Roth’s theorem on arithmetic progressions

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general framework: Ruzsa
$A \subseteq \mathbb{Z}$ ‘the’ set of integers
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solve $f(x, y, z, w, \ldots) = 0$ with $x, y, z, w, \ldots \in A$
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$x + y = z + w$ (add. quad.)
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$x + y = 2z$ (3ap)
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x + y = z + w \text{ (add. quad.)}

x + y = 2z \text{ (3ap)}

aim: show finding solutions is ‘equally’ difficult
$A \subset \mathbb{Z}$ ‘the’ set of integers

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$$x + y = 2z \text{ (3ap)}$$

aim: show finding solutions is ‘equally’ difficult

(inequivalent: $x + y = z, x - y = 7$)
\[ Q(A) := |\{(x, y, z, w) \in A^4 : x + y = z + w\}| \]

and

\[ T(A) := |\{(x, y, z) \in A^3 : x + y = 2z\}| \]
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\[ Q(A) \leq |A|^3 \quad \text{and} \quad T(A) \leq |A|^2 \]
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$Q(A) \leq |A|^3$ and $T(A) \leq |A|^2$

think $|A| \to \infty$
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\[ Q(A) \leq |A|^3 \text{ and } T(A) \leq |A|^2 \]

think \( |A| \to \infty \)

‘many’ means positive proportion of max
easy fact: ‘many 3aps $\Rightarrow$ many add. quads.’
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$$T(A) = \sum z \cdot 1_{2 \cdot A}(z) \sum_{x+y=z} 1_A(x)1_A(y)$$
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\[
T(A) = \sum_z 1_{2 \cdot A}(z) \sum_{x+y=z} 1_A(x)1_A(y)
\]

\[
\leq |2 \cdot A|^{1/2} \left( \sum_z \left( \sum_{x+y=z} 1_A(x)1_A(y) \right)^2 \right)^{1/2} = (|A|Q(A))^{1/2}
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so

\[ T(A) \geq \delta |A|^2 \Rightarrow Q(A) \geq \delta^2 |A|^3. \]
not polynomially equivalent (Behrend):

\[ Q(A) \geq \delta |A|^3 \not\Rightarrow T(A) \geq \delta^{O(1)}|A|^2. \]
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\[ Q(A) \geq \delta |A|^3 \Rightarrow T(A) \geq \exp(-O(\log^2 \delta^{-1})) |A|^2? \]
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hard fact: (Frei\v{c}man, Heath-Brown, Ruzsa, Szemerédi)

\[ Q(A) \geq \delta |A|^3 \Rightarrow T(A) \geq \exp\left(-\delta^{-O(1)}\right)|A|^2 \]
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plan: find an ‘approximate’ group \( B \) so that \( Q_B \) and \( T_B \) (strongly) polynomially equivalent
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$(A' := A \cap B$ is large and has $Q_B(A')|A'| \approx T_B(A')^2)$
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Bourgain
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so precise direct counting

Bloom, Henriot, Schoen, Shkredov