Problems and memories

András Gyárfás

Alfréd Rényi Institute of Mathematics

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…what is the smallest graph with this property?” – asked during the revenge game which I lost again.
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**Theorem**

*(Erdős, Hajnal, Moon, 1964)* The minimum number of edges in a $K_{k+2}$-saturated graph with $n$ vertices is $\binom{k}{2} + k(n - k)$. 
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Trivial or not, certainly important and rediscovered several times (Jaeger - Payan 1971, Katona 1974). The underlying idea, cross-intersecting sets, developed further and have many applications.
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- From the early 90-s the department was fortified by Béla Bollobás, who leads a Chair of Excellence in Combinatorics since then.
1. Cycles in graphs without proper subgraphs of minimum degree 3.
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- Graphs with \( n \) vertices and \( 2n - 1 \) edges must contain proper subgraphs of minimum degree 3 but this fails for graphs with \( n \) vertices and \( 2n - 2 \) edges, for example the wheel is such a graph (Erdős, Faudree, Rousseau and Schelp, 1990).
1. Cycles in graphs without proper subgraphs of minimum degree 3.

Graphs with $n$ vertices and $2n - 1$ edges must contain proper subgraphs of minimum degree 3 but this fails for graphs with $n$ vertices and $2n - 2$ edges, for example the wheel is such a graph (Erdős, Faudree, Rousseau and Schelp, 1990).

Let $G(n)$ be the family of graphs with $n$ vertices, $2n - 2$ edges and without proper subgraphs of minimum degree 3.

**Conjecture**

1. (Erdős, Faudree, Gy., Schelp, 1988) Every $G \in G(n)$ contains cycles of length $i$ for every integer $3 \leq i \leq k$ where $k$ tends to infinity with $n$. 
2. Large chordal subgraphs.
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- Any graph with \( n \) vertices and at least \( \frac{n^2}{3} \) edges contains a chordal subgraph with at least \( 2n - 3 \) edges. The complete tripartite graph shows that this is sharp.

**Conjecture**

2. (Erdős, Gy., Ordman, Zalcstein 1989) Any graph with \( n \) vertices and more than \( \frac{n^2}{3} \) edges contains a chordal subgraph with at least \( \frac{8n}{3} - 4 \) edges. The complete tripartite graph with one additional edge shows that this would be sharp.

We could prove a weaker result, that graphs with \( n \) vertices and more than \( \frac{n^2}{3} \) edges contain chordal subgraphs with at least \( \frac{7n}{3} - 6 \) edges.
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- However, there are 3-colorings such that no color can dominate more than $2n/3$ vertices of $K_n$ with any fixed number of vertices.
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**Problem**

3. (Erdős, Faudree, Gould, Gy., Rousseau, Schelp, 1990) If the edges of $K_n$ are 3-colored then in one of the colors at most 3 vertices dominate at least $2n/3$ vertices of $K_n$. Recently Kral, Liu, Sereni, Whalen and Yilma applied flag algebra to prove that in one of the colors 4 vertices dominate at least $2n/3$ vertices of $K_n$. 
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Conjecture

4. (Erdős, Gy., Pyber, 1991) The cycle partition number of any $r$-colored complete graph is at most $r$.

The case $r = 2$ in Conjecture is due to J. Lehel and was proved for large enough complete graphs by Łuczak, Rödl and Szemerédi and Allen. Then Bessy and Thomassé proved it for all complete graphs. Although Conjecture 4 for $r = 3$ is asymptotically true, Pokrovskiy found a counterexample in which three monochromatic cycles cannot cover all vertices.
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**Problem**

5. (Chen - Erdős, Gy. - Schelp, 1997) „We know that one has to be careful with conjectures in this area. That is why we only suspect that 4-critical $B + 3$-graphs on $n$ vertices must have at least $2n$ edges asymptotically and dare to conjecture only that they have significantly more than $5n/3$ edges.”
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Paul often invited us to dinner at the Merry Monks where we became regulars, one waiter have always greeted him asking „how are you and how are the prime numbers?”
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Time to time he exclaimed: „It is very annoying that we do not see this!” Sometimes we were more successful: „This is enough for a paper, don’t you think so?”
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**Problem**

6. (Erdős, Gy., Ruszinkó, 1998) Is there a positive $\epsilon$ such that $h_4(G) \leq (1 - \epsilon)n$ for every connected triangle-free graph $G$ with $n$ vertices?
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- They defined $H(n)$ as the smallest integer for which there exists a set mapping on $S$ with $|S| = n$ such that

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\bigcup_{X \subseteq T} f(X) = S
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for every $T \subset S, |T| \geq H(n)$ and they proved that $\log_2 n < H(n)$ and conjectured that $H(n) - \log_2(n)$ tends to infinity with $n$. 

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7. (Erdős, Gy., 1999) Show that $H(n) > k + 1$ for $n = 2^k$. 
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**Conjecture**

8. (Gy., 1997) If each path of a graph induces an at most 3-chromatic subgraph then for some constant $c$ the graph is $c$-colorable, perhaps with $c = 4$. 
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**Conjecture**

9. (Erdős, Gy., 1999) In every $r$-coloring of the edges of $K_{r^2+1}$ there exist $r + 1$ vertices with at least one missing color among them ($r \geq 3$, true for $r = 3, 4$).

Erdős and Hajnal asked if every subset $S$ of vertices in a graph $G$ contains an independent set of size at least $\left\lfloor \frac{|S|}{2} \right\rfloor - k$ then can one remove $f(k)$ vertices from $G$ so that the remaining graph is bipartite? This is settled in the affirmative by Bruce Reed.

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**Conjecture**

10. (Gy. 1997) A nearly bipartite graph can be made bipartite by deleting at most 5 vertices.

Molloy and Reed believed they can prove Conjecture 10 but so far they did not work out the details...
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**Problem**

11. (Erdős, Gy., 1995) Is it possible to cover the vertex set of a 2-colored $K_n$ with at most $\sqrt{n}$ monochromatic paths of the same color? This would be best possible.
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- Let \(f(n, p, q)\) denote the minimum number of colors needed for a \((p, q)\)-coloring of \(K_n\). Some of the general bounds of this paper have been improved by Sárközy and Selkow. There are much improvements on interesting small cases as well.
12. \((p, q)\)-coloring of complete graphs.

- It is sad to look at the submission date on our paper: September 15, 1996 - one day before Paul died in Warsaw.

- We called an edge coloring of a complete graph \(K_n\) a \((p, q)\)-coloring if every \(K_p \subset K_n\) spans at least \(q\) colors. Thus a \((p, 2)\)-coloring means that there is no monochromatic \(K_p\), a \((3, 3)\)-coloring means that the coloring is proper.

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- On \(f(n, 4, 3)\) the best upper bound is \(n^{o(1)}\) given in Mubayi. The lower bound of Kostochka and Mubayi is improved to \(c \log n\) by Fox and Sudakov. Lower bounds of Erdős-Gy. and upper bounds of Mubayi imply that \(f(n, 4, 4) = n^{1/2+o(1)}\).
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12. (Erdős, Gy., 1997) \(\frac{5(n-1)}{6} \leq f(n, 4, 5) \leq n\) - improve the estimates!

One of my favorite problems is to decide whether \(f(n, 5, 9)\) is linear which is equivalent to the following:
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13. (Erdős, Gy., 1997) Is there a constant \(c\) such that \(K_n\) has a proper edge coloring with \(cn\) colors, such that the union of any two color classes has no path or cycle with four edges?
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„It is not important who asked the problem, the important thing is that the problem is solved” - he said and I heartily agreed...
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Nevertheless, his results, conjectures, proofs and passionate love of mathematics are with us and carry over to the present and forthcoming generations of mathematicians.
...and a picture