Finite substructures of uncountable graphs and hypergraphs

Péter Komjáth

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Pál Erdős
(1913–1996)

A *graph* is a pair (V, X) where V (the vertices) is an arbitrary set, X (the vertices) is some set of 2-element subsets of V (the edges). (V, X) is sometimes shortened to X

The cardinalities (=sizes) of infinite sets are $\aleph_0 < \aleph_1 < \aleph_2 < \cdots$.

"Independence raised its ugly head."

(Blanche Descartes, aka W. T. Tutte) For every finite n there is a triangle-free, n-chromatic (finite) graph.

The chromatic number of a finite graph is the least number of colors in a *good coloring*: the vertices are colored $f: V \to C$ such that if $\{x, y\}$ is an edge then $f(x) \neq f(y)$.

Erdős: can we generalize this to infinite graphs?

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(Erdős–de Bruijn) If n is a natural number then an infinite graph X has chromatic number at most n iff this holds for all finite subgraphs of X.

(Erdős–Rado) If κ is an infinite cardinal, then there is a triangle-free graph X of cardinality 2^κ with $\mathrm{Chr}(X) > \kappa$. $|X| = \kappa^+$ is also possible.

(Erdős) If n, k are natural numbers, then there is a (finite) graph X with $\mathrm{Chr}(X) > n$ that does not contain $C_3, C_4, \ldots, C_k$.

(Erdős–Hajnal) If the graph X does not contain a C_4 (or any circuit of even length), then $\mathrm{Chr}(X) \leq \aleph_0$.

(Erdős–Hajnal) If κ is an infinite cardinal, n is a natural number, then there is a graph X, omitting $C_3, C_5, \ldots, C_{2n+1}$ with $\mathrm{Chr}(X) > \kappa$.

(Erdős-Hajnal) The obligatory finite graphs for uncountable chromatic number are exactly the bipartite graphs.

Same for $\mathrm{Chr}(X) > \kappa$, any infinite κ.

(Erdős–Hajnal) If (V, X) is a graph, then its *coloring number*, $\mathrm{Col}(X)$ is the least cardinal μ such that the following holds: V has a well ordering $<$ such that every vertex is joined into less than μ smaller vertices.

Then V can be good colored with μ colors with transfinite recursion by $<$, consequently $\mathrm{Chr}(X) \leq \mathrm{Col}(X)$.

(Erdős–Hajnal) If $\mathrm{Col}(X) > \aleph_0$, then X contains C_4, even every C_{2k}, even every $K_{n,\aleph_1}$.

Obligatory graph: occurs in every X with $\mathrm{Col}(X) > \aleph_0$. What are the obligatory graphs?

(K) There are a countable graph Δ and a graph Γ of cardinality $\aleph_1$ such that Δ is the largest countable obligatory graph and Γ is the largest obligatory graph (for the coloring number).

(Erdős–Hajnal) Is it true that every graph with uncountable chromatic number contains an infinitely connected (countable) subgraph?

(Erdős–Hajnal–Szemerédi) If $\mathrm{Chr}(X) > \aleph_0$ let f_X be the following function. $f_X(n)$ is the largest chromatic number of an n-vertex subgraph of X. $f_X(n) \to \infty$ by Erdős-de Bruijn.

(Erdős–Hajnal–Szemerédi) Can f_X converge to infinity arbitrarily slowly?

(Shelah) It is consistent that for every divergent function $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \geq 2$ $(n \in \mathbb{N})$ there is a graph X with $\mathrm{Chr}(X) = \aleph_1$ and $f_X(n) \leq f(n)$.

Taylor conjecture (Taylor, Erdős–Hajnal–Shelah)
If X is a graph with $\mathrm{Chr}(X) > \aleph_0$, then for every cardinal λ there is a graph Y with the same finite subgraphs and $\mathrm{Chr}(Y) > \lambda$.

(K) There consistently exists a graph X with $|X| = \mathrm{Chr}(X) = \aleph_1$ and if Y is a graph all whose finite subgraphs occur in X, then $\mathrm{Chr}(Y) \leq \aleph_2$.

(K) The following is consistent: if X is a graph with $\mathrm{Chr}(X) \geq \aleph_2$, then there are arbitrarily large chromatic graphs with the same finite subgraphs as X.

The *list-chromatic number* of a graph (V, X) is the least cardinal μ such that if $F(v)$ is an arbitrary set with $|F(v)| = \mu$ $(v \in V)$ then there is a good coloring f such that $f(v) \in F(v)$ $(v \in V)$.

For every graph X we have the inequality

$$\mathrm{Chr}(X) \leq \mathrm{List}(X) \leq \mathrm{Col}(X)$$

.

(K) It is consistent that for every graph X of cardinality $\aleph_1$

$$\mathrm{List}(X) = \aleph_1 \iff \mathrm{Chr}(X) = \aleph_1.$$

(K) It is consistent that for every graph with $\mathrm{Col}(X)$ infinite, $\mathrm{List}(X) = \mathrm{Col}(X)$ holds.

(Erdős–Hajnal) What is the situation for hypergraphs?

A hypergraph is $(V, \mathcal{H})$ where $\mathcal{H}$ consists of finite subsets of V.

The *chromatic number* of $(V, \mathcal{H})$ is the least number of colors required to color V with no member of $\mathcal{H}$ monocolored.

Restrict to triple systems.

Project: describe obligatory triple systems for uncountable chromatic number as was done for graphs.

(Erdős–Hajnal) If $(V, \mathcal{H})$ is a triple system with $|V| \leq \aleph_1$ and $|A \cap B| \leq 1$ for $A, B \in \mathcal{H}$, then $\mathrm{Chr}(\mathcal{H}) \leq \aleph_0$.

(Erdős–Hajnal–Rothschild) There is a triple system $(V, \mathcal{H})$ of cardinality c^+ with $\mathrm{Chr}(\mathcal{H}) > \aleph_0$ with no $A, B \in \mathcal{H}$ s.t. $|A \cap B| = 2$.

P. Erdős, F. Galvin, A. Hajnal: On set systems having large chromatic numbers and not containing prescribed subsystems, *Coll. Math. Soc. J. Bolyai*, **10**, Infinite and Finite Sets, Keszthely (Hungary), 1973, 425–513.

1. Describe all finite obligatory triple systems.

2. Is it true that if $\mathcal{S}_0$ and $\mathcal{S}_1$ can be separately omitted then they can be simultaneously omitted. (True for the graph case.)

3. When does $\aleph_1 \longrightarrow (\aleph_1, \mathcal{S})^3$ hold? (Read: if $\mathcal{H}$ is a triple system on a set of size $\aleph_1$ with no independent set of cardinality $\aleph_1$, then $\mathcal{S} \leq \mathcal{H}$.)

(K) A finite triple system is obligatory iff all its 2-connected components are.

(K) Every obligatory finite triple systems is tripartite but not vice versa.

Let $\mathcal{T}_0$ denote the triple system $\{A, B\}$ with $|A \cap B| = 2$.

(Hajnal–K) If $\mathrm{Chr}(\mathcal{H}) > \aleph_0$, $\mathcal{T}_0 \not\leq \mathcal{H}$ then $\mathcal{C}_7, \mathcal{C}_9, \mathcal{C}_{11}, \dots \leq \mathcal{H}$.

(K) It is consistent that there is a triple system $\mathcal{H}$ omitting $\mathcal{T}_0$, $\mathcal{C}_3$, and $\mathcal{C}_5$, with $\mathrm{Chr}(\mathcal{H}) = \aleph_1$ (and $|\mathcal{H}| = \aleph_2$).

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(Hajnal–K) It is consistent that there are finite triple systems $\mathcal{S}_0$, $\mathcal{S}_1$ such that either can be omitted but not both.

(K) It is consistent that I can describe those triple systems $\mathcal{S}$ for which $\aleph_1 \to (\aleph_1, \mathcal{S})^3$ holds.

(K) If the coloring number of some set system $\mathcal{H}$ is greater than $\aleph_0$, then $\mathcal{H}$ contains...

(K) If V is a vector space over $\mathbb{Q}$, then there is, in V, a vector set $A \subseteq V$, $|A| = \aleph_2$, which is not the union of countably many linearly independent sets, yet each $B \subseteq A$, $|B| \leq \aleph_1$, is. (Question raised by Erdős.)

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Happy birthday, Uncle Paul!