Introduction

What is Known

Preparing to Use Old Tools

Continuity of $p$ and $\alpha$

Differentiability of $p$ and $a$
Keleti’s Perimeter to Area Conjecture (\textit{PAC})

\textit{The perimeter to area ratio of the union of finitely many unit squares in a plane does not exceed 4.}
Keleti’s Perimeter to Area Conjecture (PAC)

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Problem 6 on the famous Hungarian Schweitzer Competition in 1998. Show the perimeter to area ratio is bounded.
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- Problem 6 on the famous Hungarian Schweitzer Competition in 1998. Show the perimeter to area ratio is bounded.
- Later that same year, Keleti published his Perimeter to Area Conjecture that this bound is actually 4.
- To date, the best known bound is slightly less than 5.6.
Gyenes’ Results

Theorem (Gyenes)
If the squares are oriented, the PAC is true.
**Gyenes’ Results**

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**Theorem (Gyenes)**

*If the squares have a common center, the PAC is true.*
Gyenes’ Results

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If the squares have a common center, the PAC is true.

Theorem (Gyenes)

There exist congruent convex sets, $E_1 \cong E_2 \subset \mathbb{R}^2$ such that the perimeter to area ratio for $E_1 \cup E_2$ exceeds the perimeter to area ratio for either one of them.
A Convex Counterexample
A Tantalizing Tidbit

Suppose a counterexample exists. Then there is a counterexample with a least number of squares. The Isoperimetric Inequality yields

**Theorem**

If $\mathcal{H} = \bigcup_{i=1}^{n} H_i$ is an optimal counterexample, then for each $i \leq n$, the area of $H_i \cap (\mathcal{H} \setminus H_i) > \frac{\pi}{4}$. 
Basic Notation

1. \( \mathcal{H} = \bigcup_{i=1}^{n} H_i \) is the finite union of unit squares \( H_i \) in \( \mathbb{R}^2 \).
2. \( p(\mathcal{H}) \) is the perimeter of \( \mathcal{H} \).
3. \( \alpha(\mathcal{H}) \) denotes the area of \( \mathcal{H} \).
4. square \( \equiv \) unit square in \( \mathbb{R}^2 \).
If $H \subset \mathbb{R}^2$ is a square, then $H$ can be parameterized by a point in $\mathbb{R}^3$. 
If $H \subset \mathbb{R}^2$ is a square, then $H$ can be parameterized by a point in $\mathbb{R}^3$ whose coordinates are the center of $H$ and the mod $(\pi/2)$ rotational displacement of $H$. 
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![Diagram of a square with a point labeled (s,t) and an angle labeled \(\phi\).]
Basic Notation Revisited

Suppose we are interested in unions of \( n \) unit squares \( H_i; \)
\( \mathcal{H} = \bigcup_{i=1}^{n} H_i. \) Then the associated perimeter and area are maps:

1. \( p : \mathbb{R}^{3n} \rightarrow \mathbb{R}. \)
2. \( \alpha : \mathbb{R}^{3n} \rightarrow \mathbb{R}. \)
3. \( \kappa \equiv \frac{p}{\alpha}. \)
Suppose we are interested in unions of $n$ unit squares $H_i$; $\mathcal{H} = \bigcup_{i=1}^{n} H_i$. Then the associated perimeter and area are maps:

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2. $\alpha : \mathbb{R}^{3n} \rightarrow \mathbb{R}$.
3. $\kappa \equiv \frac{p}{\alpha}$.

We’ll have a brief look at some continuity and differentiability of these maps.
CONTINUITY OF $p$ AND $\alpha$

Now, $\alpha$ IS the area function afterall, Lipschitz in each coordinate, so . . .
Continuity of $p$ and $\alpha$

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**Theorem**

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**Theorem**

$p$ is often discontinuous, but only with jump discontinuities.
Discontinuity of $p$
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Consequently, let’s first restrict the domain somewhat to avoid such unpleasantries.
1. $\mathcal{H} \subset \mathbb{R}^2$ has **distinct rotational displacement** if $\phi_i \neq \phi_j$ when $i \neq j$
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2. $\mathcal{H}$ is **vertex free** if no vertex of $H_i$ lies on the boundary of $H_j$
   whenever $i \neq j$. 
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3. $\mathcal{H}$ is **triple free** if no point lies on the boundaries of three distinct $H_i$’s.
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$\mathcal{H}$ is said to be in **standard position** provided $H$ is all three.
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**Theorem (a brief aside)**

*The set of points which are in standard position is the complement of a sparse set in the sense that it is a subset of the complement of a countable union of monotonic surfaces and so are both residual and of full measure in $\mathbb{R}^{3n}$.***
Continuity of $p$ and $\alpha$

**Theorem**

The perimeter function $p$ is continuous at every point $\mathcal{H} \in \mathbb{R}^{3n}$ which is in standard position.
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Differentiability of $p$ and $\alpha$

Here we are interested in the following questions:

1. Are $p$ and $\alpha$ differentiable at points in standard position?
2. What IS the derivative at those points?
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1. Are $p$ and $\alpha$ differentiable at points in standard position?
2. What IS the derivative at those points?

So we do the obvious:

1. compute the first partials and
2. show they’re continuous.
Theorem

The perimeter function $p : \mathbb{R}^{3n} \rightarrow \mathbb{R}^+$ is differentiable at every point $H \in \mathbb{R}^{3n}$ in standard position.

Outline of Proof.

1. Fix $H \in \mathbb{R}^{3n}$ and $1 \leq i_o \leq n$. 

Differentiability of $p$
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**Theorem**

The perimeter function $p : \mathbb{R}^{3n} \to \mathbb{R}^+ \text{ is differentiable at every point } \mathcal{H} \in \mathbb{R}^{3n} \text{ in standard position.}

**Outline of Proof.**

1. Fix $\mathcal{H} \in \mathbb{R}^{3n}$ and $1 \leq i_o \leq n$.
2. Fix a segment on the boundary of $\mathcal{H}$. 
Differentiability of $p$

**Theorem**

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**Outline of Proof.**

1. Fix $\mathcal{H} \in \mathbb{R}^{3n}$ and $1 \leq i_o \leq n$.
2. Fix a segment on the boundary of $\mathcal{H}$.
3. Compute the contribution to each of the 3 partials, $\frac{\partial p}{\partial s_{i_o}}$, $\frac{\partial p}{\partial t_{i_o}}$ and, $\frac{\partial p}{\partial \phi_{i_o}}$. 
**Differentiability of** $p$

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$\frac{\partial p}{\partial \phi_{i_o}}$ for example.
Case 1. The segment misses $H_{io}$.
In this case the contribution is 0.

Case 2. The segment lies on $H_{io}$.

Figure: $[a, b]$ and $[a^*, b^*]$
Case 2. Computation

\[ a^\ast = \left( \frac{x_1 \tan \phi_a}{\tan \phi_a - \tan \delta}, \frac{x_1 \tan \phi_a \tan \delta}{\tan \phi_a - \tan \delta} - \frac{1}{2} \right) \]

\[ b^\ast = \left( \frac{x_2 \tan \phi_b}{\tan \phi_b + \tan \delta}, \frac{x_2 \tan \phi_b \tan \delta}{\tan \phi_b + \tan \delta} - \frac{1}{2} \right). \]

Hence, with some trigonometry and limit taking:

\[ \lim_{\delta \to 0} \frac{|b^\ast - a^\ast| - |b - a|}{\delta} = |b - a|(\cot \phi_b - \cot \phi_a). \]
Case 3.

Case 3. The segment intersects $H_{i_o}$ but does not lie on it.

Again, with some trigonometry and limit taking:

$$\lim_{\delta \to 0} \frac{|a^* - a|}{\delta} = \frac{d}{\sin \phi_a}$$
Case 3. The segment intersects $H_{i_o}$ but does not lie on it.

Again, with some trigonometry and limit taking:

$$\lim_{\delta \to 0} \frac{|a^* - a|}{\delta} = \frac{d}{\sin \phi_a}$$

Oh yes, here is "d."
These cases have **congruent geometries** and are handled similarly to the case of \( \frac{\partial p}{\partial \phi_{io}} \).
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But we still have area to deal with.
Differentiability of $\alpha$

**Theorem**

The area function $\alpha : \mathbb{R}^{3n} \to \mathbb{R}^+$ is differentiable at every point $\mathcal{H} \in \mathbb{R}^{3n}$ in standard position.

**Outline of Proof.**

1. Fix $\mathcal{H} \in \mathbb{R}^{3n}$ and $1 \leq i_0 \leq n$. 

Differentiability of \( \alpha \)

**Theorem**

The area function \( \alpha : \mathbb{R}^{3n} \to \mathbb{R}^+ \) is differentiable at every point \( \mathcal{H} \in \mathbb{R}^{3n} \) in standard position.

**Outline of Proof.**

1. Fix \( \mathcal{H} \in \mathbb{R}^{3n} \) and \( 1 \leq i_o \leq n \).
2. Fix a segment on the boundary of \( \mathcal{H} \).
Differentiability of $\alpha$

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3. Compute the contribution to each of the 3 partials, $\frac{\partial \alpha}{\partial s_{i_0}}$, $\frac{\partial \alpha}{\partial t_{i_0}}$ and $\frac{\partial \alpha}{\partial \phi_{i_0}}$.

$\frac{\partial \alpha}{\partial \phi_{i_0}}$ for example.
$H$ with $H_{i_o}$ Darkened

\[ \frac{\partial \alpha}{\partial \phi_{i_o}} \]
$H$ with $H_{i_o}$ and Rotated $H_{i_o}$

\[ \frac{\partial \alpha}{\partial \phi_{i_o}} \]
Pre Computations; What are the Variables?
Pre Computations; What are the Variables?
The Computations

$$\Delta \alpha ([a, b]) =$$

$$\frac{(x_1^2 - x_2^2) \tan \phi_a \tan \phi_b \tan \delta + \tan^2 \delta (x_2^2 \tan \phi_b + x_1^2 \tan \phi_a)}{2(tan \phi_a - tan \delta)(tan \phi_b + tan \delta)}.$$
The Computations

\[ \Delta \alpha([a, b]) = \frac{(x_1^2 - x_2^2) \tan \phi_a \tan \phi_b \tan \delta + \tan^2 \delta (x_2^2 \tan \phi_b + x_1^2 \tan \phi_a)}{2 \tan \phi_a - \tan \delta)(\tan \phi_b + \tan \delta)} . \]

\[ \frac{\partial \alpha}{\partial \phi_{i_o}} \text{ at } [a,b] = \lim_{\delta \to 0} \frac{\Delta \alpha([a, b])}{\delta} = \frac{x_1^2 - x_2^2}{2} . \]
These cases again have \textit{congruent geometries} and are handled similarly.
Where We’re Pushing the Pebble

Similar ground has been plowed in other lands. For example:

**Theorem (Kneser-Paulson)**

*If a finite set of discs are rearranged so that the distance between the centers of any pair decreases, then the area and the perimeter of the union of the discs also decreases.*
THANK YOU!


