On a conjecture by Gallai
and a question by Erdős

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A (proper) \(k\)-coloring of a graph \(G = (V, E)\) is a function \(f : V \to \{1, 2, \ldots, k\}\) such that \(f(u) \neq f(v)\) for each \(uv \in E\).

A graph \(G\) is \(k\)-colorable if it has a \(k\)-coloring. The chromatic number, \(\chi(G)\), of a graph \(G\) is the smallest \(k\) such that \(G\) is \(k\)-colorable.

A graph \(G\) is \(k\)-critical if \(G\) is not \((k-1)\)-colorable, but every proper subgraph of \(G\) is \((k-1)\)-colorable.

Every \(k\)-critical graph has chromatic number \(k\) and every \(k\)-chromatic graph contains a \(k\)-critical subgraph.
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Erdős wrote in 1989:
Dirac defined a $k$-chromatic graph to be vertex critical if the omission of any vertex decreases the chromatic number and edge critical if the removal of any edge decreases the chromatic number. I immediately liked these concepts very much and in fact felt somewhat foolish that I did not think of these natural and obviously fruitful concepts before.
The only 1-critical graph is $K_1$, and the only 2-critical graph is $K_2$. The only 3-critical graphs are the odd cycles.

For every $k \geq 4$ and every $n \geq k + 2$, there exists a $k$-critical $n$-vertex graph.

Every $k$-critical graph is 2-connected and $(k - 1)$-edge-connected.
$f_k(n)$ — the minimum number of edges in a $k$-critical graph with $n$ vertices.

Since $\delta(G) \geq k - 1$ for every $k$-critical graph $G$,

$$f_k(n) \geq \frac{k - 1}{2}n$$

for all $n \geq k$, $n \neq k + 1$.

Brooks’ Theorem implies that for $k \geq 4$ and $n \geq k + 2$, the inequality in (1) is strict.
Dirac’s bound

Dirac in 1957 asked to determine $f_k(n)$ and proved that for $k \geq 4$ and $n \geq k + 2$,

$$f_k(n) \geq \frac{k - 1}{2} n + \frac{k - 3}{2}.$$  \hspace{1cm} (2)

The result is tight for $n = 2k - 1$ and yields

$$f_k(2k - 1) = k^2 - k - 1.$$
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Stiebitz and A.K. improved (2) to

$$f_k(n) \geq \frac{k - 1}{2}n + k - 3 \tag{3}$$

when $n \neq 2k - 1, k$.

This yields $f_k(2k) = k^2 - 3$ and $f_k(3k - 2) = \frac{3k(k-1)}{2} - 2$. 
Gallai’s results and conjecture

**Theorem 1** [Gallai, 1963] If $k \geq 4$ and $k + 2 \leq n \leq 2k - 1$, then

$$f_k(n) = \frac{1}{2} \left((k - 1)n + (n - k)(2k - n)\right) - 1.$$

**Theorem 2** [Gallai, 1963] For all $k \geq 4$ and $n \geq k + 2$,

$$f_k(n) \geq \frac{k - 1}{2}n + \frac{k - 3}{2(k^2 - 3)}n. \quad \text{(4)}$$
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**Theorem 2** [Gallai, 1963] For all \( k \geq 4 \) and \( n \geq k + 2 \),

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f_k(n) \geq \frac{k - 1}{2} n + \frac{k - 3}{2(k^2 - 3)} n. \tag{4}
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**Conjecture 1** [Gallai, 1963] If \( k \geq 4 \) and \( n = 1 \) (mod \( k - 1 \)), then

\[
f_k(n) = \frac{(k+1)(k-2)n-k(k-3)}{2(k-1)}.
\]
Figure: Choose a vertex \( x \) in one \( k \)-critical graph and an edge \( yz \) in the other.
Figure: Delete $yz$, split $x$ and glue the two pieces of $x$ to $y$ and $z$. Call the new graph $H(G_1, G_2)$. 
Ore’s Conjecture

Ore observed that Hajós’ construction implies

\[ f_k(n + k - 1) \leq f_k(n) + (k - 1)(\frac{k}{2} - \frac{1}{k - 1}), \]  

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which yields that \( \phi_k := \lim_{n \to \infty} \frac{f_k(n)}{n} \) exists and satisfies

\[ \phi_k \leq \frac{k}{2} - \frac{1}{k - 1}. \quad (6) \]

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Ore (1967) conjectured that for every \( n \geq k + 2 \), in (4) equality holds.
Other results


The problem of finding $f_k(n)$ is Problem 5.3 in the monograph of Jensen and Toft and Problem 12 in their list of 25 pretty graph colouring problems. It is a half of Problem P1 in the Handbook of Graph Theory.
Theorem 3 [Yancey and A.K.] If \( k \geq 4 \) and \( G \) is \( k \)-critical, then
\[
|E(G)| \geq \left\lceil \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)} \right\rceil.
\]
In other words, if \( k \geq 4 \) and \( n \geq k, n \neq k + 1 \), then
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f_k(n) \geq F(k, n) := \left\lceil \frac{(k + 1)(k - 2)n - k(k - 3)}{2(k - 1)} \right\rceil.
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This proves Conjecture 1 in full and implies that Ore’s Conjecture holds for every $n \equiv 1 \mod k - 1$ ($n > 1$).
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Corollary 1: For every $k \geq 4$ and $n \geq k + 2$,

$$0 \leq f_k(n) - F(k, n) \leq \frac{k(k-1)}{8} - 1.$$

In particular, $\phi_k = \frac{k}{2} - \frac{1}{k-1}$ and $f_4(n) = F(4, n)$ for every $n \geq 6$. 
Extremal graphs

A $k$-extremal graph is a $k$-critical graph $G$ such that

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A graph is $k$-Ore if it is obtained by a sequence of Hajos’ constructions from a set of copies of $K_k$.

By above, every $k$-Ore graph is $k$-extremal.

So, for every $k \geq 4$, there are infinitely many $k$-extremal graphs.
Theorem 4 [Yancey and A.K.] Let $k \geq 4$ and $G$ be a $k$-critical graph. Then $G$ is $k$-extremal if and only if it is a $k$-Ore graph. Moreover, if $G$ is not a $k$-Ore graph, then

$$|E(G)| \geq \frac{(k^2-k-2)|V(G)|-y_k}{2(k-1)},$$

where $y_k = \max\{2k - 6, k^2 - 5k + 2\}$. 

This gives a slightly better approximation for $f_k(n)$ and adds new cases where we now know the exact values of $f_k(n)$. In particular, we know $f_5(n)$ for every $n \geq 7$. The value of $y_k$ in Theorem 4 is best possible in the sense that, as observed by Bjarne Toft, for every $k \geq 4$, there exists an infinite family of 3-connected graphs with $|E(G)| = (k^2-k-2)|V(G)|-y_k$. 


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Planar graphs

**Grötzsch’s Theorem**  Every planar triangle-free graph is 3-colorable.

**Theorem 5** [Jensen and Thomassen] If a graph $G$ is obtained from a planar triangle-free graph $H$ by adding a vertex of degree at most 3, then $G$ is 3-colorable.
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There are infinitely many 4-critical graphs obtained from a planar triangle-free graph by adding a vertex of degree 5.
A sharpening of Theorem 5

**Theorem 6** [Borodin, A.K., Lidický and Yancey] If a graph $G$ is obtained from a planar triangle-free graph $H$ by adding a vertex of degree at most 4, then $G$ is 3-colorable.

**Proof:** Let $G$ be a smallest counter-example. Then $G$ is 4-critical and so 2-connected. Let $v$ be the vertex added to a planar triangle-free $H$. Suppose $G$ has $n$ vertices and $e$ edges. Suppose $H$ has $f$ faces, $n'$ vertices and $e'$ edges. Clearly, $n' = n - 1$ and $e' \geq e - 4$. 

Folklore observation: $H$ has no 4-faces. Then $2e' \geq 5f$. Plug this into Euler's Formula $n' - e' + f = 2$: $n' - e' + 2e' \geq 5$, i.e. $(n - 1) - 2 \geq 3(e - 4)$. So, $5n - 3 \geq 3e$, a contradiction to (6).
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By Theorem 3,

$$e \geq (5n - 2)/3.$$  \hfill (8)
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Then $2e' \geq 5f$. Plug this into Euler’s Formula $n' - e' + f = 2$:

$$n' - e' + \frac{2e'}{5} \geq 2, \quad \text{i.e.} \quad (n - 1) - 2 \geq \frac{3(e - 4)}{5}.$$

So, $5n - 3 \geq 3e$, a contradiction to (6).
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Theorem 7 [Grünbaum–Aksenov] Every planar graph with at most three triangles is 3-colorable.

$K_4$ shows that “three” in Theorem 7 cannot be replaced by “four”.

But maybe there are not many plane 4-critical graphs with exactly four triangles ($4,4$-graphs, for short)?
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But maybe there are not many plane 4-critical graphs with exactly four triangles (4, 4-graphs, for short)?

It turned out that there are many. Havel in 1969 presented a 4, 4-graph \( H_1 \) in which the four triangles had no common vertices.
Havel used the quasi-edge $H_0 = H_0(u, v)$ (on the left):

![Graphs $H_0$, $H_1$, and $H_2$](image)

**Figure:** A quasi-edge $H_0$ and 4,4-graphs $H_1$ and $H_2$. 

Sachs in 1972 asked whether it is true that in every non-3-colorable planar graph $G$ with exactly four triangles and no separating triangles, these triangles can be partitioned into two pairs so that in each pair the distance between the triangles is less than two.
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Aksenov and Mel’nikov in 1978 answered to the question in the negative by constructing a 4,4-graph $H_2$ (on the right):

 Moreover, they constructed two infinite series of 4,4-graphs.

**Figure:** A quasi-edge $H_0$ and 4,4-graphs $H_1$ and $H_2$. 
Question of Erdős

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According to a survey of Steinberg, Erdős in 1990 asked for description of 4, 4-graphs again.

Thomas and Walls constructed an infinite family $\mathcal{TW}$ of 4, 4-graphs that have no 4-faces (we will call 4, 4-graphs with no 4-faces 4, 4, 4f-graphs):

![Graphs](image_url)

**Figure**: Smallest Thomas-Walls graphs.

Note that $H_1$ is a 4, 4, 4f-graph but is not in $\mathcal{TW}$. Graph $H_2$ is not a 4, 4, 4f-graph.
4-Ore graphs

First we describe the family $P_{4,4,4}$ of all $4, 4, 4f$-graphs.

Recall that a graph is 4-Ore if it is obtained from a set of copies of $K_4$ by a sequence of the above Hajos’ constructions. Every 4-Ore graph is 4-critical.

By Euler’s Formula, every $4, 4, 4f$-graph is a 4-Ore graph.
4-Ore graphs

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**Theorem 8** [Borodin, Dvořák, A. K., Lidický, and Yancey] Every 4-Ore graph has at least four triangles. Moreover, a 4-Ore graph $G$ has exactly four triangles if and only if $G$ is a $4, 4, 4f$-graph.
A characterization of $4, 4, 4f$-graphs

**Theorem 9** [B-D-K-L-Y] Every $4, 4, 4f$-graph is either in $\mathcal{TW}$ or is obtained from a graph in $\mathcal{TW}$ by replacing one or both diamond edges by the Havel’s quasi-edge $H_0$. 
A characterization of $4, 4, 4f$-graphs

**Theorem 9** [B-D-K-L-Y] Every $4, 4, 4f$-graph is either in $\mathcal{T}\mathcal{W}$ or is obtained from a graph in $\mathcal{T}\mathcal{W}$ by replacing one or both diamond edges by the Havel’s quasi-edge $H_0$. 

\[ \begin{array}{c}
\text{\includegraphics[width=0.8\textwidth]{diagram.png}}
\end{array} \]
Example: A 4, 4-graph with no 5-faces
A patch $P$ is a subgraph of a plane graph such that
a) the boundary of $P$ is a 6-cycle $C_P = (x, z', y, x', z, y')$, all vertices on $C_P$ and inside $C_P$ are in $P$;
b) vertices $x', y', z'$ have no neighbors outside of $P$,
c) all faces inside $C_P$ are 4-faces.

**Observation:** If $G$ is a 4,4-graph and a vertex $x \in V(G)$ has exactly 3 neighbors, $x, y$ and $z$, then the graph $G_v$ obtained from $G - v$ by inserting a patch $P$ with $C_P = (x, z', y, x', z, y')$ where $x, y, z$ are old and $x', y', z'$ are new vertices is again a 4,4-graph.
Examples

\[ H_0 \quad \xrightarrow{\text{example}} \quad H_3 \]

\[ H_1 \quad H_2 \]
Main result

**Theorem 10** [B-D-K-L-Y] A plane 4-critical graph has exactly four 3-cycles if and only if it is obtained from a 4, 4, 4f-graph by replacing several (maybe zero) non-adjacent 3-vertices with patches.
Main result

**Theorem 10** [B-D-K-L-Y] A plane 4-critical graph has exactly four 3-cycles if and only if it is obtained from a $4, 4, 4f$-graph by replacing several (maybe zero) non-adjacent 3-vertices with patches.

This fully answers the question of Erdős from 1990.

So, Sachs had right intuition in 1972: his question has positive answer if we replace “less than two” with “at most two”.

Aksenov and Mel’nikov in 1979 conjectured, in particular, that $H_1$ is the unique smallest 4, 4-graph with the minimum distance 1 between triangles and $H_2$ is the unique smallest 4, 4-graph with the minimum distance 2 between triangles. Our description confirms this.