The sum-free set constant is $\frac{1}{3}$

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Sum-free sets

A set $A \subset \mathbb{N}$ is *sum-free* if there do not exist $x, y, z \in A$ with $x + y = z$.

Theorem (Erdős, 1965)

Let A be a set of n positive integers. Then A contains a sum-free subset of size at least $n/3$.

Proof. For any $\theta \in \mathbb{R}/\mathbb{Z}$, the set $A'_\theta := \{a \in A : \theta a (\bmod 1) \subset [\frac{1}{3}, \frac{2}{3})\}$ is sum-free.

But if a is fixed and θ is selected uniformly at random then θa is uniformly distributed modulo 1, so the probability that $\theta a (\bmod 1) \in [\frac{1}{3}, \frac{2}{3})$ is $\frac{1}{3}$.

Hence the expected size of A'_θ is $n/3$.

IMPROVING $n/3$.

Let A be a set of n natural numbers.

Erdős (1965): A contains a sum-free subset of size $\geqslant n/3$.

Alon and Kleitman (1990): A contains a sum-free subset of size $\geqslant (n+1)/3$.

Bourgain (1997): A contains a sum-free subset of size $\geqslant (n+2)/3$.

Open question: Does A contain a sum-free subset of size $\geqslant (n+1000)/3$, for large enough n?

I believe the answer is yes.

The Inverse Littlewood Problem

Let A be a set of n natural numbers.
Bourgain (1997): A contains a sum-free subset of size at least

$$\frac{n}{3} + \frac{c\|A\|}{\log n},$$

where

$$\|A\| := \int_0^1 |\sum_{a\in A} e^{2\pi i a\theta}| d\theta.$$

Theorem (Konyagin, McGehee-Pigno-Smith 1981)

$\|A\| \gg \log n$ *for all sets* A *of* n *integers.*

Open question: Suppose that $A \subseteq \mathbb{Z}$ is a set of size n and that $\|A\| \leqslant K \log n$. What structure does A have? Is A a union of $O_K(\log n)$ progressions*?

$\frac{1}{3}$ IS BEST POSSIBLE

Theorem (Eberhard-G.-Manners, 2013)

Let $\epsilon > 0$ be arbitrary. Then there is a set A of n natural numbers which does not contain a sum-free set of size greater than $(\frac{1}{3} + \epsilon)n$.

Previously: $\frac{11}{28}n$ (Erdős, Lewko). Note $\frac{11}{28} \approx 0.393$.

$(\frac{11}{28} - 10^{-50000})n$ (Alon, unpublished).

The largest sum-free subset of $A = \{1, \dots, n\}$ has size essentially $n/2$: take $\{1, 3, 5, \dots\}$ or $\{a : n/2 < a \leqslant n\}$.

Ways of making large sum-free subsets

Let A be a set of n integers.

- (mod Q) constructions. The sets $\{a \in A : a \equiv 1 \pmod 2\}$ and $\{a \in A : a \equiv 2, 3 \pmod 5\}$ are always sum-free.
- $\mathbb{R}$-constructions. The set $A \cap (X, 2X]$ is sum-free for any X.

When $A = \{1, \dots, n\}$:

$|\{a \in A : a \equiv 1 \pmod 2\}| \sim n/2;$

$|\{a \in A : a \equiv 2, 3 \pmod 5\}| \sim 2n/5;$

$|\{a \in A : a \equiv 3, 4, 5 \pmod 8\}| \sim 3n/8;$

$|\{a \in A : n/2 < a \leqslant n\}| \sim n/2.$

We must defeat all these "local" methods of construction.

The local problem

Definition

Let $Q \in \mathbb{N}$. Say that a probability measure ν on $\mathbb{Z}/Q\mathbb{Z} \times [0,1]$ is δ-good if whenever $S \subset \mathbb{Z}/Q\mathbb{Z} \times [0,1]$ is open and sum-free then $\nu(S) \leqslant \frac{1}{3} + \delta$.

Proposition

There is a δ-good probability measure on $\mathbb{Z}/Q\mathbb{Z} \times [0,1]$ for every $\delta > 0$. Furthermore $\nu(S) = \int_S w(x,y)dxdy$ for some $O_\delta(1)$-Lipschitz weight function $w : \mathbb{Z}/Q\mathbb{Z} \times [0,1] \to (0,\infty)$.

Local implies global: Let N be large, and choose a set $A \subset \{1,\dots,N\}$ at random by selecting a to lie in A with probability proportional to $w(a (\mathrm{mod}\, Q), a/N)$, where $Q = Q(\epsilon)$ is some highly composite number and w is associated to some $\epsilon/2$-good measure.

The details of checking that such an A almost surely works are not trivial.

Solving the local problem for $[0,1]$

Say that a probability measure ν on $[0,1]$ is δ-good if whenever $S \subset [0,1]$ is open and sum-free then $\nu(S) \leqslant \frac{1}{3} + \delta$.

Crucial idea: open sum-free subsets of $[0,1]$ with uniform measure $> \frac{1}{3}$ are "repelled from zero" and so we should choose ν to be concentrated near zero.

Proposition (Repulsion from 0)

Suppose that $S \subset [0,1]$ is open and sum-free and that $\mu(S) \geqslant \frac{1}{3} + \epsilon$. Then $S \cap [0, \epsilon'] = \emptyset$ for some $\epsilon' \gg_\epsilon 1$.

Easy observation: The uniform measure μ is $\frac{1}{6}$-good.

Constructing good measures on $[0, 1]$

Let ν be a δ-good measure and set

$$\nu' = \tfrac{3}{4}\pi_*\nu + \tfrac{1}{4}\mu$$

where $\pi : [0,1] \rightarrow [0,\epsilon']$ is the contraction map. $(\pi_*\nu(S) := \nu(\pi^{-1}(S)))$.

We claim ν' is δ'-good, where $\delta' = \frac{3}{4}\delta + \frac{1}{4}\epsilon$. Let $S \subset [0,1]$ be open and sum-free: we must show that $\nu'(S) \leqslant \frac{1}{3} + \delta'$.

Case 1. $\mu(S) \geqslant \frac{1}{3} + \epsilon$. Then S is repelled from 0, i.e. is disjoint from $[0,\epsilon']$. So $\pi_*\nu(S) = 0$, and $\nu'(S) = \frac{1}{4}\mu(S) < \frac{1}{3}$.

Case 2. $\mu(S) < \frac{1}{3} + \epsilon$. Then, since $\pi^{-1}(S)$ is sum-free, we have $\pi_*\nu(S) \leqslant \frac{1}{3} + \delta$. Thus

$$\nu'(S) \leqslant \tfrac{3}{4}(\tfrac{1}{3} + \delta) + \tfrac{1}{4}(\tfrac{1}{3} + \epsilon) = \tfrac{1}{3} + \delta',$$

as required.

Iterating, we can take δ as close to ϵ as we like.

Proposition (Repulsion from 0)

Suppose that $S \subset [0,1]$ is open, sum-free and $\mu(S) \geqslant \frac{1}{3} + \epsilon$. Then $S \cap [0, \epsilon'] = \emptyset$ for some $\epsilon' \gg_\epsilon 1$.

S is sum-free implies $(S - S) \cap S = (S - S) \cap (-S) = \emptyset$.
($S - S := \{s_1 - s_2 : s_1, s_2 \in S\}$.) Note $S - S \subset [-1, 1]$. Thus

$$\mu(S - S) \leqslant 2 - 2\mu(S) \leqslant \frac{4}{3} - 2\epsilon < (4 - \epsilon)\mu(S).$$

Sets of doubling less than 4

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Sets of doubling less than 4

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Theorem (Sets of doubling less than 4)

Suppose that $S \subset [0,1]$ is open, $\mu(S) \geqslant \epsilon$ and $\mu(S - S) \leqslant (4 - \epsilon)\mu(S)$. Then $S - S$ contains $[0, \epsilon']$.

Sets of doubling less than 4

Proposition (Repulsion from 0)

Suppose that $S \subset [0,1]$ is open, sum-free and $\mu(S) \geqslant \frac{1}{3} + \epsilon$. Then $S \cap [0, \epsilon'] = \emptyset$ for some $\epsilon' \gg_\epsilon 1$.

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$$\mu(S - S) \leqslant 2 - 2\mu(S) \leqslant \frac{4}{3} - 2\epsilon < (4 - \epsilon)\mu(S).$$

Theorem (Sets of doubling less than 4)

Suppose that $S \subset [0,1]$ is open, $\mu(S) \geqslant \epsilon$ and $\mu(S - S) \leqslant (4 - \epsilon)\mu(S)$. Then S has density $> \frac{1}{2}$ on some interval of length at least ϵ'.

Structure theorem for arbitrary open sets $S \subset [0,1]$.

Rough definition: a set $B \subset [0,1]$ is *of Bohr type* if there is a homomorphism $\pi : \mathbb{R} \to (\mathbb{R}/\mathbb{Z})^d$,

$$\pi(t) = (X_1 t, \dots, X_d t) (\mathrm{mod}\, 1)$$

with the X_i large and highly independent over $\mathbb{Q}$, and an M such that

$$B = \pi^{-1}(\Sigma_i) \quad \text{on} \quad [\tfrac{i}{M}, \tfrac{i+1}{M}), \quad i = 0, 1, \dots, M-1,$$

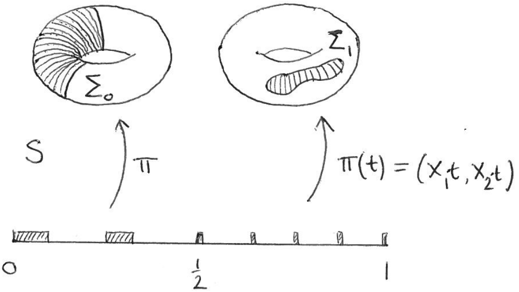
where $\Sigma_i \subset (\mathbb{R}/\mathbb{Z})^d$ is open.

An arbitrary open set $S \subset [0,1]$ is, up to set of measure $< \epsilon$, extremely well-approximated by sets of Bohr type with d, M and the complexity of each open set Σ_i being $O_\epsilon(1)$.

Sets of doubling less than 4

Suppose that $S \subset [0,1]$ is open, $\mu(S) \geqslant \epsilon$ and $\mu(S-S) \leqslant (4-\epsilon)\mu(S)$. Then S has density $> \frac{1}{2}$ on some interval of length at least $\epsilon' \gg_{\epsilon} 1$.

Applying the regularity lemma, we may assume that S is of Bohr type.

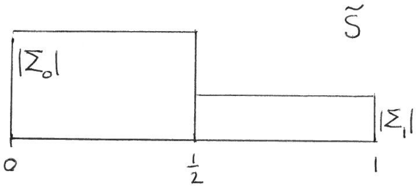


Density of S on $[\frac{i}{M}, \frac{i+1}{M})$ is $\approx |\Sigma_i|$. $M = 2$ in the picture.

If $|\Sigma_i| > \frac{1}{2}$ for any i then the theorem holds (with $\epsilon' = 1/M$).

Theorem (Macbeath, Math. Proc. Camb. Phil. Soc. 1953)

Suppose that Σ_i, Σ_j are open subsets of a torus $(\mathbb{R}/\mathbb{Z})^d$ with $|\Sigma_i|, |\Sigma_j| \leqslant \frac{1}{2}$. Then $|\Sigma_i - \Sigma_j| \geqslant |\Sigma_i| + |\Sigma_j|$.



$\tilde{S} \subset \mathbb{R}^2$.
$\tilde{S}$ is a compression of S.
$\mu(S) = \mu_{\mathbb{R}^2}(\tilde{S})$ and, by Macbeath's Theorem, $\mu(S - S) \geqslant \mu_{\mathbb{R}^2}(\tilde{S} - \tilde{S})$.

Theorem (Brunn-Minkowski)

Let $X, Y \subset \mathbb{R}^D$. Then $\mu_{\mathbb{R}^D}(X + Y)^{1/D} \geqslant \mu_{\mathbb{R}^D}(X)^{1/D} + \mu_{\mathbb{R}^D}(Y)^{1/D}$.

With $X = \tilde{S}$, $Y = -\tilde{S}$ we get $\mu_{\mathbb{R}^2}(\tilde{S} - \tilde{S}) \geqslant 4\mu_{\mathbb{R}^2}(\tilde{S})$. Thus $\mu(S - S) \geqslant 4\mu(S)$, contrary to assumption.

Open problems

Problem

We showed that if $n > n_0(\epsilon)$ then there is a set of positive integers of size n with no sum-free subset of size $(\frac{1}{3} + \epsilon)n$. Find a reasonable dependence of $n_0(\epsilon)$ on n.

Problem

Do sets like $A := \bigcup_{j=1}^{J} \{j!, 2j!, \dots, Nj!\}$ have sum-free subsets of density much more than $\frac{1}{3}$?

Problem

Suppose that $A \subseteq [0,1]$ is open. If $\mu(A) > \frac{1}{3}$, is it true that A has a solution to $xy = z$? Is the measure $\nu_N(S) := \int_S e^{-Nt} dt / \int_0^1 e^{-Nt} dt$ on $[0,1]$ δ-good for sufficiently large N?

Some more open problems

Problem

If G is a group, what is the largest product-free subset $A \subset G$?

$G = \mathrm{Alt}(n)$. Edward Crane's example: A consists of all even permutations π of $\{1, \dots, n\}$ for which $\pi(1) \in \{2, \dots, m\}$ and $\pi(2), \dots, \pi(m) \in \{m + 1, \dots, n\}$ with $m \sim \sqrt{n/2}$ optimised to make $|A| \sim (2en)^{-1/2}|G|$ as big as possible. Is this optimal for large n?

Attack using representation theory and Gowers notion of quasirandomness (with Ellis, Menzies).

Kedlaya (1997) proved that every group G has a product-free subset of size at least $c|G|^{11/14}$. Can the constant $\frac{11}{14}$ be improved?

Just one further open problem

Question of Erdős and Moser (1965):

Problem (Sum-avoiding sets)

Let A be a set of n positive integers. What is the size of the largest $A' \subset A$ with no solutions to $x + y = z$ with $x, y \in A'$, $z \in A$?

Sudakov, Szemerédi, Vu (2005): at least $\log n (\log\log\log\log\log n)^{1-o(1)}$.

Jehanne Dousse (2012): at least $\log n (\log\log\log n)^c$.

Ruzsa (2005): need not be more than $e^{C\sqrt{\log n}}$.