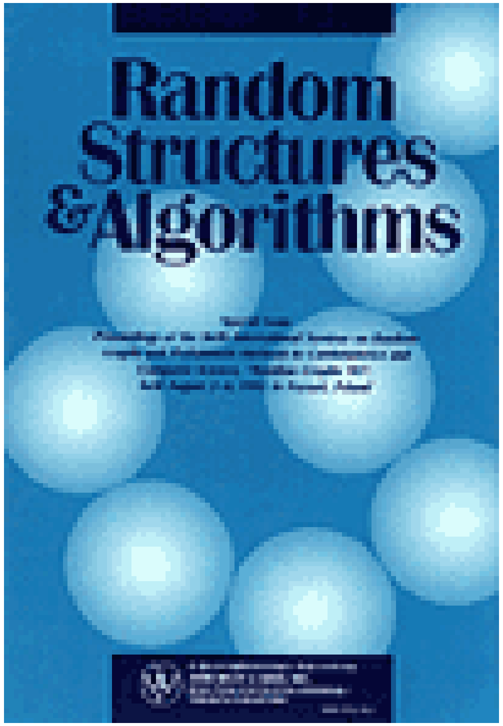
Paul Erdős and Probabilistic Reasoning

Noga Alon, Tel Aviv U.

Budapest, July 2013

The Probabilistic Method

To prove that a structure with certain desired properties exists, define an appropriate probability space of structures and show that the desired properties hold in this space with positive probability.

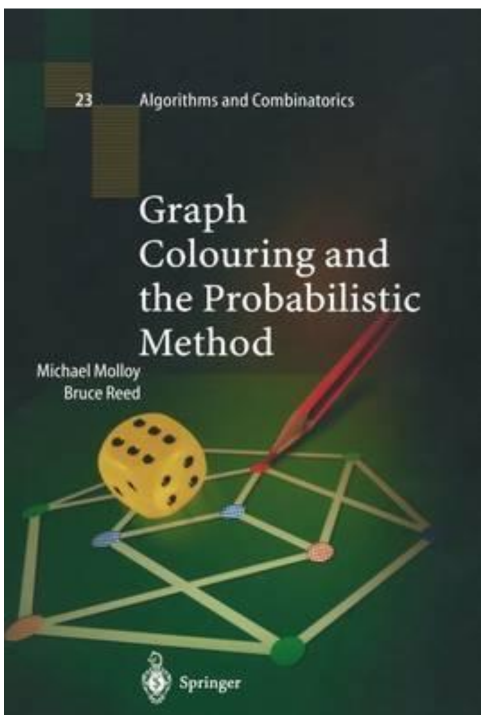

Random
Structures
&Algorithms

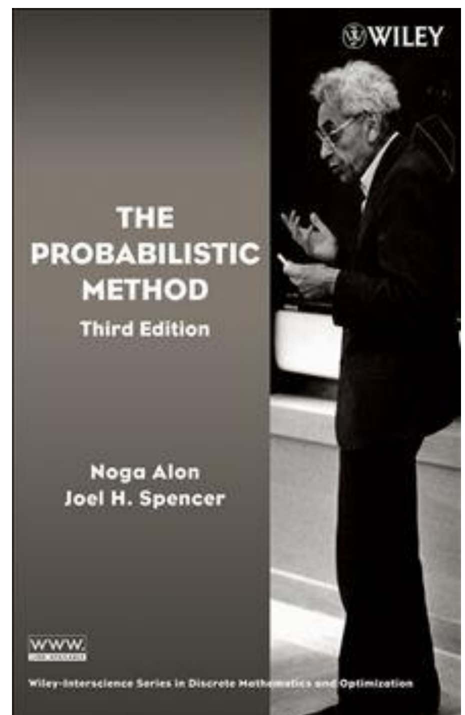

Combinatorics,
Probability &
Computing

Erdős/Spencer

Probabilisitic Methods in Combinatorics

Akadémiai Kiadó, Budapest





Ramsey Theory

Informally: every sufficiently large system contains a well organized large subsystem.

Applications exist in:

Combinatorics and Graph Theory

Number Theory

Functional Analysis

Game Theory

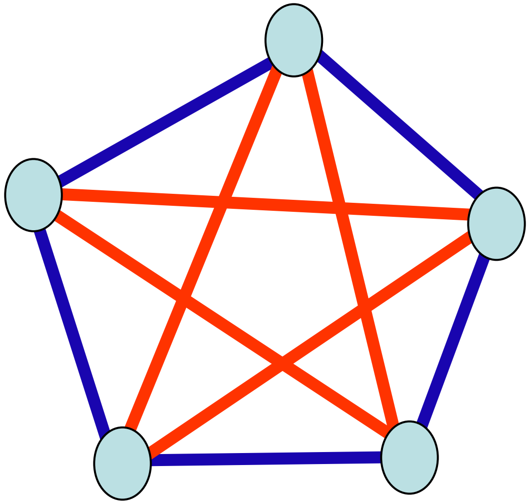
Theoretical Computer Science

Geometry

…

Def: For graphs $H_1, H_2, \ldots, H_k$, the **Ramsey Number** $r(H_1, H_2, \ldots, H_k)$ is the minimum n such that in every k-edge-coloring of the complete graph K_n on n vertices, there is a **monochromatic** copy of H_i in color i, for some i between 1 and k.

Example: $r(K_3, K_3) = 6$



Theorem [Ramsey (1930)]: For every k graphs H_1 , H_2 , … , H_k , the **Ramsey number** $r(H_1,H_2, \ldots ,H_k)$ is finite.

In particular:

$r(K_3,K_3)=6$, $r(K_3,K_4)=9$

$r(K_3,K_5)=14$, $r(K_3,K_6)=18$

$r(K_3,K_7)=23$, $r(K_3,K_8)=28$

$r(K_3,K_9)=36$

Theorem [Erdős (47), Erdős and Szekeres (35)]:

$$\boldsymbol{2^{k/2} \leq r(K_k, K_k) \leq 4^k}$$

In fact:

$$(1 - o(1))\frac{k}{e\sqrt{2}}2^{k/2} \leq r(K_k, K_k) \leq \binom{2k-2}{k-1}$$

Spencer (75), using the Lovász Local Lemma, improved the lower bound by a factor of 2.

Conlon (2009) improved the upper bound by a factor of $k^{-C \log k / \log \log k}$

Erdős (47):

Let $N \leqq 2^{k/2}$. Clearly the number of different graphs of N vertices equals $2^{N(N-1)/2}$. (We consider the vertices of the graph as distinguishable.) The number of different graphs containing a given complete graph of order k is clearly $2^{N(N-1)/2}/2^{k(k-1)/2}$. Thus the number of graphs of $N \leqq 2^{k/2}$ vertices containing a complete graph of order k is less than

$$(1) \qquad C_{N,k}\frac{2^{N(N-1)/2}}{2^{k(k-1)/2}} < \frac{N^k}{k!}\,\frac{2^{N(N-1)/2}}{2^{k(k-1)/2}} < \frac{2^{N(N-1)/2}}{2}$$

since by a simple calculation for $N \leqq 2^{k/2}$ and $k \geqq 3$

$$2N^k < k!2^{k(k-1)/2}.$$

But it follows immediately from (1) that there exists a graph such that neither it nor its complementary graph contains a complete subgraph of order k, which completes the proof of Theorem I.

Turán (54):

I had conjectured for a while that the graph $D(n,[\sqrt{n}]+2)$ is the extremum graph of the problem, and hence every graph P of order n or its complementary graph $\bar{P}$ contains a complete subgraph of order "about $\sqrt{n}$". Now denoting by $h(n)$ the maximum integer l such that in every graph P of order n or in its complementary graph $\bar{P}$ there is a complete subgraph of order l, Erdös[7]) showed that for $n \geqq 64$

$$(3) \qquad h(n) \leqq \frac{2}{\log 2} \log n,$$

i. e. the order of $h(n)$ is much less than expected. His proof is purely an existence proof.

Thm [Ajtai, Komlós and Szemerédi (80), Kim (95)]:

There are positive constants c_1, c_2 so that for all t

$$c_1 \frac{t^2}{\log t} \leq r(K_3, K_t) \leq c_2 \frac{t^2}{\log t}$$

Shearer (83): $c_2 \leq 1+o(1)$

The proof uses subtle probabilistic arguments based on the semi-random method.

Upper bound: random greedy
Lower bound: the triangle free process

The triangle-free process [Bollobás-Erdős (90)]: starting with the empty graph, keep adding random edges without creating a triangle.

Bohman (09): analysis using the differential equation method of Wormald

Fiz-Pontiveros, Griffiths, Morris (13+):

$$r(3,t) \geq \left(\frac{1}{4} - o(1)\right)\frac{t^2}{\log t}$$

Conjecture (Erdős-Sós 79): the limit of the ratio

$$r(K_3,K_t) \;/\; r(K_3,K_3,K_t)$$

as t tends to infinity, is 0.

Thm [A and Rödl (05)]:

There are positive constants c_1, c_2 so that

$$c_1 \frac{t^2}{\log^2 t} \le r(C_4, C_4, C_4, K_t) \le c_2 \frac{t^2}{\log^2 t}$$

More generally: For all p>1 and q≥(p-1)!+1 there are positive c_1 , c_2 so that for all t

$$c_1 \frac{t^p}{\log^p t} \le r(K_{p,q}, K_{p,q}, K_{p,q}, K_t) \le c_2 \frac{t^p}{\log^p t}$$

The proof uses the spectral properties of the Projective Norm Graphs [A-Rónyai and Szabó (99), following Kollár, Rónyai and Szabó(96)].

These imply that they don't have many independent sets of size t, and each of the first three color classes is a random shift of such a graph.

Similar ideas suffice to settle the Erdős-Sós conjecture and show that

$$r(K_3,K_3,K_t) = \tilde{O}(t^3),$$

whereas

$$r(K_3,K_t) = \tilde{O}(t^2)$$

Sum-Free subsets

A subset A of an abelian group is sum-free if there are no x,y,z in A with x+y=z.

Erdős (65): every set of n positive reals contains a sum-free subset of size at least n/3.

A-Kleitman (90): indeed so (in fact > n/3).

Eberhard, Green, Manners (13): the constant 1/3 is tight.

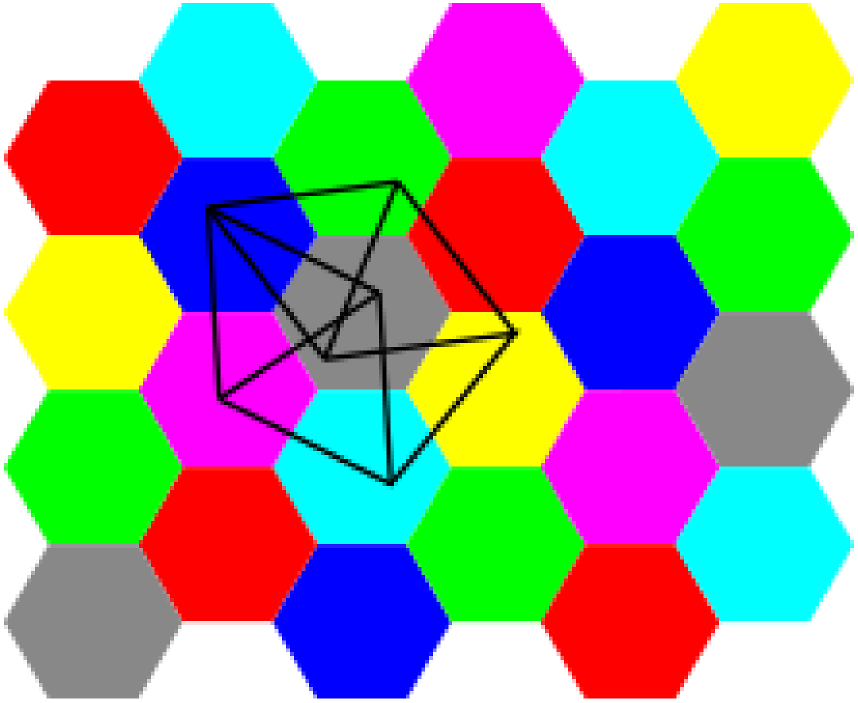
Euclidean Ramsey Theory

The Hadwiger-Nelson Problem: what is the minimum number of colors required to color the points of the Euclidean plane with no two points of distance 1 with the same color ?

Equivalently: what is the chromatic number of the unit distance graph in the plane ?

Nelson (1950): at least 4

Isbell (1950): at most 7



Both bounds have also been proved by Hadwiger (1945)

Erdős, Graham, Montgomery, Rothschild, Spencer and Straus (73,75,75):

For a finite set K in the plane, let H_K be the hypergraph whose set of vertices is R^2, where a set of |K| points forms an edge iff it is an isometric copy of K.

Problem: What's the chromatic number $x(H_K)$ of H_K ?

(The case |K|=2 is the Hadwiger-Nelson problem)

Fact (EGMRSS): If K is the set of 3 vertices of an equilateral triangle, then $x(H_K)=2$

Conjecture 1 (EGMRSS): For any set of 3 points K $x(H_K) \leq 3$

Conjecture 2 (EGMRSS): For any set of 3 points K which is not the set of vertices of an equilateral triangle, $x(H_K) \geq 3$.

List Coloring

[Vizing (76), Erdős, Rubin and Taylor(79)]

Def: G=(V,E) - graph or hypergraph, the list chromatic number $x_L(G)$ is the smallest k so that for every assignment of lists L_v for each vertex v of G, where $|L_v|=k$ for all v, there is a coloring f of V satisfying $f(v) \varepsilon L_v$ for all v, with no monochromatic edge

Clearly $x_L(G) \geq x(G)$ for all G, strict inequality is possible

Question 1: x_L**(Unit Distance Graph)=?**

Question 2: **For a given finite K in the plane,** $x_L(H_K)$**=?**

Thm [A+Kostochka (11)]: For any finite K in the plane $x_L(H_K)$ is **infinite** !

That is: for any finite K in the plane and for any positive integer s, there is an assignment of a list of s colors to any point of the plane, such that in any coloring of the plane that assigns to each point a color from its list, there is a **monochromatic isometric copy of K**

The reason is combinatorial:

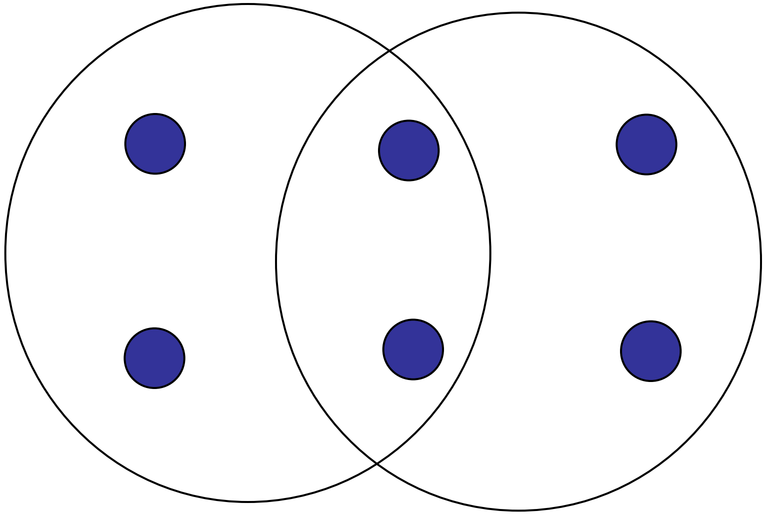
Thm 1 (A-00): For any positive integer s there is a finite d=d(s) such that for any (simple, finite) graph G with **average degree** at least d, $x_L(G)>s$.

Thm 2 (A+Kostochka): For any positive integers r,s there is a finite d=d(r,s) such that for any **simple** (finite) r-uniform hypergraph H with **average (vertex)- degree** at least d, $x_L(H)>s$.

(Special case proved independently by Haxell+ Verstaete)

A hypergraph is simple if it contains no two edges sharing more than one common vertex.

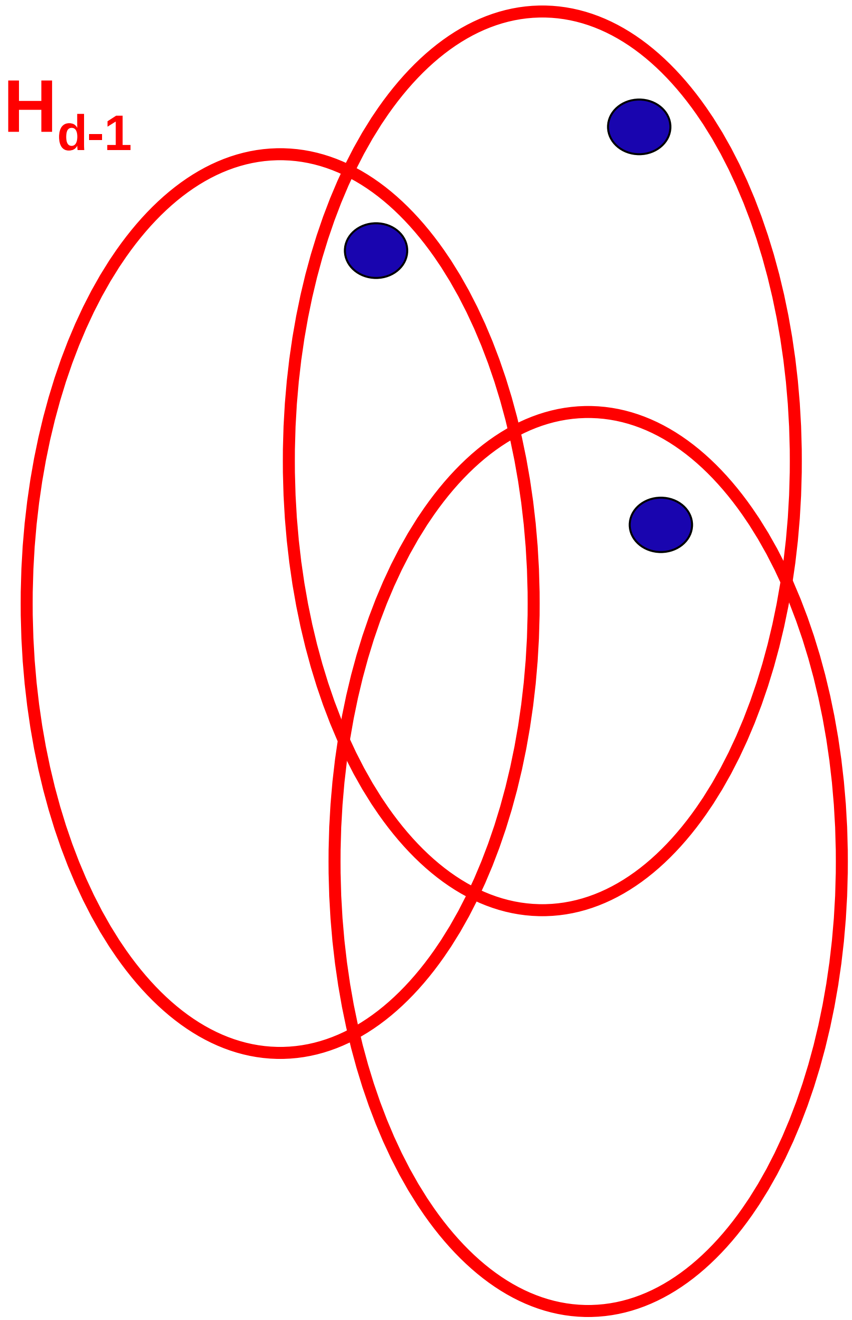
E.g., for r=4, the following is not allowed



Deriving the geometric result from the combinatorial one:

Given a finite K in the plane, prove, by induction on d, that there exists a finite, simple d-regular hypergraph H_d whose vertices are points in the plane in which every edge is an isometric copy of K.

H_d is obtained by taking |K| copies of H_{d-1}, obtained according to a random rotation of K.



Example: $|K|=3$, H_d is obtained from 3 copies of H_{d-1}

The proofs of the combinatorial results are probabilistic.

Theorem 1 (graphs with large average degree have high list chromatic number) is proved by assigning to each vertex, randomly and independently, a uniform random s-subset of the set [2s]={1,2,…,2s}.

It can be shown that with high probability there is no proper coloring using the lists, provided the average degree is sufficiently large.

Note: **if the graph is a complete bipartite graph with d vertices in each vertex class, where d is arbitrarily large, there are at least s^{2d} potential proper colorings (with colors in {1,2,..,s} to the first vertex class, and colors in {s+1,s+2,…,2s} to the second). Each such potential coloring will come from the lists with probability $1/2^{2d}$, hence the expected number of colorings from the lists is at least $(s/2)^{2d}$ which is far bigger than 1 !**

This means that a naïve computation does not suffice.

The proof is obtained by revealing the random lists in several steps. The hypergraph case is more complicated, combining careful induction with a certain decomposition result.

Saxton+Thomason (13+): a better proof, based on their hypergraph containers, supplying improved quantitative bounds

The proofs do not give an efficient (deterministic) algorithm that finds, for a given input simple (hyper-)graph with sufficiently large minimum degree, lists L_v, each of size s, for the vertices, so that there is no proper coloring using the lists

Can one give such lists and a witness showing there is no proper coloring using them ?

Some algorithmic aspects

- The probabilistic method is closely related to randomized algorithms
- The search for explicit constructions is the subject of derandomization
- The method of conditional expectations has been initiated by Erdős and Selfridge (73)
- The first question on graph property testing is due to Erdős
- The Erdős-Rado sunflower (=Δ system) method is used in complexity and algorithms
- …

Conclusion:

The probabilistic way of thinking is a major contribution, which has reshaped the way of reasoning in many areas of mathematics, and beyond.