Coloring a graph arising from a lacunary sequence, Diophantine approximation, and constructing a Kakeya set: Applications of the probabilistic method

Yuval Peres

Based on joint works with

Wilhelm Schlag;

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1 Microsoft Research


Fine properties of Brownian paths, following Paul Lévy and continued by S.J. Taylor, J.F. Le Gall and many others.
Does Brownian motion have points of increase?
From the review by J. Lamperti:

Let $X$ be the standard Brownian motion in one dimension. It is well-known that, with probability 1, a path of this process is nowhere differentiable; the present paper establishes the more delicate fact that almost all Brownian paths have no points of increase. The proof is quite intricate . . .

Simple proofs:
According to S. Kakutani (1990), DEK first found a "proof" that points of increase do exist, by a fancy version of the reflection principle . . .

Some echoes of 3AM can be found in the original paper . . .
For more information on Brownian sample paths:

- *Ecole d’Été de Probabilités de Saint-Flour XX – 1990* by Mark I. Freidlin and Jean-François Le Gall
- *Brownian Motion* by Peter Mörters and Yuval Peres
Define a graph $G_S$ with vertex set $\mathbb{Z}$, where the pair $\{n, m\}$ is an edge iff $|n - m| \in S$.

**Example:** $n_k = k^d$ where $2 < d \in \mathbb{N}$.

- $G_S$ has no triangles by FLT
- Furstenberg (1977) and Sárközy (1978) showed that $\forall A \subset \mathbb{Z}$ of positive upper density, $\exists x, y \in A$ and $k \in \mathbb{N}$ such that $x - y = k^d$.
- Thus every independent set in $G_S$ has zero density $\Rightarrow$
- The chromatic number $\chi(G_S) = \infty$. 
Two problems of Erdős on lacunary sequences

- The chromatic number $\chi(G)$ of a graph $G$ is the minimal number of colors in a proper vertex coloring (neighbors assigned distinct colors.)

**Problem A (Erdős, 1987)**

Fix $\varepsilon > 0$ and suppose $S = \{n_j\}_{j=1}^\infty$ is a lacunary sequence of positive integers, where $n_{j+1} > (1 + \varepsilon)n_j$ for all $j \geq 1$. Is the chromatic number $\chi(G_S)$ necessarily finite?

**Problem B (Erdős, 1975)**

Let $\varepsilon > 0$ and $S$ be as in Problem A. Is there a number $\theta \in (0, 1)$ so that the sequence $\{n_j\theta\}_{j=1}^\infty$ is not dense modulo 1?

The relation between Problem A and Problem B was discovered by Katznelson in 1987, and published in 2001.
• Let $\delta > 0$ and $\theta \in (0, 1)$ be such that $\inf_j \|\theta n_j\| > \delta$, where $\| \cdot \|$ is distance to the closest integer.

• Partition $\mathbb{T} = [0, 1)$ into $k = \lceil \delta^{-1} \rceil$ disjoint intervals $I_1, \ldots, I_k$ of length $\frac{1}{k} \leq \delta$.

• Let $G$ be the graph from Problem A and assign the vertex $n \in \mathbb{Z}$ the color $j$ iff $n\theta \in I_j \pmod{1}$.

• Any two vertices connected by an edge must have different colors. Therefore, $\chi(G) \leq k = \lceil \delta^{-1} \rceil$. 
Previous Works

- Problem B was solved by Pollington (1979), de Mathan (1980) and Katznelson (2001); 
- As noted by Moshchevitin (2010), problem B was already raised and solved in 1926 by Khinchin, but this was forgotten . . .
- Khinchin (1926) and Katznelson (2001) showed that there exists a $\theta$ such that
  \[
  \inf_{j \geq 1} \| \theta n_j \| > c \varepsilon^2 |\log \varepsilon|^{-1}.
  \]
Theorem (P., Schlag; Bull. London Math. Soc. 42 (2010))

Suppose \( S = \{n_j\} \) satisfies \( n_{j+1}/n_j \geq 1 + \varepsilon \), where \( 0 < \varepsilon < 1/4 \). Then there exists \( \theta \in (0, 1) \) such that

\[
\inf_{j \geq 1} \|\theta n_j\| > c\varepsilon \log \varepsilon^{-1},
\]

where \( c > 0 \) is a universal constant. Therefore, the graph \( G = G_S \) described in Problem A satisfies \( \chi(G) \leq c^{-1} |\log \varepsilon|/\varepsilon \).

- Up to the \( |\log \varepsilon|^{-1} \) factor, (1) is optimal. Indeed, let \( n_j = j \) for \( j = 1, 2, \ldots, \lfloor \varepsilon^{-1} \rfloor \) and continue this as a lacunary sequence with ratio \( 1 + \varepsilon \). In this case \( \chi(G) > \lfloor \varepsilon^{-1} \rfloor \).
The following quantitative result on Problem B extends the previous theorem.

**Theorem (P., Schlag 2010)**

Suppose $S = \{n_j\}$ satisfies $n_{j+1}/n_j \geq 1 + \varepsilon$ for all $j$. Define

$$E_j = \left\{ \theta \in \mathbb{T} : \|n_j \theta\| < \frac{c_0 \varepsilon}{|\log_2 \varepsilon|} \right\}$$

for $j \geq 1$. If $240 c_0 \leq 1$, then

$$\bigcap_{j=1}^{\infty} E_j^c \neq \emptyset.$$
Proof ingredient: Lovász local lemma

Lemma

Let \( \{A_j\}_{j=1}^N \) be events in a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( \{x_j\}_{j=1}^N \) be a sequence of numbers in \((0, 1)\). Assume that for every \( i \leq N \), there is an integer \( 0 \leq m(i) < i \) so that

\[
\mathbb{P}(A_i \mid \bigcap_{j<m(i)} A_j^c) \leq x_i \prod_{j=m(i)}^{i-1} (1 - x_j).
\] (4)

Then for any integer \( n \in [1, N] \), we have

\[
\mathbb{P}\left(\bigcap_{i=1}^{n} A_i^c\right) \geq \prod_{\ell=1}^{n} (1 - x_\ell).
\] (5)

The lemma is applied to Lebesgue measure in \([0, 1]\) and to sets \( \{A_j\} \), where \( A_j \) is the union of all binary intervals of length \( \frac{c_0 \varepsilon}{n_j \log_2 \varepsilon} \) that intersect \( E_j \).
Further applications of the method

**MR2770060 Y. Bugeaud and N. Moshchevitin (2011)**
Badly approximable numbers and Littlewood-type problems.

**From Math Reviews:**

*The Littlewood conjecture states that, for any given pair \((\alpha, \beta)\) of real numbers, we have* \[
\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0,
\]
*where \(\|\cdot\|\) denotes the distance to the nearest integer. The authors prove, with a method introduced by Y. Peres and W. Schlag, that the set of pairs \((\alpha, \beta) \in \mathbb{R}^2\) such that*

\[
\lim_{q \to +\infty} q \cdot (\log q)^2 \cdot \|q\alpha\| \cdot \|q\beta\| > 0
\]

*has full Hausdorff dimension in \(\mathbb{R}^2\).*
A subset $S \subseteq \mathbb{R}^2$ is called a Kakeya set if it contains a unit segment in every direction.
A subset $S \subseteq \mathbb{R}^2$ is called a **Kakeya** set if it contains a unit segment in every direction.

Kakeya’s question (1917): Is the three-pointed deltoid shape a Kakeya set of minimal area?
Besicovitch (1919) gave the first *deterministic* construction of a Kakeya set of zero area.

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(Figures due to Terry Tao)
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New connection to game theory and probability

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We do so by relating these sets to a game of pursuit on the cycle $\mathbb{Z}_n$ introduced by Adler et al.
A. S. Besicovitch.  
On Kakeya’s problem and a similar one. 

Roy O. Davies.  
Some remarks on the Kakeya problem.  

Micah Adler, Harald Räcke, Naveen Sivadasan, Christian Sohler, and Berthold Vöcking.  
Randomized pursuit-evasion in graphs.  

Yakov Babichenko, Yuval Peres, Ron Peretz, Perla Sousi, and Peter Winkler.  
Hunter, Cauchy Rabbit and Optimal Kakeya Sets.  
Definition of the game $G_n$

Two players

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Definition of the game $G_n$

Two players
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Two players

Hunter
Definition of the game $G_n$

Two players

Hunter

Rabbit
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Hunter

Rabbit
Definition of the game $G_n$

Two players

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Rabbit

Where?
Two players

Hunter

Rabbit

Where?

On $\mathbb{Z}_n$
Definition of the game

When?

At night – they cannot see each other...
Definition of the game

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At night – they cannot see each other....
Definition of the game $G_n$

**Rules**

At time 0 both hunter and rabbit choose initial positions. At each subsequent step, the hunter either moves to an adjacent node or stays put. Simultaneously, the rabbit may leap to any node in $\mathbb{Z}^n$.

When does the game end? At “capture time”, when the hunter and the rabbit occupy the same location in $\mathbb{Z}^n$ at the same time.

**Goals**

Hunter: Minimize “capture time”

Rabbit: Maximize “capture time”
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Define a **zero sum** game $G_n^*$ with payoff 1 to the hunter if he captures the rabbit in the first $n$ steps, and payoff 0 otherwise.
The $n$-step game $G_n^*$

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- We will estimate $p_n$, and construct a Kakeya set of area $\asymp p_n$, that consists of $4n$ triangles.
Examples of strategies

- If the rabbit chooses a random node and stays there, the hunter can sweep the cycle, so expected capture time is $\leq n$. 

- What if the rabbit jumps to a uniform random node in each step? Then, for any hunter strategy, he will capture the rabbit with probability $\frac{1}{n}$ at each step, so expected capture time is $n - 1$.

- Zig-Zag hunter strategy: He starts in a random direction, then switches direction with probability $\frac{1}{n}$ at each step.

- Rabbit counter-strategy: From a random starting node, the rabbit walks $\sqrt{n}$ steps to the right, then jumps $2\sqrt{n}$ to the left, and repeats. The probability of capture in $n$ steps is $\approx \frac{n}{2}$, so mean capture time is $n^{3/2}$. 

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It turns out the best the hunter can do is start at a random point and continue at a random speed. More formally...

Let $a, b$ be independent uniform on $[0, 1]$. Let the position of the hunter at time $t$ be $H_t = \lceil an + bt \rceil \mod n$.

What capture time does this yield?

Let $R_\ell$ be the position of the rabbit at time $\ell$ and $K_n$ the number of collisions, i.e. $K_n = n - 1 \sum_{i=0}^{1} (R_i = H_i)$.

Use second moment method – calculate first and second moments of $K_n$. 

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K_n = \sum_{i=0}^{n-1} 1(R_i = H_i).
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We will show that $P(K_n > 0) \gtrsim 1 \log n$.

Recall $K_n = \sum_{n-1}^{0} i = \sum_{i=0}^{n-1} (R_i = H_i)$

$H_t = \lceil an + bt \rceil \mod n$

$E[K_n] = \sum_{i=0}^{n-1} P(H_i = R_i) = 1$

$E[K_{2n}] = E[K_n] + \sum_{i \neq \ell} P(H_i = R_i, H_\ell = R_\ell)$

Suffices to show $E[K_{2n}] \asymp \log n$

Then by Cauchy-Schwartz

$P(K_n > 0) \geq \frac{E[K_n]}{E[K_{2n}]} \gtrsim \frac{1}{\log n}$.
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\[ \text{Need to prove} \]

\[ P(H_i = R_i, H_{i+1} = R_{i+1}) \lessapprox 1/n. \]

This is equivalent to showing that for fixed \( r, s \),

\[ P(a_n + b_i \in (r - 1, r], na + b_i (i + j) \in (s - 1, s]) \lessapprox 1/n. \]

Subtract the two constraints to get

\[ b_j \in [s - r - 1, s - r + 1] - \text{this has measure at most} \frac{2}{j}. \]

After fixing \( b \), the choices for \( a \) have measure \( 1/n \).
Hunter’s optimal strategy

Need to prove

$$\mathbb{P}(H_i = R_i, H_{i+j} = R_{i+j}) \lesssim \frac{1}{jn}.$$
Hunter’s optimal strategy

Need to prove

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With the hunter’s strategy above $r$,

Recall

$K_n = \sum_{i=0}^{n-1} (H_i = R_i) P(K_n > 0) \gtrsim \log n.$

This gave expected capture time at most $n \log n$.

What about the rabbit? Can he escape for time of order $n \log n$?

Looking for a rabbit strategy with $P(K_n > 0) \ll \log n$.

Extend the strategies until time $2n$ and define $K_{2n}$ analogously.

Obviously $P(K_n > 0) \leq E[K_{2n}]$.

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\[ \mathbb{P}(K_n > 0) \leq \frac{\mathbb{E}[K_{2n}]}{\mathbb{E}[K_{2n} \mid K_n > 0]}. \]
If the rabbit starts at a uniform point and the jumps are independent, then $E[K^2_n] = 2r$. Recall $K^2_n = 2^n - \sum_{i=0}^{1} (R_i^2)$. 

Idea: Need to make $E[K^2_n | K_n > 0]$ "big" so $P(K_n > 0) \leq (\log n)^{-1}$. This means that given the rabbit and hunter collided, we want them to collide "a lot". The hunter can only move to neighbours or stay put. So the rabbit should also choose a distribution for the jumps that favors short distances, yet grows linearly in time. This suggests a Cauchy random walk.
If the rabbit starts at a uniform point and the jumps are independent, then
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So the rabbit should also choose a distribution for the jumps that favors short distances, yet grows linearly in time. This suggests a Cauchy random walk.
By time $i$, the hunter can only be in the set $\{ -i \mod n, \ldots, i \mod n \}$. We are looking for a distribution for the rabbit so that $P(R_i = \ell) \succ 1/i$ for $\ell \in \{ -i \mod n, \ldots, i \mod n \}$.

Then by the Markov property $E[K_{2n} | K_n > 0] \geq n - 1 \sum_{i=0}^{\infty} P_0(H_i = R_i) \succ \log n$.

Intuition: If $X_1, \ldots$ are i.i.d. Cauchy random variables, i.e. with density $\left( \pi \left( 1 + x^2 \right) \right)^{-1}$, then $X_1 + \ldots + X_n$ is spread over $(-n, n)$ and with roughly uniform distribution. This is what we want—but in the discrete setting...
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*Let’s imitate that in the discrete setting:*
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**Let's imitate that in the discrete setting:**
Let \((X_t, Y_t)_t\) be a simple random walk in \(\mathbb{Z}^2\). Define hitting times
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T_i = \inf\{t \geq 0 : Y_t = i\}
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- With probability \(1/4\), SRW exits the square via the top side.

Of the \(2^i + 1\) nodes on the top, the middle node is the most likely hitting point: subdivide all edges, and condition on the (even) number of horizontal steps until height \(i\) is reached; the horizontal displacement is a shifted binomial, so the mode is the mean. Thus the hitting probability at \((0, i)\) is at least \(1/(8i + 4)\).
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- Thus the hitting probability at \((0, i)\) is at least \(1/(8i + 4)\).
Suppose $0 < k < i$. With probability $1/4$, SRW exits the square $[-k, k]^2$ via the right side. Repeating the previous argument, the hitting probability at $(k, i)$ is at least $c/i$. 

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![Diagram of Cauchy Rabbit]
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$$\frac{1}{n} \quad \frac{1}{n}$$

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In these triangles we can find a unit segment in all directions that have an angle in $[0, \pi/4]$
If the rabbit employs the Cauchy strategy, then

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Kakeya sets from the Cauchy process

Motivated by the Cauchy strategy, let’s see a continuum analog of the probabilistic Kakeya construction of the hunter and rabbit.
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\(\Lambda\) is a quarter of a *Kakeya set* – it contains all directions from 0 up to 45° degrees. *Take four rotated copies of \(\Lambda\) to obtain a Kakeya set.*
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\(\text{Leb}(\Lambda) = 0\) and most importantly the \(\varepsilon\)-neighbourhood satisfies almost surely

\[
\text{Leb}(\Lambda(\varepsilon)) \asymp \frac{1}{|\log \varepsilon|}
\]
Keich in 1999 showed there is no Kakeya set which is a union of $n$ triangles with area of smaller order than $1/\log n$. Bourgain earlier noted that the $\varepsilon$ neighborhood of any Kakeya set has area at least $1/|\log \varepsilon|$. 
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General graphs

Consider a graph on \( n \) vertices.
Pick a spanning tree.
Depth first search yields

This is a closed path of length \( 2n - 2 \).

The hunter can now employ his previous strategy on this path.
This will give \( O(n \log n) \) capture time.
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On any graph the hunter can catch the rabbit in time $O(n \log n)$. Open Question: If the hunter and rabbit both walk on the same graph, is the expected capture time $O(n)$?