ORDERINGS OF SPARSE GRAPHS

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ERDÖS 100
BUDAPEST
ORDERINGS OF SPARSE GRAPHS

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(REMEMBERING DĚDEČEK)

ERDŐS 100
BUDAPEST
A WARM UP

EVERY ORIENTATION OF A GRAPH $G$
$\chi(G) \geq n+1$ CONTAINS A MONOTONE
PATH $\vec{P}_n$ OF LENGTH $n$.

(GALLAI, HASSE, VITAVER, ROY)
A Warm Up

Every orientation of a graph $G$ $\chi(G) \geq n + 1$ contains a monotone path $P_n$ of length $n$.

(Gallai, Hasse, Vitaver, Roy)

For every oriented tree $\vec{T}$, there exists oriented graph $\vec{H}$ such that

$\vec{T} \rightarrow G$ iff $G \rightarrow \vec{H}$

For every oriented graph $G$. 
Every orientation of a graph $G$ with $\chi(G) \geq n + 1$ contains a monotone path $\overrightarrow{P}_n$ of length $n$.

(GALLAI, HASSE, VITAVER, ROY)

For every oriented tree $\overrightarrow{T}$, there exists oriented graph $\overrightarrow{H}$ such that $\overrightarrow{T} \rightarrow G$ iff $G \rightarrow \overrightarrow{H}$ for every oriented graph $G$.

(N., C. TARDIF)

Finite dualities studied by many

(HUBIČKA, KANTOR, P. ERDŐS, J. FONIOK, G. TARDOS, ...)
IN DUALITIES: TREES ONLY

IN RAMSEY CONTEXT: ALL ORDERINGS (~ACYCLIC ORIENTATIONS)

THM (ORDERING LEMMA)
FOR EVERY GRAPH $F = (V, E)$
THERE EXISTS GRAPH $G = (V', E')$
SUCH THAT:
FOR ARBITRARY ORDERINGS $(V_i \leq)$
$(V'_i \leq')$
THERE EXISTS A MONOTONNE (w.r.t. $\leq, \leq'$)
EMBEDDING $F \hookrightarrow G$.

("ALL GRAPHS HAVE ORDERING PROPERTY")

(N. RÖDL)
I. \textbf{THM} \textsc{holds} for graphs with given girth:

\textbf{THM} (ordering lemma)

For every graph $F = (V,E), \text{girth}(F) > \ell$, there exists graph $G = (V',E'), \text{girth}(G) > \ell$, such that:

For arbitrary orderings $(V \leq) \quad (V' \leq')$

There exists a \textbf{monotonne} (w.r.t. $\leq, \leq'$) embedding $F \hookrightarrow G$.


Random replacement constr.
I. THM holds for graphs with given girth:

**THM** (Ordering Lemma)

For every graph $F = (V, E), \text{girth}(F) > \ell$, there exists graph $G = (V', E'), \text{girth}(G) > \ell$, such that:

For arbitrary orderings $(V_i \leq) (V'_i \leq')$

There exists a **monotone** (w.r.t. $\leq, \leq'$) embedding $F \hookrightarrow G$.


Random replacement constr.

("All graphs have ordering property")

Sparse
**COROLLARY**

1. \[ F = \underbrace{\cdots}_{n+1} \Rightarrow \chi(G) \geq n+1 \quad \text{(Erdős)} \]

2. \[ F = \underbrace{\cdots}_{n+1} \Rightarrow \text{Girth}(G) > n \quad \text{G NOT DIAGRAM OF A POSET} \quad \text{(Erdős-Ore Problem)} \]

More corollaries later
II. A class $\mathcal{C}$ of graphs has ordering property if

holds for $\mathcal{C}$:

for every $F \in \mathcal{C}$ there exists $G \leq \mathcal{C}$

$$G \xrightarrow{\text{ord}} F$$

"almost every" class $\mathcal{C}$ has ordering property

(e.g. every $\mathcal{C}$ determined by forbidding finitely many $2$-connected graphs)
Consequently orderings are natural obstacle for Ramsey properties.

(The original motivation of ordering property)
REMARK

(SHELAH) ORDERING PROPERTY OF \( \mathcal{C} \):
FOR EVERY \( n \) THERE EXISTS \( G \in \mathcal{C} \)
SUCH THAT IN \( G \) ONE CAN FO-DEFINE
A LINEAR ORDERING OF \( n \)-TUPLE

\( \mathcal{C} \) IS [STABLE] IF \( \mathcal{C} \) HAS NO
ORDERING PROPERTY
THM

For a monotone class $\mathcal{C}$

1. $\mathcal{C}$ is stable

2. $\mathcal{C}$ is nowhere dense

(N., Possona de Mendez)

Adler, Adler
THM FOR A MONOTONE CLASS $\mathcal{C}$

1. $\mathcal{C}$ is stable
2. $\mathcal{C}$ is nowhere dense

(N., Possona de Mendez)
Adler, Adler

Nowhere dense $\equiv$ doesn't contain all $K_n$ with $s$ & subdivision points on every edge.
NOWHERE DENSE $\supseteq$ BOUNDED EXPANSION

THM

ANY BOUNDED EXPANSION CLASS $\mathcal{E}$ HAS ALL RESTRICTED DUALITIES (CONNECTED)

FOR EVERY $F_1, \ldots, F_t \in \mathcal{E}$ THERE EXIST $D$ SUCH THAT

1) $F_i \rightarrow D$, $i = 1, \ldots, t$

2) FOR EVERY $G \in \mathcal{E}$

$F_i \rightarrow G$, $i = 1, \ldots, t$ $\iff$ $G \rightarrow D$. 
NOWHERE DENSE  $\supset$ BOUNDED EXPANSION

**THM**

ANY BOUNDED EXPANSION CLASS $\mathcal{C}$ HAS **ALL RESTRICTED DUALITIES**

(CONNECTED)

FOR EVERY $F_1, \ldots, F_t \in \mathcal{C}$ THERE EXIST $D$ SUCH THAT

1) $F_i \rightarrow D \quad i = 1, \ldots, t$

2) FOR EVERY $G \in \mathcal{C}$

$F_i \rightarrow G, \quad i = 1, \ldots, t \iff G \rightarrow D.$

"ABSENCE OF ORDERINGS LEADS TO DUALITIES"
\[ \mathcal{N} = \{1, 2, \ldots, N\} \]

\[ k \ll N \]

\[ \pi_1, \pi_2, \ldots, \pi_k! \]

**Fixed enumeration of all permutations of \([k]\).**

\( \sigma \) - permutation of \([N]\).

\( k \)-profile of \( \sigma \) - \((s_1^\sigma, \ldots, s_k^\sigma)\)

**k-statistics**

\[ s_i = \left| \{ k \in \mathcal{N} \setminus \mathcal{N}_k | \pi_k = \pi_i \} \right| \]

\[ \frac{1}{N^k} \]
For every 2-connected $F$, $|F| = k$, there exists $G$ such that

(1) For any $\sigma$ on $V(G) = [N]$:

$$\text{Emb}((F_\alpha, \sigma), (G, \sigma)) = (S_\alpha^\sigma + o(1)).$$

$$\text{Emb}(F, G)$$

(2) $\text{girth}(F) = \text{girth}(G)$.

"Sparsification Lemma"
For every 2-connected $F$, $|F| = k$, there exists $G$, $V(G) = [N]$, such that for any $\sigma$ on $[N]$

$$\text{Emb}((F, \pi_i)_1 (G, \sigma)) = \left(\frac{1}{k!} + o(1)\right).$$

$$\text{Emb}(F, G)$$

and $\text{girth}(F) = \text{girth}(G)$.

N. Rödl

Generalizes O. Angel, A. Kechris, R. Lyons

(in context of topological dynamics)
FOR STRUCTURES WITH CANONICAL ORDERINGS?

(N., PROMEL, RODL, VOIGT)

HOMOMORPHISMS OF ORDERED GRAPHS?

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THANK YOU
FOR YOUR ATTENTION