/Q(n) \in \mathbb{Z}$ for all sufficiently large integers $n$ then $r(x) = P(x)/Q(x)$ is an integer-valued polynomial.

This is easy.
Proof of Theorem 1

**Lemma 3.** If $3^\nu \parallel 2N$, where $N = n, n + 1$ or $n + \frac{1}{2}$ and $\nu \geq 1$, then for every positive integer $m$

$$3^{\nu-1} \mid S_{2m}(n + 1).$$

**Lemma 4.** If $2^\nu \parallel N$, where $N = n$ or $n + 1$ and $\nu \geq 1$, then for every positive integer $r > 2$

$$2^{\nu-1} \mid S_{2r}(n + 1).$$

Proofs of both lemmas are tedious.
Lemma 5. If a prime $p$ satisfies $p - 1 \nmid k$, then $p$ does not divide the denominator of $B_k$. If $p - 1 | k$, then $p$ occurs in the denominator of $B_k$ in the first power only.

This is the von Staudt theorem.
Proof of Theorem 1. Necessity. Since (1) holds for \( n = 2 \) we obtain \( m \equiv 1 \pmod{2} \). Consider now \( k \) even. By Lemma 1 we have

\[
S_k(n) = \frac{1}{k+1} B_{k+1}(n), \quad S_{km}(n) = \frac{1}{km+1} B_{km+1}(n),
\]

hence, for all integers \( n > 1 \), \( B_{k+1}(n) > 0 \) and

\[
\frac{k+1}{km+1} \frac{B_{km+1}(n)}{B_{k+1}(n)} \in \mathbb{Z}.
\]
Proof of Theorem 1

By Lemma 2

\[ r(x) = \frac{k + 1}{km + 1} \frac{B_{km+1}(x)}{B_{k+1}(x)} \]

is an integer-valued polynomial and, since \( r(0) = B_{km}/B_k \), (2) follows.

Consider next \( k \geq 3 \) odd. We have by Lemma 1

\[ S_k(n) = \frac{1}{k + 1} (B_{k+1}(n) - B_{k+1}), \]
\[ S_{km}(n) = \frac{1}{km + 1} (B_{km+1}(n) - B_{km+1}), \]

hence, for all integers \( n > 1 \), \( B_{k+1}(n) > B_{k+1} \) and

\[ \frac{k + 1}{km + 1} \frac{B_{km+1}(n) - B_{km+1}}{B_{k+1}(n) - B_{k+1}} \in \mathbb{Z}. \]
Proof of Theorem 1

By Lemma 2

\[ r(x) = \frac{k + 1}{km + 1} \frac{B_{km+1}(x) - B_{km+1}}{B_{k+1}(x) - B_{k+1}} \]

is an integer-valued polynomial and, since

\[ r(0) = mB_{km-1}/B_{k-1}, \]

(2) follows.

Proof of sufficiency for \( k \leq 3 \) is tedious.
**Proof of Theorem 1**

*Insufficiency for k = 4.* Take $m$ to be a prime $\equiv 17 \pmod{30}$. The condition (2) is fulfilled, since $B_{4m}/B_4 = -30B_{4m} \in \mathbb{Z}$. Indeed, by Lemma 5, $B_{4m}$ has in the denominator only the first powers of primes $p$ such that $p - 1 | 4m$. The divisibility gives $p = 2, 3, 5, 2m + 1$ or $4m + 1$. Now, $2 \cdot 3 \cdot 5 = 30$, $2m + 1$ is divisible by 5 and $4m + 1$ by 3. It follows from Theorem 2 that $S_4(n + 1) \nmid S_{4m}(n + 1)$ for a positive integer $n$. 
The proof of Theorem 2 is based on four lemmas.

**Lemma 6.** If $p$ is a prime, $k' \equiv k \not\equiv 0 \pmod{p-1}$ and $n' \equiv n \pmod{p}$, then

$$S_{k'}(n') \equiv S_k(n) \pmod{p}.$$ 

**Lemma 7.** If $p > 2$ is a prime, $k \geq \alpha \geq 2$, $k' \geq \alpha$, $k \not\equiv 0 \pmod{p(p-1)}$, $k' \equiv k \pmod{p^{\alpha-1}(p-1)}$ and $n' \equiv n \pmod{p^{\alpha+1}}$, then

$$S_{k'}(n') \equiv S_k(n) \pmod{p^\alpha}.$$
Lemma 8. If \( n \equiv 58966743 \pmod{11251^2} \), then \( S_{4m}(n + 1) \equiv 0 \pmod{11251} \) only if \( m \equiv 1 \pmod{5625} \).

Proof. The number \( p = 11251 \) is a prime and \( n \equiv 252 \pmod{p} \), \( \left\lfloor \frac{n}{p} \right\rfloor \equiv 5241 \pmod{p} \). If \( 4m \equiv 0 \pmod{p - 1} \), then

\[
S_{4m}(n + 1) \equiv n - \left\lfloor \frac{n}{p} \right\rfloor \equiv -4989 \not\equiv 0 \pmod{p}.
\]

If \( 4m \not\equiv 0 \pmod{p - 1} \), it suffices by Lemma 6 to verify the congruence \( S_{4m}(252) \equiv 0 \pmod{p} \) for \( m \) in the interval \([1, 11249]\). The verification has been performed by J. Browkin.
Lemma 9. If $n \equiv 58966743 \pmod{5^6}$, then $S_{4m}(n + 1) \equiv 0 \pmod{5^5}$ only if $m = 1$ or $m \equiv 501 \pmod{625}$.

Proof. We have $58966743 \equiv 13618 \pmod{5^6}$. If $m \equiv 0 \pmod{5}$, then

$$S_{4m}(n+1) \equiv n - \left\lfloor \frac{n}{5} \right\rfloor \equiv 13618 - 2723 = 10895 \not\equiv 0 \pmod{25}.$$ 

If $m \not\equiv 0 \pmod{5}$, it suffices by Lemma 7 to verify the congruence $S_{4m}(13619) \equiv 0 \pmod{5^5}$ for $m$ in the interval $[1, 626]$. The verification has been performed by J. Browkin.
Proof of Theorem 2. Since for $n \equiv 58966743 \pmod{5^6 \cdot 11251^2}$ we have

$$S_4(n + 1) \equiv 0 \pmod{5^5 \cdot 11251}$$

the theorem follows from Lemmas 8 and 9.
The proof of Theorem 3 is based on

**Lemma 10.** *For every positive integer $k$*

$$(2^k, 1 + 2^k + 3^k) \leq 4,$$

$$(3^{k+1}, 1 + 2^k + 3^k) \leq 3k.$$  

This is easy.
Proof of Theorem 3. We have

\[1 + 2^{3k} + 3^{3k} - 2^k \cdot 3^{k+1} = (1 + 2^k + 3^k)(1 + 2^{2k} + 3^{2k} - 2^k - 3^k - 6^k),\]

thus if (1) holds, then

\[1 + 2^k + 3^k \mid 2^k \cdot 3^{k+1}.\] (3)

By Lemma 10 \((2^k \cdot 3^{k+1}, 1 + 2^k + 3^k) \leq 12k\), thus by (3)

\[1 + 2^k + 3^k \leq 12k,\]

which implies \(k \leq 3\).