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On a Lattice Problem of Steinhaus: Simultaneous tilings of the Plane

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Steinhaus (1950's) : *(1) Are there subsets* $A, S \subset \mathbb{R}^2$ *such that*

$$\textit{card}(S \cap T(A)) = |S \cap T(A)| = 1,$$

for all isometries T *of* $\mathbb{R}^2$?

Let's call such a set S a Steinhaus set for A. The trivial case where $A = \mathbb{R}^2$ and $|S| = 1$ is ruled out.

T.: Sierpinski (1958), Erdös (1985) Yes.

Steinhaus (1950's): *What if the set* A *is specified? In particular,* $A = \mathbb{Z}$ *or* $A = \mathbb{Z}^2$, i.e., Can S be a fundamental domain simultaneously for all rotations of $\mathbb{Z}^2$?

Steinhaus' questions appeared in Sierpinski's 1958 paper on this subject.

T.: Komjath (1992) S *exists if* $A = \mathbb{Z}$ *or* $A = \mathbb{Q} \times \mathbb{Q}$.

The problem remained: what if $A = \mathbb{Z}^2$? To get a feeling for the problem, let's check some other dimensions.

T.: *There is a Borel set which is a Steinhaus set for $\mathbb{Z}^1$ in $\mathbb{R}^1$, namely* $[0,1)$. *There is no Steinhaus set of any kind for $\mathbb{Z}^4$ in $\mathbb{R}^4$.*

Let $z_1 = (a_1, ..., a_4) \in \mathbb{Z}^4$ and $z_2 = (b_1 + \frac{1}{2}, ..., b_4 + \frac{1}{2}) \in \mathbb{Z}^4 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then $||z_1 - z_2||^2 \in \mathbb{Z}_+$ and is therefore the sum of four squares. Thus, $z_1, z_2 \in T(\mathbb{Z}^4)$, for some isometry T.

T.: (Jackson, Mauldin (2002)) *There is a "Steinhaus set," a set $S \subseteq \mathbb{R}^2$ such that for every isometric copy L of the integer lattice $\mathbb{Z}^2$ we have $|S \cap L| = 1$.*

Equivalent to showing: There is some $S \subseteq \mathbb{R}^2$ such that:

(i) For every isometric copy L of $\mathbb{Z}^2$ we have $S \cap L \neq \emptyset$.

(ii) For all distinct $z_1, z_2 \in S$, $||z_1 - z_2||$ is not a lattice distance, i.e., $||z_1 - z_2||^2$ is not the sum of two squares.

We actually show something stronger: there is some S such that
(ii') if $z_1, z_2 \in S$, $z_1 \neq z_2$, then $||z_1 - z_2||^2 \notin \mathbb{Z}$.

NOTE: From this point on a set satisfying (i) and (ii') will be called an "S set". Let $d_1 = 1 < d_2 = \sqrt{2} < d_3$... be the increasing sequence of lattice distances and note that the gaps $d_{n+1} - d_n \to 0$ as $n \to \infty$.

T.: (Jackson, Mauldin (2003)(a variation of Croft's argument for measure) No S set for any lattice in $R^d, d > 1$ can be a Borel set; or, more generally can have Baire property.

Indication for $\mathbb{Z}^2$. Suppose S is an S set and has the Baire property.

(i) S is not meager: otherwise $\mathbb{R}^2 = \cup_{z \in \mathbb{Z}^2} (S + z)$

(ii) $\mathbb{R}^2 \setminus S$ is not meager: otherwise $(S + 1) \cap S \neq \emptyset$.

(iii) $\exists$ ball in which S is co-meager. So, S is essentially bounded: there is a ball outside of which S is meager.

(iv) There is a category boundary point P and a lattice L containing P such that all other points of the lattice are either category density points of S or of $\mathbb{R}^2 \setminus S$.

(v) All points of $L \setminus \{P\}$ are category density points of $\mathbb{R}^2 \setminus S$.

(vi) There is a lattice L' which misses S.

SOME RESULTS AND UNSOLVED PROBLEMS

1. Is there a Lebesgue measurable set S in $\mathbb{R}^2$? NOTE: Then $m(S) = 1$.

T.: (Croft (1982), Beck (1989)) No Lebesgue measurable S set for $\mathbb{Z}^2$ can be (essentially) bounded.

T.: (Kolountzakis, Wolff (1999)) If S is a Lebesgue measurable S set in $\mathbb{R}^2$, then $\int_S |x|^\alpha dx = \infty$, for all $\alpha > 46/27$. Also, there is no Lebesgue measurable S set for $\mathbb{Z}^d, d > 2$. (This is a deep paper and I unfortunately will not have time to discuss it very much.)

T.: (Chan, Mauldin (2008)) There is no Lebesgue measurable S set for any rational lattice in $\mathbb{R}^d, d > 2$. Unsolved for other lattices.

These last 3 results use Fourier transform methods.

2. Is there a S set for $\mathbb{Z}^3$? This seems unlikely, but remains unsolved.

3. Is there a Steinhaus set for the rectangular lattices in $\mathbb{R}^2$?

For rational rectangular lattices the answer is yes.

4. Can a Steinhaus set for $\mathbb{Z}^2$ be bounded?

5. As far as I know nothing much is known about Steinhaus sets for lattices in other geometries.

Construction of S?

Usual approach: (1) Well order all the lattices - isometric copies of $\mathbb{Z}^2$:

$$L_0, L_1, ..., L_\alpha, ...$$

(2) Carefully choose a point from each lattice with no two points at a lattice distance apart.

There are several reasons why this approach might fail. The first of which is:

There are finite obstructions

T.: There is a 17 point partial S set with each point at a lattice distance from some point of $\mathbb{Z}^2$:

$(216/5, 2/5)$	$(107/5, 4/5)$	$(283/5, 1/5)$	$(174/5, 3/5)$
$(677/13, 5/13)$	$(340/13, 10/13)$	$(744/13, 2/13)$	$(407/13, 7/13)$
$(70/13, 12/13)$	$(474/13, 4/13)$	$(137/13, 9/13)$	$(541/13, 1/13)$
$(204/13, 6/13)$	$(712/13, 11/13)$	$(271/13, 3/13)$	$(779/13, 8/13)$
$(2601/65, 57/65)$			

We were unable to find a smaller example. Does a smaller example exist?

Repairing finite Obstructions: Retreat and be more careful

Let $d > 1$ be an integer. Let $X_d = \{(a,b) \in \mathbb{Z}^2 : 0 \leq a, b < d\}$. S is an *d-partial S set means*

$$\textit{(i)}\ |S \cap (\tfrac{x}{d} + \mathbb{Z}^2)| = 1, \quad \forall x \in X_d$$

$$\textit{(ii')}\ ||x - y||^2 \notin \mathbb{Z}, \qquad \forall x, y \in X_d, x \neq y$$

i.e., S is a partial S set for the translations of $\mathbb{Z}^2$ by rationals with denominator d.

Given d, for any $v \in \mathbb{Z}^2$, we write

$$v = y(v) + d\epsilon(v),$$

where $\epsilon_i(v) \in \mathbb{Z}$ is the quotient and $y_i(v) \in X_d$ is the remainder when v_i is divided by d.

T.: There is an d-partial Steinhaus set if and only if there is $L : X_d \to X_d$ such that $\forall x, z \in X_d, x \neq z$,

$$(+)\ ||(\tfrac{z}{d} + L(z)) - (\tfrac{x}{d} + L(x))||^2 \notin \mathbb{Z}.$$

Moreover, if L has (+), then $S = \{\frac{x}{d} + L(x)\}$ is an d-partial S set.

Expanding (+) we see that if $||z - x||^2 \notin d\mathbb{Z}$, then (+) holds.

In particular,

T.: If p is a prime and $p \equiv 3 \pmod 4$, then any function L produces a p-partial S set.

Consider a prime $p \equiv 1 \pmod 4$ and $n \geq 1$ and $d = p^n$. Suppose

$$x = (i_1, j_1), \quad z = (i_2, j_2) \in X_{p^n}$$

and

$$||z - x||^2 = (i_2 - i_1)^2 + (j_2 - j_1)^2 \equiv 0 \pmod{p^n}.$$

Let $\lambda^2 \equiv -1 \pmod{p^n}$ with

$$j_2 - j_1 = \lambda(i_2 - i_1).$$

Let $b \in \{0, 1, ..., p^n - 1\}$ with $z = y(i_2, b + \lambda i_2)$ and $x = y(i_1, b + \lambda i_1)$.

Define the function $\pi_b^\lambda : \{0, 1, ..., p^n - 1\} \to \{0, 1, ..., p^n - 1\}$ by

$$\pi_b^\lambda(i) \equiv i\Big(\frac{1 + \lambda^2}{2p^n}\Big) + \langle (1, \lambda) \cdot [L(y((i, b + i\lambda))) - \epsilon(i, b + i\lambda)] \rangle \pmod{p^n}.$$

We find that

$$||\Big(\frac{z}{p^n} + L(z)\Big) - \Big(\frac{x}{p^n} + L(x)\Big)||^2 \notin \mathbb{Z}$$

if and only if

$$\Big(i_2 - i_1\Big)\Big(\pi_b^\lambda(i_2) - \pi_b^\lambda(i_1)\Big) \not\equiv 0 \pmod{p^n}.$$

Good families of permutations = Partial S sets for prime powers

T.: Let p be a prime, $p \equiv 1 \pmod 4$, and $n \geq 1$. Let $L : X_{p^n} \to X_{p^n}$. TFAE:

(i) $\forall x, z \in X_{p^n}$ with $x \neq z$,

$$||\Big(\frac{z}{p^n} + L(y)\Big) - \Big(\frac{x}{p^n} + L(x)\Big)||^2 \notin \mathbb{Z}.$$

(ii) For each $b \in p^n$, each λ with $\lambda^2 = -1 \pmod{p^n}$ and all distinct $i, j \in p^n$:

$$\Big(j - i\Big)\Big(\pi_b^\lambda(j) - \pi_b^\lambda(i)\Big) \not\equiv 0 \pmod{p^n}.$$

(iii) $\forall b \in p^n$ and λ with $\lambda^2 \equiv 0$, π_b^λ is a permutation of p^n and is 'good': if $0 \leq i \neq j < p^n$ and $i - j = p^r u$ where $(p, u) = 1$, then $\pi_b^\lambda(i) \neq \pi_b^\lambda(j) \pmod{p^{n-r}}$.

T.: *There is a good permutation of length p^n.*

Proof. For $n = 1$ take $\pi = (0, 1, \ldots, p-1)$. For $n > 1$, if $i = b_0 + b_1 p + b_2 p^2 + \cdots + b_{n-1} p^{n-1}$ where $0 \le b_i < p$, set $\pi(i) = b_0 p^{n-1} + b_1 p^{n-2} + \cdots + b_{n-1}$, the base p digit reversal permutation. This easily works. $\square$

We were able to continue with this procedure which becomes more technically involved to prove the following existence

T.: For each $d \in \mathbb{Z}_+$, there is a function $L : X_d \to X_d$ such that

$$(*)_d : \{\frac{x}{d} + L(x) : x \in X_d\}$$

forms a partial Steinhaus set.

We also showed the following extension property is true.

T.: Let $d|d'$ and assume $L : X_d \to X_d$ satisfies $(*)_d$. Then L may be extended to a function $L' : X_{d'} \to X_{d'}$ satisfying $(*)_{d'}$.

We immediately get:

T.: LEMMA [A] Let $\mathcal{L}_{\mathbb{Q}}$ denote the set of rational translations of $\mathbb{Z}^2$, that is, lattices of the form $\mathbb{Z}^2 + (r,s)$ where $r,s \in \mathbb{Q}$. Then there is a set $S \subseteq \mathbb{R}^2$ satisfying the following.

(i) For every lattice $L \in \mathcal{L}_{\mathbb{Q}}$, $S \cap L \neq \emptyset$.

(ii) For all distinct $z_1, z_2 \in S$, $\rho(z_1, z_2)^2 \notin \mathbb{Z}$.

PROBLEM. If S is a partial Steinhaus set for all the rational translations of $\mathbb{Z}^2$, then must S be unbounded?

One nice thing: We automatically get a partial Steinhaus set for the rational rotations (meaning here the matrix has rational entries) from one for rational translations.

DEF. We say $L \sim L'$ if L' is obtained from L by rational translations/rotations (in the coordinate system of L).

Another foiled Construction of S

Next approach: (1) Well order all the equivalence classes of lattices:

$$\{\mathcal{L}_\alpha\}_{\alpha<2^\omega}$$

(2) Successively build partial Steinhaus sets

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_\alpha \subseteq ...$$

such that at step α, $S_\alpha \cap L \neq \emptyset$ for all $L \in \mathcal{L}_\alpha$. At limit stages we would take unions and there would be no problem. But, there is

Another Geometric Obstruction

Problem. It may be that every point on some $L \in \mathcal{L}_\alpha$ lies at a lattice distance from some point of $\cup_{\beta<\alpha} S_\beta$, in which case the extension is impossible.

To investigate this, suppose $z_1 \in L$, $c_1 \in S_\alpha$, and $\rho^2(c_1, z_1) \in \mathbb{Z}$. Let us assume that c_1 does not have rational coordinates with respect to L (we comment on the general case below). The following lemma is easily verified.

T.: LEMMA [C] Let L be a lattice and suppose c_1 does not have rational coordinates with respect to L. Then there is a line $l_1 = l(c_1, L)$ such that if $w \in L$ and $\rho^2(c_1, w) \in \mathbb{Q}$, then $w \in l_1$.

Thus, the point $c_1 \in S_\alpha$ can only rule out a line $l_1 = l(c_1, L)$ of points on L. Choose $z_2 \in L \setminus l_1$. Suppose there is a $c_2 \in S_\alpha$ with $\rho^2(c_2, z_2) \in \mathbb{Z}$. Suppose again that c_2 does not have rational coordinates with respect to L. Let $l_2 = l(c_2, L)$, and let $z_3 \in L \setminus (l_1 \cup l_2)$. Finally, suppose there is a $c_3 \in S_\alpha$ with $\rho^2(c_3, z_3) \in \mathbb{Z}$. Let $r_1 = \rho(c_1, z_1)$ and likewise for r_2, r_3. Let C_1 be the circle with center c_1 and radius r_1, and likewise for C_2, C_3.

Eliminating the obstruction.

We have the three circles with centers the points c_1, c_2, c_3 of S_α each with square radius in $\mathbb{Q}$.

We would like to assert that there are only finitely many lattices L with points z_1, z_2, $z_3 \in L$ and with $z_1 \in C_1, \ldots, z_3 \in C_3$. This would then be a contradiction if we assume the $\mathcal{L}_\alpha$ is sufficiently closed and L is not definable from the points of S_α. (Again, note that at most one point of L can lie in S_α as L is definable from any two of its points).

Obvious exception to the above assertion. Namely, the case where $r_1 = r_2 = r_3$ and $\triangle z_1 z_2 z_3 \cong \triangle c_1 c_2 c_3$. This exceptional case does not arise in the argument though, as in this case we would have $\rho(c_1, c_2) = \rho(z_1, z_2)$ is a lattice distance, contradicting S_α being a partial Steinhaus set. The following geometric lemma says that this is the only exceptional case to our assertion.

4-bar linkages and coupler curves

T.: LEMMA [B] Let c_1, c_2, c_3 be three distinct points in the plane, and let r_1, r_2, $r_3 > 0$ be real numbers. Let C_1 be the circle in the plane with center at c_1 and radius r_1, and likewise for C_2 and C_3. Let p_1, p_2, p_3 be three distinct points in the plane. Then, except for the exceptional case described afterwards, there are only finitely many (≤ 48 ((6?)) triples of points (z_1, z_2, z_3) in the plane such that

(i) $z_1 \in C_1$, $z_2 \in C_2$, and $z_3 \in C_3$.

(ii) The triangle $p_1p_2p_3$ is isometric with the triangle $z_1z_2z_3$ (we allow the degenerate case where the points $z_1z_2z_3$ are colinear).

The exceptional case is when $r_1 = r_2 = r_3$ and the triangle $p_1p_2p_3$ is isometric with $c_1c_2c_3$.

So, if we ensure that our families of lattices in the inductive family are sufficiently closed under lattices determined by 4-bar linkages, we can continue the transfinite induction to produce a Steinhaus set.

MOVIE

Construction of S in ZFC

The particular method used to prove the existence of an S set is perhaps unusual. It is useful when one is closing up some objects under some geometric, alegraic, combinatorial or logical operations.

First, we built a particular enumeration of sets of equivalence classes of lattices. To begin, let $\kappa(\emptyset) = 2^\omega$, and let

$$\{\mathcal{M}_{\alpha_0} : \alpha_0 < \kappa(\emptyset)\}$$

be an increasing family of sets of equivalence classes with

$$\mathcal{M}_0 = \emptyset, \quad |\mathcal{M}_{\alpha_0}| < \kappa(\emptyset)$$

and such that

$$\text{every } [L]_\sim \text{ is in some } \mathcal{M}_{\alpha_0}.$$

We proceed inductively to build a well founded subtree T of $ON^{<\omega}$ and functions $\kappa : T \to$ cardinals and $\mathcal{M} : T \to$ sets of equivalence classes satisfying

(i) If $(\alpha_0, ..., \alpha_k) \in T$, then (i) $\kappa(\alpha_0, ..., \alpha_{k-1})$ is an uncountable cardinal, (ii) $\mathcal{M}_{\alpha_0,...,\alpha_{k-1},\beta}$ is defined $\iff \beta < \kappa(\alpha_0, ..., \alpha_{k-1})$

(ii) $(\alpha_0, ..., \alpha_k)$ is a terminal node in $T \iff \mathcal{M}_{\alpha_0,...,\alpha_k+1} \setminus \mathcal{M}_{\alpha_0,...,\alpha_k}$ is countable.

(iii) If $\text{card}(\mathcal{M}_{\alpha_0,...,\alpha_k+1} \setminus \mathcal{M}_{\alpha_0,...,\alpha_k}) := \kappa(\alpha_0, ..., \alpha_k) > \omega_0$, then

$$\mathcal{M}_{\alpha_0,...,\alpha_k+1} \setminus \mathcal{M}_{\alpha_0,...,\alpha_k} = \cup \mathcal{M}_{\alpha_0,...,\alpha_k,\alpha_{k+1}}$$

with

$$\text{card}(\mathcal{M}_{\alpha_0,...,\alpha_k,\alpha_{k+1}}) < \kappa(\alpha_0, ..., \alpha_k).$$

(iv) If $c_1, c_2, c_3 \in \cup\{L : [L] \in \mathcal{M}_{\alpha_0,...,\alpha_k}\}$, with $\rho(c_i, c_j)^2 \notin Q$, then the finitely many equiv classes determined by the linkage are in $\mathcal{M}_{\alpha_0,...,\alpha_k}$.

We construct by transfinite induction partial S-sets $S_{\vec{\alpha}}$, where $\vec{\alpha}$ is a terminal node of T and these nodes have the lexicographic well-ordering. Finally, we set

$$S = \cup S_{\vec{\alpha}}$$

Hueristic argument why there should not be an S set for Z^3

Again, let p be a prime and let

$$X_p = \{(a,b,c) \in \mathbb{Z}^3 : 0 \le a,b,c < d\}.$$

Let $L : X_p \to X_p$. For each $\lambda \in X_p$, with $||\lambda||^2$ is divisible by p, and for each $b \in \{0, ..., p-1\}$, define

$$\pi_b^\lambda(t) = \frac{td(\lambda)}{2} + \lambda \cdot [L(y(x+\lambda t)) - \epsilon(x+\lambda t)] \pmod{p},$$

where $d = d(\lambda) \in \{0, ..., p-1\}$ with $||\lambda||^2 \equiv dp \pmod{p}^2$

T.: TFAE:

(i) $\forall x, z \in X_{p^n}$ with $x \neq z$,

$$||\Big(\frac{z}{p} + L(y)\Big) - \Big(\frac{x}{p} + L(x)\Big)||^2 \notin \mathbb{Z}.$$

(ii) There is a set $\Lambda \subset X_p$ with $card(\Lambda) = p+1$ such that for each $\lambda \in \Lambda$, $||\lambda||^2$ is divisible by p and there is a subset X_λ of X_p with $card(X_\lambda) = p^2$, such that for each $b \in X_\lambda$, π_b^λ is a permutation of $GF(p)$.

For p large, perhaps $p = 11$ or even maybe $p = 5$ it doesn't appear this is possible:

(i) $card(X_p) = p^3$.

(ii) There are p^{3p^3} functions L from X_p to itself

(iii) The probability that a random function from $\{0, ..., p-1\}$ is a permutation is $p!/p^p$.

(iv) So, if the $p^2(p+1)$ functions associated to L were random and independent (which they are not),the expected number N_p of p-partial Steinhaus functions would be

$$N_p = p^{3p^3}(\frac{p!}{p^p})^{(p+1)p^2} \to 0 \text{ as } p \to \infty.$$

For $p = 11$, N_p is already close to 0. Partial 3-Steinhaus sets have been constructed, but none have been found for $p = 5$.

Fourier transforms and Lebesgue measuable Steinhaus sets

T.: (Kolountzakis, Wolff (1999)) *There does not exist a Lebesgue measurable set $S \subset \mathbb{R}^3$ such that for every (orientation preserving) isometry T of $\mathbb{R}^3$*

$$\textit{card}(S \cap T(\mathbb{Z}^3)) = |S \cap T(\mathbb{Z}^3)| = 1.$$

In fact, their method has been refined and generalized a bit as follows. Let A be a non-singular matrix and consider the lattice $\Lambda = \Lambda_A = A(\mathbb{Z}^d)$.

T.: (Chan, Mauldin (2008)) *Let Λ be a rational lattice in $R^d, d > 2$. There does not exist a Lebesgue measurable set $S \subset \mathbb{R}^d$ such that for every (orientation preserving) isometry T of $\mathbb{R}^d$*

$$\textit{card}(S \cap T(\Lambda)) = |S \cap T(\Lambda)| = 1.$$

REMARK. This Fourier transform approach completely fails in $\mathbb{R}^2$. The problem in the plane seems to involve some as yet unknown aspects of planar geometric measure theory and/or insufficient estimates of convergence rates.

Observe that this property implies the following:

$$\sum_{n\in A\mathbb{Z}^d} \mathbf{1}_{TS}(x-n) = 1, \quad \text{(a.e.) } x \in \mathbb{R}^d, \text{ (a.e.) rotation T.} \tag{1}$$

Integrating both sides of this equation over the fundamental domain $D = A([0,1)^d)$ of the lattice, we find $\mu(S)$, the Lebesgue measure of such a set S:

$$\begin{aligned} |\det A| &= \int_D 1 dx = \int_D \sum_{n\in A\mathbb{Z}^d} \mathbf{1}_{TS}(x-n)dx = \sum_{n\in A\mathbb{Z}^d} \int_D \mathbf{1}_{TS}(x-n)dx \\ &= \sum_{n\in A\mathbb{Z}^d} \int_{n+D} \mathbf{1}_{TS}(x)dx = \int_{R^d} \mathbf{1}_{TS}(x)dx = \mu(T(S)) = \mu(S). \end{aligned} \tag{2}$$

So, if S is even "an almost sure measurable Steinhaus set for the lattice $A\mathbb{Z}^d$," the Lebesgue measure of S is $|\det A|$. More importantly, there is a characterization of almost sure measurable Steinhaus sets by Fourier transform methods.

Let $L_A^* = A^{-T}\mathbb{Z}^d$ be the dual lattice to L_A.

T.: (Kolountzakis, Wolff (1999)) *Let f be an L^1 function. Then there is a constant C with*

$$\sum_{\lambda \in L_A} f(x-\lambda) = C, \quad \text{a.e. } x \tag{3}$$

if and only if the Fourier transform $\widehat{f}$ of f satisfies:

$$\widehat{f}(\lambda) = 0, \quad \forall \lambda : \lambda \in L_A^* \setminus \{0\}. \tag{4}$$

Moreover, if (3) holds, then by integrating both sides of (3) over D, the fundamental domain or parallelepiped spanned by the columns of A, we find that $C = \int f(x)dx / |\det(A)|$.

Thus, we can characterize an (even almost) Steinhaus set S for a lattice L in terms of the properties of its Fourier transform.

T.: *A measurable set S has the almost sure Steinhaus property for the lattice L_A if and only if it has Lebesgue measure $\mu(S) = |\det(A)|$ and the Fourier transform $\widehat{\mathbf{1}_S}$ vanishes on all points x, such that $\|x\| = \|\lambda\|$ for some $\lambda \in L_A^*$, $\lambda \neq 0$.*

Sufficient conditions under which there is no measurable set with the almost sure Steinhaus property for the lattice L_B.

DEF. Given a matrix M, let

$$\mathcal{D}(M) = \{\|Mx\|^2 : x \in \mathbb{Z}^d\},$$

If

$$\mathcal{D}(A) \subseteq \mathcal{D}(B),$$

we say B norm dominates A, and write $B \succ A$ or $A \prec B$. If $B \succ A$ and we have $\det(A)/\det(B)$ not an integer, we say B weakly norm dominates A, and write $B \succ_w A$. With this terminology in place, we have the following theorem.

T.: *Let $B \in GL(d, \mathbb{R})$ and suppose there exists a matrix $A \in GL(d, \mathbb{R})$, where $B^{-T} \succ_w A^{-T}$. Then there is no measurable set with the almost sure Steinhaus property on the lattice L_B.*

Proof. Suppose by way of contradiction, that there is a measurable set S with the almost sure Steinhaus property on L_B. By our calculations,

$$\int \mathbf{1_S}(x)dx = |\det(B)|$$

and $\widehat{\mathbf{1}_S}$ vanishes on all nonzero points with norm square in $\mathcal{D}(B^{-T})$. So, $\widehat{\mathbf{1}_S}$ vanishes on all nonzero points with norm square in $\mathcal{D}(A^{-T})$. In particular, $\widehat{\mathbf{1}_S}$ vanishes on $L_A^* \setminus \{0\}$. Again, by our calculations,

$$\sum_{\lambda \in \Lambda_A} f(x - \lambda) = \frac{\int \mathbf{1_S}(x)dx}{|\det A|} = \frac{|\det(B)|}{|\det(A)|} = \frac{|\det(A^{-T})|}{|\det(B^{-T})|}$$

for almost all x. However, the left side must be an integer, whereas we have supposed that the right side is not. $\square$

T.: (Chan, Mauldin (2008)) *Let $d > 2$, $B \in GL(d, \mathbb{R})$ and suppose $B(\mathbb{Z}^d)$ is a rational lattice. Then there is a matrix $A \in GL(d, \mathbb{R})$, where $B^{-T} \succ_w A^{-T}$. Thus, there is no Lebesgue measurable set with the (even almost sure) Steinhaus property on the lattice L_B.*

As an immediate corollary let us have the following theorem.

T.: *There is no measurable Steinhaus set for the lattices Z^d for $d > 2$.*

For $d = 3$ Kolountzakis and Papadimitrakis gave a simple example. They showed

$$B^{-T} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \succ_w \begin{bmatrix} \sqrt{2} & & \\ & \sqrt{11} & \\ & & \sqrt{6} \end{bmatrix} = A^{-T}.$$

So, there can be no measurable Steinhaus set for the lattice Z^3.

Growth rate for simultaneous tiles of finitely many lattices.

T.: (Kolountzakis and Wolff) For each $d \geq 1$, $\exists C = C(d)$ for which the following is true: Suppose the lattices $A_i(\mathbb{Z}^d) = \Lambda_i, i = 1, ..., n$ have volume 1 and

$$\Lambda_i \cap \Lambda_j = \{0\}, \quad i \neq j.$$

Suppose S is "an almost simultaneous S set for these lattices":

$$\sum_{\lambda \in \Lambda_i} 1_S(x - \lambda) = 1$$

Then

$$diam(\text{support}(f)) \geq Cn^{1/d}.$$

In particular, for $d = 2$, if S is a measurable partial S set for these lattices, then

$$\lambda(S \setminus B(0, C\sqrt{n}) > 0.$$

THANK YOU FOR YOUR ATTENTION.

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