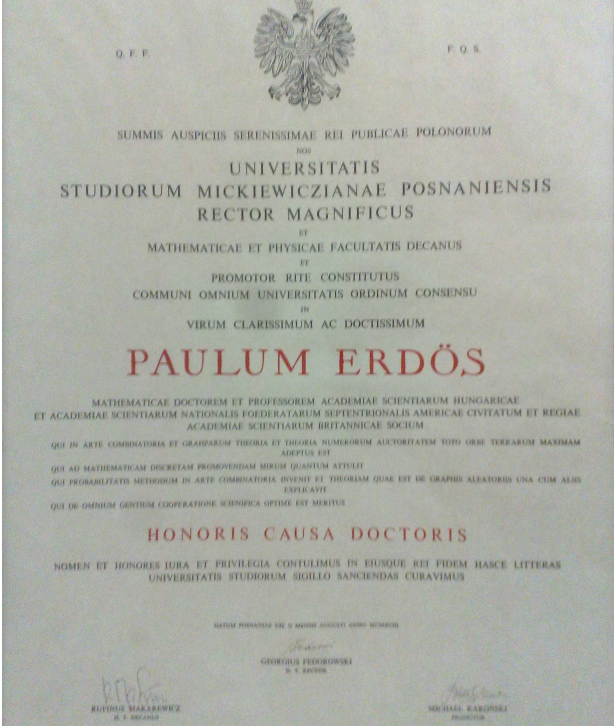
Erdős Centennial Conference
Budapest, 5th June 2013

# Threshold functions: a historical overview

Tomasz Łuczak
Adam Mickiewicz University,
Poznań, Poland
and
Emory University
Atlanta, Georgia



Q. F. F. F. Q. S.

SUMMIS AUSPICIIS SERENISSIMAE REI PUBLICAE POLONORUM

NOS

UNIVERSITATIS
STUDIORUM MICKIEWICZIANAE POSNANIENSIS
RECTOR MAGNIFICUS

ET

MATHEMATICAE ET PHYSICAE FACULTATIS DECANUS

ET

PROMOTOR RITE CONSTITUTUS
COMMUNI OMNIUM UNIVERSITATIS ORDINUM CONSENSU

IN

VIRUM CLARISSIMUM AC DOCTISSIMUM

PAULUM ERDÖS

MATHEMATICAE DOCTOREM ET PROFESSOREM ACADEMIAE SCIENTIARUM HUNGARICAE
ET ACADEMIAE SCIENTIARUM NATIONALIS FOEDERATARUM SEPTENTRIONALIS AMERICAE CIVITATUM ET REGIAE
ACADEMIAE SCIENTIARUM BRITANNICAE SOCIUM

QUI IN ARTE COMBINATORIA ET GRAHPARUM THEORIA ET THEORIA NUMERORUM AUCTORITATEM TOTO ORBE TERRARUM MAXIMAM ADEPTUS EST

QUI AD MATHEMATICAM DISCRETAM PROMOVENDAM MIRUM QUANTUM ATTULIT

QUI PROBABILITATIS METHODUM IN ARTE COMBINATORIA INVENIT ET THEORIAM QUAE EST DE GRAPHIS ALEATORIIS UNA CUM ALIIS EXPLICAVIT

QUI DE OMNIUM GENTIUM COOPERATIONE SCIENIFICA OPTIME EST MERITUS

HONORIS CAUSA DOCTORIS

NOMEN ET HONORES IURA ET PRIVILEGIA CONTULIMUS IN EIUSQUE REI FIDEM HASCE LITTERAS UNIVERSITATIS STUDIORUM SIGILLO SANCIENDAS CURAVIMUS

GEORGIUS FEDOROWSKI
H. T. RECTOR

RUFINUS MAKAREWICZ
H. T. DECANUS

MICHAEL KAROŃSKI
PROMOTOR

. . .

virum clarissimum ac doctissimum
PAULUM ERDÖS

. . .

qui in arte combinatoria et grapharum et theoria numerorum auctoritatem toto orbe terrarum maximam adeptus est

qui ad mathematicam discretam promovendam mirum quantum attulit

qui probabilitatis methodum in arte combinatoria invenit et theoriam quae est de graphis aleatorris una cum aliis explicavit

qui de omnium gentium cooperatione scienifica optime est meritus

. . .

. . .

virum clarissimum ac doctissimum
PAULUM ERDÖS

. . .

qui in arte combinatoria et grapharum et theoria numerorum auctoritatem toto orbe terrarum maximam adeptus est

qui ad mathematicam discretam promovendam mirum quantum attulit

qui probabilitatis methodum in arte combinatoria invenit et theoriam quae est de graphis aleatorris una cum aliis explicavit

qui de omnium gentium cooperatione scienifica optime est meritus

. . .

# Graphorum aleatoriorum definitio

### Definition of $G(n,p)$

$G(n,p)$ is a random graph with vertex set $\{1,2,\ldots,n\}$ in which each edge is generated with probability $p$, independently for each of $\binom{n}{2}$ pairs.

### Definition of $G(n,M)$

$G(n,M)$ is a random graph with vertex set $\{1,2,\ldots,n\}$ chosen uniformly at random from the family of all graphs with $M$ edges.

# Graphorum aleatoriorum definitio

## Definition of $G(n,p)$

$G(n,p)$ is a random graph with vertex set $\{1,2,\ldots,n\}$ in which each edge is generated with probability $p$, independently for each of $\binom{n}{2}$ pairs.

## Definition of $G(n,M)$

$G(n,M)$ is a random graph with vertex set $\{1,2,\ldots,n\}$ chosen uniformly at random from the family of all graphs with $M$ edges.

# Random graphs

$$\Pr(G = G(n,p)) = p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}.$$

$$\Pr(G = G(n,M)) = \begin{cases} \binom{\binom{n}{2}}{M}^{-1} & \textit{if} \quad e(G) = M \\ 0 & \textit{if} \quad e(G) \neq M. \end{cases}$$

# Threshold functions

**Definition** Erdős, Renyi'60

$f(n,p)$ is a threshold function for a (monotone) property $A$ if

$$\lim_{n\to\infty} Pr(G(n,p) \text{ has } A) = \begin{cases} 1 & \text{if } \frac{p}{f(n,p)} \to \infty, \\ 0 & \text{if } \frac{p}{f(n,p)} \to 0. \end{cases}$$

# Threshold functions

**Definition Erdős, Renyi'60**

$f(n,p)$ is a threshold function for a (monotone) property $A$ if

$$\lim_{n\to\infty} Pr(G(n,p) \text{ has } A) = \begin{cases} 1 & \text{if } \frac{p}{f(n,p)} \to \infty, \\ 0 & \text{if } \frac{p}{f(n,p)} \to 0. \end{cases}$$

We will call this type of thresholds as ‘coarse’ thresholds.

# Threshold functions: example

**Theorem Erdős, Renyi'60**

$$\lim_{n\to\infty} Pr(G(n,p) \supseteq K_r) = \begin{cases} 1 & \text{if} \quad n^r p^{\binom{r}{2}} \to \infty, \\ 0 & \text{if} \quad n^r p^{\binom{r}{2}} \to 0. \end{cases}$$

# Threshold functions: example

**Theorem** Erdős, Renyi'60

$$\lim_{n\to\infty} Pr(G(n,p) \supseteq K_r) = \begin{cases} 1 & \text{if} \quad n^r p^{\binom{r}{2}} \to \infty, \\ 0 & \text{if} \quad n^r p^{\binom{r}{2}} \to 0. \end{cases}$$

Thus, $p(n) = n^{-2/(r-1)}$ is the threshold function for the property that a graph contains $K_r$.

# Threshold functions: example

**Theorem** Erdős, Renyi'60

$$\lim_{n\to\infty} Pr(G(n,p) \supseteq K_r) = \begin{cases} 1 & \text{if} \quad n^r p^{\binom{r}{2}} \to \infty, \\ 0 & \text{if} \quad n^r p^{\binom{r}{2}} \to 0. \end{cases}$$

Thus, $p(n) = n^{-2/(r-1)}$ is the threshold function for the property that a graph contains $K_r$.

Bollobás'81 found the (coarse) thresholds for the property that a graph contains copy of a given graph $H$.

# EARLY PERIOD OF RANDOM GRAPH THEORY

*It was the best of times, it was the worst of times.*
*"The tales of two models"*

Main Objective:
*Take your favorite property, guess the threshold function, prove that it is the threshold.*

# EARLY PERIOD IN RANDOM GRAPH THEORY

*Take your favorite property, guess the threshold function, prove that it is the threshold.*

# Early period in random graph theory

*Take your favorite property, guess the threshold function, prove that it is the threshold.*

Two ways of identifying the threshold function

- find a necessary local condition;
- compute the expectation of an appropriate random variable.

# CONNECTIVITY THRESHOLD

**THEOREM ERDŐS, RENYI'59**

If $p(n) = \frac{\log n}{n} + \frac{\omega(n)}{n}$, then

$$\lim_{n\to\infty} Pr(G(n,p) \text{ is connected})$$

$$= \lim_{n\to\infty} Pr(\delta(G(n,p)) \geq 1) = \begin{cases} 1 & \text{if} \quad \omega(n) \to \infty, \\ 0 & \text{if} \quad \omega(n) \to -\infty. \end{cases}$$

# Connectivity threshold

**Theorem Erdős, Renyi'59**

If $p(n) = \frac{\log n}{n} + \frac{\omega(n)}{n}$, then

$$\lim_{n\to\infty} Pr(G(n,p) \text{ is connected})$$

$$= \lim_{n\to\infty} Pr(\delta(G(n,p)) \geq 1) = \begin{cases} 1 & \text{if} \quad \omega(n) \to \infty, \\ 0 & \text{if} \quad \omega(n) \to -\infty. \end{cases}$$

Note that this is a sharp threshold function, where the term which affects the limit probability $\omega/n$ is smaller than the leading term $\log n/n$.

# Hamiltonicity threshold

**Theorem** Komlós, Szemerédi'83, Bollobás'84, Ajtai, Komlós, Szemerédi'85

The hitting times for Hamiltonicity and for the property that the minimum degree is at least 2 a.a.s. coincide.

# HAMILTONICITY THRESHOLD

**THEOREM KOMLÓS, SZEMERÉDI'83, BOLLOBÁS'84, AJTAI, KOMLÓS, SZEMERÉDI'85**

The hitting times for Hamiltonicity and for the property that the minimum degree is at least 2 a.a.s. coincide.

$G(n, M)$ can be viewed as a state of the random graph process

$$\mathcal{G} = \left\{ G(n, M) : 0 \leq M \leq \binom{n}{2} \right\}.$$

Thus, for instance, we may view $G(n, M)$ as a subgraph of $G(n, M + 1)$.

# HAMILTONICITY THRESHOLD

**THEOREM** KOMLÓS, SZEMERÉDI'83, BOLLOBÁS'84, AJTAI, KOMLÓS, SZEMERÉDI'85

The hitting times for Hamiltonicity and for the property that the minimum degree is at least 2 a.a.s. coincide.

$G(n, M)$ can be viewed as a state of the random graph process

$$\mathcal{G} = \left\{ G(n, M) : 0 \leq M \leq \binom{n}{2} \right\}.$$

Thus, for instance, we may view $G(n, M)$ as a subgraph of $G(n, M + 1)$.

In fact $G(n, p)$ can also be considered as a state of Markov process $\hat{\mathcal{G}} = \{G(n, M) : 0 \leq p \leq 1\}$.

# What have been left

Two ways of guessing the threshold function

- find a necessary local condition;
- compute the expectation of an appropriate random variable.

The most difficult properties are those without obvious local necessary condition for which it is hard to compute the expectation, such as $k$-colorability (Bollobás'88, Łuczak'90, Achlioptas, Naor'05), or Ramsey properties (Rödl, Ruciński'95, Friedgut, Hán, Person, Schacht'+13)

# Towards a theory of threshold functions

**Theorem** Bollobás, Thomason'87

Each monotone property $A$ has a threshold function. More precisely, for every $\epsilon > 0$, there exists $k$ such that if

$$\Pr(G(n,p) \text{ has } A) \geq \epsilon\,,$$

then

$$\Pr(G(n,kp) \text{ has } A) \geq 1 - \epsilon\,.$$

# Towards a theory of threshold functions

**Theorem** Bollobás, Thomason'87

Each monotone property $A$ has a threshold function. More precisely, for every $\epsilon > 0$, there exists $k$ such that if

$$\Pr(G(n,p) \text{ has } A) \geq \epsilon\,,$$

then

$$\Pr(G(n,kp) \text{ has } A) \geq 1 - \epsilon\,.$$

**Proof** $G(n,kp)$ is roughly the union of $k$ independent copies of $G(n,p)$, so

$$\Pr(G(n,kp) \text{ has } A) \geq 1 - (1-\epsilon)^k \geq 1 - \epsilon\,. \quad \square$$

# Towards a theory of threshold functions

**Theorem** Bollobás, Thomason'87

Each monotone property $A$ has a threshold function. More precisely, for every $\epsilon > 0$ there exists $k$ such that if

$$\Pr(G(n,p) \text{ has } A) \geq \epsilon\,,$$

then

$$\Pr(G(n,kp) \text{ has } A) \geq 1 - \epsilon\,.$$

Note that $k$ depends only of $\epsilon$ not on the property $A$.

# Classifications of properties

## Question 1

Given a function $p = p(n)$, can we decide if it is a threshold function for some property $A$ in $G(n,p)$?

## Question 2

Given a property $A$, can we decide if its threshold function in $G(n,p)$ is sharp?

# 0-1 Laws – what (some) people have been doing back in 1990's

Main Objective:
*For your favourite class of properties $\mathcal{A}$ identify all functions $p = p(n)$ for which the 0-1 law holds, i.e.*

$$\lim_{n\to\infty} \Pr(G(n,p) \textit{ has } A) = 0 \textit{ or } 1.$$

# 0-1 LAWS – WHAT (SOME) PEOPLE HAVE BEEN DOING BACK IN 1990'S

Main Objective:
*For your favourite class of properties $\mathcal{A}$ identify all functions $p = p(n)$ for which the 0-1 law holds, i.e.*

$$\lim_{n\to\infty} \Pr(G(n,p) \text{ has } A) = 0 \text{ or } 1.$$

Typically, $\mathcal{A}$ is characterized by the language in which properties are described. The first order language, where we can quantify only over vertices (and not over sets of vertices) has been the most popular choice.

# 0-1 Laws: how it works

Main Objective: *For your favourite class of properties $\mathcal{A}$ identify all functions $p = p(n)$ for which the 0-1 law holds, i.e.*

$$\lim_{n\to\infty} \Pr(G(n,p) \textit{ has } A) = 0 \textit{ or } 1.$$

To this end, for a given $p = p(n)$:

- Find an axiom system $A_1, A_2, \ldots,$ which is complete (i.e. for each formula $\phi$ one can prove either $\phi$ or $\neg\phi$);
- Show that each of yours axioms holds a.a.s. for $G(n,p)$.

# 0-1 Laws: examples

**Theorem Shelah, Spencer'88**

If $p(n) = n^{-\alpha+o(1)}$, where $\alpha \in (0, 1)$ is irrational, then the probability of each first order property of graphs tends either to 0, or to 1.

# 0-1 LAWS: EXAMPLES

**THEOREM SHELAH, SPENCER'88**

If $p(n) = n^{-\alpha+o(1)}$, where $\alpha \in (0,1)$ is irrational, then the probability of each first order property of graphs tends either to 0, or to 1.

What about $p(n) = n^{-\alpha+o(1)}$, with rational $\alpha$?

# 0-1 Laws: examples

**Theorem Łuczak, Spencer'91**

There exists a function $p(n) = n^{-1/7+o(1)}$, $p(n) < n^{-1/7}$, such that the probability of each first order property of graphs tends either to 0, or to 1.

**Theorem Łuczak, Spencer'91**

For every recursive function $p(n) = n^{-1/7+o(1)}$, $p(n) < n^{-1/7}$, there exists a first order property $\psi$ such that

$$\liminf_{n\to\infty} \Pr(G(n,p) \text{ has } \psi) = 0,$$

but

$$\limsup_{n\to\infty} \Pr(G(n,p) \text{ has } \psi) = 1.$$

# 0-1 LAWS: EXAMPLES

**THEOREM ŁUCZAK, SPENCER'91**

There exists a function $p(n) = n^{-1/7+o(1)}$, $p(n) < n^{-1/7}$, such that the probability of each first order property of graphs tends either to 0, or to 1.

**THEOREM ŁUCZAK, SPENCER'91**

For every recursive function $p(n) = n^{-1/7+o(1)}$, $p(n) < n^{-1/7}$, there exists a first order property $\psi$ such that

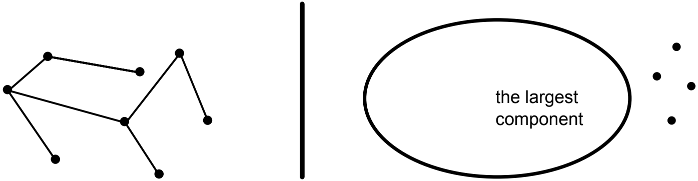
$$\liminf_{n\to\infty} \Pr(G(n,p) \text{ has } \psi) = 0,$$

but

$$\limsup_{n\to\infty} \Pr(G(n,p) \text{ has } \psi) = 1.$$

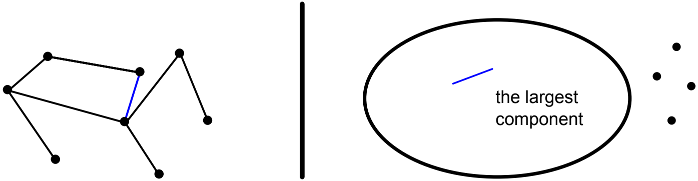
# WHICH PROPERTIES HAS SHARP THRESHOLDS?

Why the threshold for having a triangle is ‘coarse’,
while the connectivity threshold is sharp?



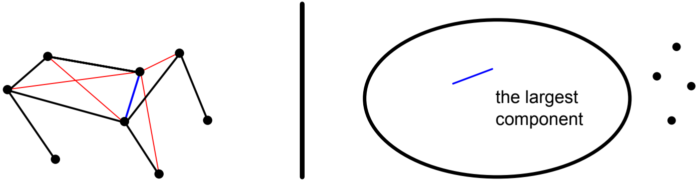
# WHICH PROPERTIES HAS SHARP THRESHOLDS?

Why the threshold for having a triangle is ‘coarse’,
while the connectivity threshold is sharp?



# WHICH PROPERTIES HAS SHARP THRESHOLDS?

Why the threshold for having a triangle is ‘coarse’,
while the connectivity threshold is sharp?



# Sharp thresholds

## (Meta)Theorem

If $A$ has a coarse threshold in $G(n, p)$, then, basically, the minimum graphs with property $A$ are small.

If thresholds are coarser, then the graphs from the critical family can be made smaller.

The upper bounds for sizes of the critical graphs are better for properties $A$ which is highly symmetric.

# Sharp thresholds

**(Meta)Theorem**

If $A$ has a coarse threshold in $G(n, p)$, then, basically, the minimum graphs with property $A$ are small.

If thresholds are coarser, then the graphs from the critical family can be made smaller.

The upper bounds for sizes of the critical graphs are better for properties $A$ which is highly symmetric.

Kahn, Kalai, Linial'88,
Bourgain, Kahn, Kalai, Katznelson, Linial'92
Friedgut, Kalai'96
Bourgain, Kalai'97
Friedgut with Bourgain'99

# Sharp thresholds

**(Meta)Theorem**

If $A$ has a coarse threshold in $G(n, p)$, then, basically the minimum graphs with property $A$ are small.

Coarser thresholds result in smaller critical graphs.

The symmetry of $A$ helps.

The bounds for the ‘sharpness’ of the thresholds which follows from a general theory are usually quite weak (typically logarithmic). But how sharp ‘natural’ thresholds can be?

# Sharp thresholds

**(Meta)Theorem**

If $A$ has a coarse threshold in $G(n,p)$, then, basically the minimum graphs with property $A$ are small.

Coarser thresholds result in smaller critical graphs.

The symmetry of $A$ helps.

The bounds for the 'sharpness' of the thresholds which follows from a general theory are usually quite weak (typically logarithmic). But how sharp 'natural' thresholds can be?

And does it really matter?!?

# Sometimes the 'sharpness' does matter!

**Theorem** Bollobás'84, Łuczak'90

Let $A$ be the property that $G(n,p)$ contains a subgraph with more edges than vertices. If $np = 1 + \omega(n)n^{-1/3}$, then

$$\lim_{n\to\infty} Pr(G(n,p) \text{ has } A) = \begin{cases} 1 & \text{if} \quad \omega(n) \to \infty, \\ 0 & \text{if} \quad \omega(n) \to -\infty. \end{cases}$$

# Sometimes the 'sharpness' does matter!

**Theorem Bollobás'84, Łuczak'90**

Let $A$ be the property that $G(n, p)$ contains a subgraph with more edges than vertices. If $np = 1 + \omega(n)n^{-1/3}$, then

$$\lim_{n\to\infty} Pr(G(n,p) \text{ has } A) = \begin{cases} 1 & \text{if} \quad \omega(n) \to \infty, \\ 0 & \text{if} \quad \omega(n) \to -\infty. \end{cases}$$

It matters because this determines the width of 'the phase transition window'.

# Two remarks

## Remark 1

Note that there are no theory of monotone first order properties. Thus, we do not know if, say, $p(n) = n^{-1/2} + \omega(n)n^{-2/\pi}$, is a threshold function for such a property.

## Remark 2

Are properties of type *G has more than M edges* sharpest among all properties? It is **not** the case in other models.

# Two remarks

### Remark 1

Note that there are no theory of monotone first order properties. Thus, we do not know if, say, $p(n) = n^{-1/2} + \omega(n)n^{-2/\pi}$, is a threshold function for such a property.

### Remark 2

Are properties of type *G has more than M edges* sharpest among all properties? It is **not** the case in other models.

# CLUSTER-SCALED MODEL $G_q(n,p)$

$$\Pr\left(G(n,p) = G\right) = p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}$$

# CLUSTER-SCALED MODEL $G_q(n,p)$

$$\Pr\left(G(n,p) = G\right) = p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}$$

$$\Pr\left(G_q(n,p) = G\right) = q^{c(G)}p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}/Z(n,p),$$

where $c(G)$ denotes the number of components of $G$ and $Z(n,p)$ is a scaling factor.

# AN EXAMPLE OF A VERY SHARP THRESHOLD

**THEOREM** LUCZAK, ŁUCZAK'06, GANDOLFO, RUIZ, WOUTS'10

For every $q > 2$ there exists a constant $c_q$ such that the following holds. Let $A$ be the property that $G_q(n, p)$ contains a subgraph with more edges than vertices. If $p = c_q/n + a/n^2$, then

$$\lim_{n\to\infty} Pr(G(n,p) \text{ has } A) = p_a;$$

where

$$\lim_{a\to-\infty} p_a = 0 \quad \text{while} \quad \lim_{a\to\infty} p_a = 1.$$

# BACK TO THE RANDOM PROCESS

The random graph process

$$\mathcal{G} = \left\{ G(n, M) : 0 \leq M \leq \binom{n}{2} \right\}.$$

is a Markov chain which starts with the empty graph and each time we add to a graph a randomly chosen edge.

# Back to the random process

The random graph process

$$\mathcal{G} = \left\{ G(n, M) : 0 \le M \le \binom{n}{2} \right\}.$$

is a Markov chain which starts with the empty graph and each time we add to a graph a randomly chosen edge.

There are properties which can be defined for the random graph process and does not make sense for $G(n, M)$.

# Back to the random process

The random graph process

$$\mathcal{G} = \left\{ G(n, M) : 0 \leq M \leq \binom{n}{2} \right\}.$$

is a Markov chain which starts with the empty graph and each time we add to a graph a randomly chosen edge.

There are properties which can be defined for the random graph process and does not make sense for $G(n, M)$.

One of Erdős favorite properties of this kind was the race of components

# Race of components

## Definition

Let $u(n, M)$ denote the minimum number $u$ such that the largest component $L$ of $G(n, M + u)$ does not contain the largest component of $G(n, M)$. If such $u$ does not exists we put $u(n, M) = 0$.

Thus, $u(n, M)$ is the time we need to wait until the largest component of a graph $G(n, M)$ will be overpassed by some other component of subsequent graphs.

# Race of components

**Theorem** Erdős, Łuczak'94

For every function $M = M(n)$ a.a.s. we have

$$u(n, M) = O(n/\ln n).$$

Furthermore, a.a.s. $u(n, n/4) = \Omega(n/\ln n)$.

**Theorem** Erdős, Łuczak'94

A.a.s. the random graph process is such that

$$\max_M u(n, M) = \Theta\left(\frac{n \ln \ln n}{\ln n}\right).$$

# Race of components

**Theorem Erdős, Łuczak'94**

For every function $M = M(n)$ a.a.s. we have

$$u(n, M) = O(n/\ln n)\,.$$

Furthermore, a.a.s. $u(n, n/4) = \Omega(n/\ln n)$.

**Theorem Erdős, Łuczak'94**

A.a.s. the random graph process is such that

$$\max_M u(n, M) = \Theta\left(\frac{n\ln\ln n}{\ln n}\right).$$

It is an 'iterated logarithm' type result!

# Law of the Iterated Logarithm

**Law of iterated logarithm**

Let $v(n)$ denote the difference between the number of tails and heads after $n$ flips of a coin. Then a.a.s. for every $n$ we have

$$u(n) = O(\sqrt{n}),$$

but a.a.s.

$$\max_{m \leq n} u(m) = O(\sqrt{n} \ln \ln n).$$

# Race of components

**Theorem Erdős, Łuczak'94**

For every function $M = M(n)$ a.a.s. we have

$$u(n, M) = O(n/\ln n) .$$

Furthermore, a.a.s. $u(n, n/4) = \Omega(n/\ln n)$.

**Theorem Erdős, Łuczak'94**

A.a.s. the random graph process is such that

$$\max_M u(n, M) = \Theta\left(\frac{n \ln\ln n}{\ln n}\right).$$

# Bootstrap percolation on $G(n, M)$

For a graph $G(n, M)$ let us define the epidemic process in the following way.
First we infect vertices $1, 2, \ldots, \ln n$.
Then, in each step a vertex which is adjacent to at least two infected vertices gets infected.
There are no cure for the disease, so everybody infected remains infected until the end of the process.

# Bootstrap percolation on $G(n, M)$

Let $\tau(n, M)$ be the number of steps the infection takes on $G(n, M)$.

**Theorem** Janson, Łuczak, Turova, Vallier'12

For every $M$ a.a.s. $\tau(n, M) = \tilde{O}(n^{1/4})$, and the estimate is sharp.

**Conjecture**

A.a.s. the random process is such that

$$max_M \tau(n, M) = \tilde{\Theta}(n^{1/3}).$$

# FUTURE OF RANDOM STRUCTURES

I will briefly present two possible directions where the random graph fans may want to look at

- geometric (random groups)
- topological (random simplicial complexes)

# Group presentations

$$G = \langle S | R \rangle$$

is a group which consists of words with letters $a_1, a_2, \dots$ (as well as $a_1^{-1}, a_2^{-1}, \dots$) from an alphabet $S$ in which all words from set $R$ are equivalent to the empty word.

# Examples

## Example 1

$$G = \langle \{a, b\} | aba^{-1}b^{-1} \rangle = \{a^n b^m : a, b \in \mathbb{Z}\} \cong \mathbb{Z}^2 .$$

since $aba^{-1}b^{-1} = e$ implies $ab = ba$.

## Example 2

$$\langle \{a, b, c\} | aba^{-1}cb^{-1}a^2 \rangle \cong \langle \{a, b\} | \emptyset \rangle$$

since in every word we may replace $c$ by $ab^{-1}a^{-3}b$.

# Examples

## Example 1

$$G = \langle \{a, b\} | aba^{-1}b^{-1} \rangle = \{a^n b^m : a, b \in \mathbb{Z}\} \cong \mathbb{Z}^2 \,.$$

since $aba^{-1}b^{-1} = e$ implies $ab = ba$.

## Example 2

$$\langle \{a, b, c\} | aba^{-1}cb^{-1}a^2 \rangle \cong \langle \{a, b\} | \emptyset \rangle$$

since in every word we may replace $c$ by $ab^{-1}a^{-3}b$.

Note that both above groups are infinite

# Random groups

In late 80's, Gromov proposed to define random groups using their presentations.

**Definition**

$\Gamma(k,\ell,d)$ is a group $\langle\{a_1,a_2,\ldots,a_k\}|R\rangle$ where $R$ is a set of $(2k-1)^{d\ell}$ words chosen at random from all irreducible words of length $\ell$.
The parameter $d$, $0\leq d\leq 1$ , is called the density of $\Gamma(k,\ell,d)$.

# RANDOM GROUPS

In late 80's, Gromov proposed to define random groups using their presentations.

**DEFINITION**

$\Gamma(k,\ell,d)$ is a group $\langle\{a_1,a_2,\ldots,a_k\}|R\rangle$ where $R$ is a set of $(2k-1)^{d\ell}$ words chosen at random from all irreducible words of length $\ell$.
The parameter $d$, $0 \leq d \leq 1$ , is called the density of $\Gamma(k,\ell,d)$.

Gromov's model: $k$ is fixed, $\ell \to \infty$,

Żuk's model: $\ell$ is fixed, $k \to \infty$.

# Random graphs and random groups

*I feel, random groups altogether may grow up as healthy as random graphs, for example.*

*M.Gromov, Spaces and questions*

# Random graphs and random groups

*I feel, random groups altogether may grow up as healthy as random graphs, for example.*

*M.Gromov, Spaces and questions*

The definition of a random group appeared in early 1980's. Note by the way that 'the Stone Age period' of random graphs 1959-1984 lasted 25 years.

# Random groups: critical densities

## Known critical densities for random groups

The critical density for $\Gamma(k, \ell, d)$ to be trivial is 1/2, in both Gromov and Żuk models.

The critical density for having Kazdan's property (T) is 1/3 for Żuk's model.

For the critical density $d_T$ for having property (T) in Gromov's model we have $1/5 \leq d_T \leq 1/3$ (at this moment it is conjectured that $d_T = 1/4$).

# Random groups: critical densities

## Known critical densities for random groups

The critical density for $\Gamma(k, \ell, d)$ to be trivial is 1/2, in both Gromov and Żuk models.

The critical density for having Kazdan's property (T) is 1/3 for Żuk's model.

For the critical density $d_T$ for having property (T) in Gromov's model we have $1/5 \leq d_T \leq 1/3$ (at this moment it is conjectured that $d_T = 1/4$).

All of the above are density thresholds, which are even 'coarser' than the coarse thresholds for random graphs. Only a handful of sharp thresholds for $\Gamma(k, l, d)$ are known so far (cf. Antoniuk, Łuczak, Świątkowski'13+).

# Random groups: 0-1 laws

**Question**

Do there exist any natural group properties with irrational critical densities?

**QUESTION**

Do there exist any natural group properties with irrational critical densities?

Comments: It is well known that in general the problem of deciding if a word from $\langle S|R\rangle$ is empty is undecidable, but this is not the case for random groups. Each such group is a.a.s. hyperbolic. It means that a modification of Dehn's algorithm can decide it basically in a linear time.

# Future of random structures

I will briefly present two possible directions where the random graph fans may want to look at

- geometric (random groups)
- topological (random simplicial complexes)

# Random simplicial complexes

## Main Idea

Study topological properties of a random hypergraph $G^k(n, p)$, where we generate $k$-element subsets of $\{1, 2, \ldots, n\}$ independently with probability $p$.

# Example

**Theorem Babson, Hoffman, Kahle'11**

If

$$p(n) = \sqrt{(3\ln + \omega(n))/n}$$

where $\omega(n) \to \infty$, then a.a.s. $\pi_1(G^2(n,p))$ is trivial.
On the other hand, if for some $\epsilon > 0$,

$$p(n) \leq n^{-1/2-\epsilon},$$

then a.a.s. $\pi_1(G^2(n,p))$ is infinite and hyperbolic.

# EXAMPLE

**THEOREM BABSON, HOFFMAN, KAHLE'11**

If

$$p(n) = \sqrt{(3\ln + \omega(n))/n}$$

where $\omega(n) \to \infty$, then a.a.s. $\pi_1(G^2(n, p))$ is trivial.
On the other hand, if for some $\epsilon > 0$,

$$p(n) \leq n^{-1/2-\epsilon},$$

then a.a.s. $\pi_1(G^2(n, p))$ is infinite and hyperbolic.

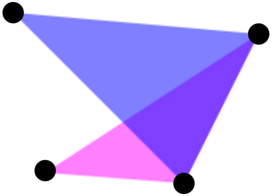
Note that the density 1/2 was critical for random groups to collapse to trivial groups. Possibly, there is a common generalization for both these statements.

Instead of random hypergraph $G^k(n,p)$ we may consider its incidence matrix $M^k(n,p)$ whose columns are the indicator functions of its $(k+1)$-element sets and rows correspond to all subsets of $\{1,2,\ldots,n\}$ size $k$.
Note that the number of columns is a random variable.

Instead of random hypergraph $G^k(n, p)$ we may consider its incidence matrix $M^k(n, p)$ whose columns are the indicator functions of its $(k+1)$-element sets and rows correspond to all subsets of $\{1, 2, \ldots, n\}$ size $k$.
Note that the number of columns is a random variable.

Thus, for example, $M^1(n, p)$ is the incidence matrix of $G(n, p)$.

# Another Example



$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

# QUESTION ON VANISHING TOP HOMOLOGY

## QUESTION ON VANISHING TOP HOMOLOGY GROUP

Find the threshold for the property that some subset columns of $M^k(n,p)$ sum up to a zero vector in group $\mathbb{Z}/2\mathbb{Z}$.

# Question on vanishing top homology

## Question on vanishing top homology group

Find the threshold for the property that some subset columns of $M^k(n,p)$ sum up to a zero vector in group $\mathbb{Z}/2\mathbb{Z}$.

A short dictionary of topological terms:
homology – we look at the dependence of columns
top homology – rows of the matrix are labeled by $d-1$ dimensional subsets

# Question about collapsibility

## Question on collapsibility

Find the threshold for the property that there exists a subset of columns of $M^k(n,p)$ in which no row contains exactly one non-zero term.

# Question about collapsibility

## Question on collapsibility

Find the threshold for the property that there exists a subset of columns of $M^k(n,p)$ in which no row contains exactly one non-zero term.

Collapsibility implies vanishing of homology group.

# Random graphs

## Question on vanishing top homology group

Find the threshold for property that $G(n,p)$ contains a subgraph with all degrees even.

## Question on collapsibility

Finb the threshold for the property that there exists a subgraph of $G(n,p)$ with all degrees larger than one.

# Random graphs

## Question on vanishing top homology group

Find the threshold for property that $G(n, p)$ contains a subgraph with all degrees even.

## Question on collapsibility

Finb the threshold for the property that there exists a subgraph of $G(n, p)$ with all degrees larger than one.

These properties are equivalent in dimension one (and mean that a graph contains a cycle); for larger dimensions it is not the case.

# Results and conjectures

## Aronshtam, Linial, Luczak, Meshulam

The collapsibility of $G^k(n,p)$ has a sharp threshold for $p = (c_k + o(1))/n$, where the constant $c_k$ can be explicitly computed.

For every $c > 0$ there exists $d > 0$ such that a.a.s. $G^k(n, c/n)$, contains no collapsible subcomplexes of size smaller than $dn^k$; i.e. each subset of columns of $M^k(n,p)$ of size smaller than $dn^k$ contains a row with only one 1.

It is conjectured that the threshold for vanishing top homology is of the order $p = (c'_k + o(1))/n$, with $c'_k > c_k$.

# Applications

Both questions on vanishing top homology and collapsibility can have potential applications in constructions of error correcting codes.
Indeed, let us recall that the parity check matrix of error correcting codes should not contain small sets of dependent columns, precisely as it is the case of $M^k(n,p)$.