DEFINITION. (Hewitt, 1943, Pearson, 1963)

– A topological space $X$ is $\kappa$-resolvable iff it has $\kappa$ disjoint dense subsets. (resolvable $\equiv$ 2-resolvable)

– $X$ is maximally resolvable iff it is $\Delta(X)$-resolvable, where

$$\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open in } X\}.$$ 

EXAMPLES:

– $\mathbb{R}$ is maximally resolvable.

– Compact Hausdorff, metric, and linearly ordered spaces are maximally resolvable.

QUESTION. What happens if these properties are relaxed?
Malychin’s problem

EXAMPLE. (Hewitt, ’43) There is a countable $T_3$ space $X$ that is
– crowded (i.e. $\Delta(X) = |X| = \aleph_0$) and
– irresolvable ($\equiv$ not 2-resolvable).

PROBLEM. (Malychin, 1995)

Is a Lindelöf $T_3$ space $X$ with $\Delta(X) > \omega$ resolvable?

NOTE. Malychin constructed Lindelöf irresolvable Hausdorff (= $T_2$) spaces, and Pavlov Lindelöf irresolvable Uryson (= $T_{2.5}$) spaces.

THEOREM. (Filatova, 2004)

YES, every Lindelöf $T_3$ space $X$ with $\Delta(X) > \omega$ is 2-resolvable.

This is the main result of her PhD thesis. It didn’t work for 3!
Pavlov’s theorems

\[ s(X) = \sup \{|D| : D \subset X \text{ is discrete} \} \]
\[ e(X) = \sup \{|D| : D \subset X \text{ is closed discrete} \} \]

**THEOREM. (Pavlov, 2002)**

(i) Any \( T_2 \) space \( X \) with \( \Delta(X) > s(X)^+ \) is maximally resolvable.
(ii) Any \( T_3 \) space \( X \) with \( \Delta(X) > e(X)^+ \) is \( \omega \)-resolvable.

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**THEOREM. (J-S-Sz, 2007)**

Any space \( X \) with \( \Delta(X) > s(X) \) is maximally resolvable.

**THEOREM. (J-S-Sz, 2012)**

Any \( T_3 \) space \( X \) with \( \Delta(X) > e(X) \) is \( \omega \)-resolvable. In particular, every Lindelöf \( T_3 \) space \( X \) with \( \Delta(X) > \omega \) is \( \omega \)-resolvable.
THEOREM. (J-S-Sz, 2007)
If $\Delta(X) \geq \kappa = \text{cf}(\kappa) > \omega$ and $X$ has no discrete subset of size $\kappa$ then $X$ is $\kappa$-resolvable.

THEOREM. (J-S-Sz, 2012)
If $X$ is $T_3$, $\Delta(X) \geq \kappa = \text{cf}(\kappa) > \omega$ and $X$ has no closed discrete subset of size $\kappa$ then $X$ is $\omega$-resolvable.

NOTE. For $\Delta(X) > \omega$ regular these suffice. If $\Delta(X) = \lambda$ is singular, we need to do extra work.

For $\Delta(X) = \lambda > s(X)$ we automatically get that $X$ is $< \lambda$-resolvable.

But now $\Delta(X) = \lambda > s(X)^+$, so we may use Pavlov’s Thm (i).

For $\Delta(X) = \lambda > e(X)^+$ we may use Pavlov’s Thm (ii).
THEOREM. (J-S-Sz, 2006)
For any $\kappa \geq \lambda = \text{cf}(\lambda) > \omega$ there is a dense $X \subset D(2)^{2\kappa}$ with $\Delta(X) = \kappa$ that is $< \lambda$-resolvable but not $\lambda$-resolvable.

NOTE. This solved a problem of Ceder and Pearson from 1967. We used the general method of constructing $D$-forced spaces.

THEOREM. (Illanes, Baskara Rao)
If $\text{cf}(\lambda) = \omega$ then every $< \lambda$-resolvable space is $\lambda$-resolvable.

PROBLEM.
Is this true for each singular $\lambda$? How about $\lambda = \aleph_\omega$?
DEFINITION.

The space $X$ is monotonically normal (MN) iff it is $T_1$ (i.e. all singletons are closed) and it has a monotone normality operator $H$ that "halves" neighbourhoods:

$H$ assigns to every $\langle x, U \rangle$, with $x \in U$ open, an open set $H(x, U)$ s. t.

(i) $x \in H(x, U) \subset U$,

and

(ii) if $H(x, U) \cap H(y, V) \neq \emptyset$ then $x \in V$ or $y \in U$.

FACT. Metric spaces and linearly ordered spaces are MN.

QUESTION. Are MN spaces maximally resolvable?
DEFINITION.

(i) $D \subset X$ is strongly discrete if there are pairwise disjoint open sets $\{U_x : x \in D\}$ with $x \in U_x$ for $x \in D$.

EXAMPLE: Countable discrete sets in $T_3$ spaces are SD.

(ii) $X$ is an SD space if every non-isolated point $x \in X$ is an SD limit.

THEOREM. (Sharma and Sharma, 1988)
Every $T_1$ crowded SD space is $\omega$-resolvable.

THEOREM. (DTTW, 2002)
MN spaces are SD, hence crowded MN spaces are $\omega$-resolvable.

PROBLEM. (Ceder and Pearson, 1967)
Are $\omega$-resolvable spaces maximally resolvable?
J-S-Sz


DEFINITION. X is a DSD space if every dense subspace of X is SD. Clearly, MN spaces are DSD.

Main results of [J-S-Sz]

- If $\kappa$ is measurable then there is a MN space $X$ with $\Delta(X) = \kappa$ that is $\omega_1$-irresolvable.
- If $X$ is DSD with $|X| < \aleph_\omega$ then $X$ is maximally resolvable.
- From a supercompact cardinal, it is consistent to have a MN space $X$ with $|X| = \Delta(X) = \aleph_\omega$ that is $\omega_2$-irresolvable.

This left a number of questions open.
decomposability of ultrafilters

**DEFINITION.**
- An ultrafilter $\mathcal{F}$ is $\mu$-descendingly complete iff for any descending $\mu$-sequence $\{A_\alpha : \alpha < \mu\} \subseteq \mathcal{F}$ we have $\bigcap\{A_\alpha : \alpha < \mu\} \in \mathcal{F}$. $\mu$-descendingly incomplete is called $\mu$-decomposable.
- $\Delta(\mathcal{F}) = \min\{|A| : A \in \mathcal{F}\}$.
- $\mathcal{F}$ is maximally decomposable iff it is $\mu$-decomposable for all $\mu$ with $\omega \leq \mu \leq \Delta(\mathcal{F})$.

**FACTS.**
- Any "measure" is countably complete, hence $\omega$-indecomposable.
- [Donder, 1988] If there is a not maximally decomposable ultrafilter then there is a measurable cardinal in some inner model.
- [Kunen - Prikry, 1971] Every ultrafilter $\mathcal{F}$ with $\Delta(\mathcal{F}) < \aleph_\omega$ is maximally decomposable.
Main results of [J-M]

(1) TFAEV

– Every DSD space (of cardinality $< \kappa$) is maximally resolvable.
– Every MN space (of cardinality $< \kappa$) is maximally resolvable.
– Every ultrafilter $\mathcal{F}$ (with $\Delta(\mathcal{F}) < \kappa$) is maximally decomposable.

(2) TFAEC

– There is a measurable cardinal.
– There is a MN space that is not maximally resolvable.
– There is a MN space $X$ with $|X| = \Delta(X) = \aleph_\omega$ that is $\omega_1$-irresolvable.