§1. Introduction

Many important linear operators $P : X \to U$ of a linear space $X$ of functions onto a linear subspace $U$ of $X$ are defined by the minimum problem

$$\|f - Pf\| = \min\{\|f - u\| : u \in U\}, \quad f \in X,$$

where the semi-norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ on $X$ is induced by some semi-definite inner product $\langle \cdot, \cdot \rangle$. Such operators $P$ are linear projectors, i.e. they satisfy $Pu = u$, for all $u \in U$.

In my talk, $X = C[a, b]$ and $\| \cdot \|_\infty$ is the uniform norm on $[a, b]$. I am interested in the $L_\infty$-norm of the projectors $P$,

$$\|P\|_\infty := \sup\{\|Pf\|_\infty : f \in C[a, b], \quad \|f\|_\infty = 1\}.$$

The Lebesgue function $\Lambda_P \in C[a, b]$,

$$\Lambda_P(x) := \sup\{|Pf(x)| : f \in C[a, b], \quad \|f\|_\infty = 1\}, \quad x \in [a, b],$$

will play an essential role. It provides the local error estimate

$$|f(x) - Pf(x)| \leq (1 + \Lambda_P(x)) \text{dist}(f, U)_\infty,$$

for all $f \in C[a, b]$ and all $x \in [a, b]$.

Moreover,

$$\|P\|_\infty = \max_{x \in [a, b]} \Lambda_P(x)$$

and

$$\|f - Pf\|_\infty \leq (1 + \|P\|_\infty) \text{dist}(f, U)_\infty,$$
§2. Examples

Example 1.

Let $[a, b] = [-\pi, \pi]$, $X = \{f \in C[-\pi, \pi], \ f(-\pi) = f(\pi)\}$,

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(t)g(t)dt, \ f, g \in X.$$ 

Let $U = \mathcal{T}_n$ be the trigonometric polynomials of degree $\leq n$,
$Pf = s_n(f)$ is the $n$-th partial sum of the Fourier series of $f$.

The Lebesgue function is the constant function

$$\Lambda_{s_n}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((n+1/2)t)}{\sin(t/2)} \right| dt, \ \text{for all } x \in [-\pi, \pi],$$

and one has (L. Fejér)

$$\|s_n\|_\infty = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((n+1/2)t)}{\sin(t/2)} \right| dt = \frac{4}{\pi^2} \log(n+1) + O(1).$$

Example 2.

Let $[a, b] = [-1, 1]$ and $X = C[-1, 1]$.

Let $n \in \mathbb{N}$, $U = \Pi_n$ be the algebraic polynomials of degree $\leq n$.

For fixed $n + 1$ points $\Delta : -1 \leq t_0 < t_1 < \ldots < t_n \leq 1$,

let $\langle \cdot, \cdot \rangle$ be defined by $\langle f, g \rangle := \sum_{j=0}^n f(t_j)g(t_j)$.

The corresponding linear projector $P : C[-1, 1] \to \Pi_n$ is the Lagrange interpolation operator. Its Lebesgue function

$$\Lambda_P(x) = \sum_{j=0}^n \left| \prod_{k=0, k\neq j}^n \frac{x-t_k}{t_j-t_k} \right|, \ x \in [-1, 1],$$

has been studied for various point sets $\Delta$ by many authors. Paul Erdős has contributed essentially to this important topic, partly in cooperation with his Hungarian colleagues Paul Turan, Jozsef Szabados, Peter Vertesi, Andras Kroo.
Example 3.

Let $X = C[a, b]$,

$$\langle f, g \rangle = \int_a^b f(t)g(t)dt, \quad \|f\|_2 : \sqrt{\langle f, f \rangle}, \quad f, g \in X.$$  

Let $m \in \mathbb{N}$ and

$$\Delta : a = t_0 < t_1 < \cdots < t_n = b$$

be fixed. Let

$$S_m(\Delta) := \{S \in C^{m-1}[a, b] : S_{[t_i, t_{i+1}]} \in \Pi_m, \ 0 \leq i \leq n-1\}.$$  

The $L_2$-spline projector $P = P_m(\Delta) : C[a, b] \to S_m(\Delta)$ is defined by

$$\|f - Pf\|_2 = \min\{\|f - S\|_2 : S \in S_m(\Delta)\}, \quad f \in C[a, b].$$

We want to study the Lebesgue function $\Lambda_P$ and thus bounds for the $L_\infty$-norm. Moreover,

$$\|P\|_\infty = \max_{x \in [a, b]} \Lambda_P(x)$$

of the $L_2$-spline projector $P = P_m(\Delta)$.

It is obvious that $\|P_1(\Delta)\|_\infty \leq K_1$ for some constant $K_1$ independent of $\Delta$.

Carl de Boor proved in 1968 that $\|P_2(\Delta)\|_\infty \leq K_2$ for some constant $K_2$ independent of $\Delta$ and formulated his famous conjecture in 1973 that

$$\|P_m(\Delta)\|_\infty \leq K_m$$  \hspace{1cm} (1)

holds for all $m$ where the constant $K_m$ depend only on $m$. He proved his conjecture for $m = 3$ in 1979.

Finally, in 2001, S. Shadrin proved de Boor’s conjecture for all $m \in \mathbb{N}$.

A. Shadrin

“The $L_\infty$-norm of the $L_2$-spline projector $P_m(\Delta)$

is bounded independently of the knot sequence:

a proof of de Boor’s conjecture”,

§3. An alternative proof of Shadrin’s theorem

My new proof consists of two steps: $P = P_m(\Delta)$.

1. Prove that the Lebesgue function $\Lambda_P$ is bounded at the endpoint $a$ by a constant $C_m$ depending only on $m$.

2. Prove that there exists $C_m^*$ depending only on $m$ such that

$$\max_{x \in [a,b]} \Lambda_P(x) \leq C_m^* \Lambda_P(a)$$

so that $\|P\|_\infty \leq C_m^* C_m$.

It is not difficult to derive Step 2.

I will focus on the proof in Step 1, which was much harder to find, but is easier to explain.

Let $B = \{N_k\}_{k=1}^{n+m}$ be the B-spline basis of $S_m(\Delta)$ for the extended knot sequence

$$\Delta_e : t_m = \cdots = t_0 < t_1 < \cdots < t_n = \cdots = t_{n+m}.$$

The $L_1$-normalized basis $\{M_k\}_{k=1}^{n+m}$ of $S_m(\Delta)$ is defined by

$$M_k := \frac{(m+1)N_k}{t_k - t_{k-m-1}}, \quad k = 1, \ldots, n+m.$$

They have the properties (de Boor, 1976)

$$\text{supp}(N_k) = \text{supp}(M_k) = [t_{k-m-1}, t_k],$$

$$N_k \geq 0, \quad M_k \geq 0,$$

$$\sum_{k=1}^{n+m} N_k = 1, \quad \int_a^b M_k(t) \,dt = 1.$$

In Shadrin’s proof, a spline $\phi \in S_m(\Delta)$ plays the main role. It has the following properties:

**Theorem $\Phi$ (Shadrin [2001]).** There exist positive numbers $c_{min}$ and $c_{max}$ depending only on $m$ such that

$$\phi(a) = m!$$

$$\text{sign}(\langle \phi, M_j \rangle) = (-1)^{j+1}, \quad j = 1, 2, \ldots, n + m \quad (A_1)$$

$$|\langle \phi, M_j \rangle| \geq c_{min}, \quad j = 1, 2, \ldots, n + m \quad (A_2)$$

$$\|\phi\|_\infty \leq c_{max} \quad (A_3)$$

The proof of (A1) and (A2) is short and not difficult, while the proof of (A3) is very difficult. Fortunately, I need only (A1) and (A2) in my proof of de Boor’s conjecture.
Proof of Step 1.

**Definition 3.1.** We define the spline \( Q_a \in S_m(\Delta) \) by
\[
v(a) = \langle v, Q_a \rangle, \quad \text{for all } v \in S_m(\Delta)
\]

The importance of \( Q_a \) follows from

**Lemma 3.2.** For \( P = P_m(\Delta) \),
\[
\Lambda_P(a) = \|Q_a\|_1 \tag{3.1}
\]

**Proof:** Recall that
\[
\Lambda_P(a) := \sup\{ |Pf(a)| : f \in C[a,b], \|f\|_\infty = 1 \}.
\]

Let \( f \in C[a,b], \|f\|_\infty = 1 \).
Definition 3.1 and the orthogonality relations imply
\[
P f(a) = \langle Pf, Q_a \rangle = \langle f, Q_a \rangle.
\]
Taking the supremum for \( f \in C[a,b], \|f\|_\infty = 1 \), we obtain (3.1) for \( f = \text{sign}(Q_a) \). \( \square \)

**Lemma 3.3.** Let
\[
Q_a = \sum_{k=1}^{n+m} c_k M_k,
\]
then
\[
c_{k+1}c_k < 0, \quad k = 1, \ldots, n + m - 1
\]

Now we complete the proof of **Step 1** as follows:

Using Definition 3.1 for \( v = \phi \), Lemma 3.3, (A1) and (A2), we obtain
\[
m! = \phi(a) = \langle \phi, Q_a \rangle = \sum_{k=1}^{n+m} c_k \langle \phi, M_k \rangle
\]
\[
= \sum_{k=1}^{n+m} |c_k| \|\phi, M_k \| \geq c_{\min} \sum_{k=1}^{n+m} |c_k|
\]
so that
\[
\|Q_a\|_1 = \left\| \sum_{k=1}^{n+m} c_k M_k \right\|_1 \leq \sum_{k=1}^{n+m} |c_k| \leq \frac{m!}{c_{\min}} =: C_m
\]
and thus by Lemma 3.2, \( \Lambda_P(a) = \|Q_a\|_1 \leq C_m \) \( \square \)
4. Local lower bounds

Recall that for $\Delta : a = t_0 < t_1 < \cdots < t_n = b$
and the $L_2$-spline projector $P := P_m(\Delta) : C[a,b] \to S_m(\Delta)$,
the Lebesgue function $\Lambda_P \in C[a,b]$ is defined by

$$\Lambda_P(x) := \sup\{|Pf(x)| : f \in C[a,b], \|f\|_{\infty} = 1\}, \ x \in [a,b].$$

It satisfies (by Shadrin’s theorem)

$$\|P\|_{\infty} = \max_{x \in [a,b]} \Lambda_P(x) \leq K_m$$

for some constant $K_m$ depending only on $m$.

Shadrin [2001] and later also his student Simon Foucart [JAT 140, 2006] proved that at the endpoint $a$

$$\sup_{\Delta} \Lambda_{P_m(\Delta)}(a) \geq 2m + 1$$

which implies that $K_m \geq 2m + 1$.

Shadrin [2001] conjectures that even

$$K_m = \sup_{\Delta} \Lambda_{P_m(\Delta)}(a) = 2m + 1$$

is true.

Recently I obtained further results which seem to support Shadrin’s conjecture:

**Definition.** Let $1 \leq \rho < \infty$. We say that a knot sequence $\Delta : a = t_0 < t_1 < \cdots < t_n = b$ belongs to the set $\Omega_{\rho,n}$ of knot sequences if

$$\min_{0 \leq i \leq n-2} \frac{t_{i+2} - t_{i+1}}{t_{i+1} - t_i} = \rho.$$ 

**Lemma 4.1.** There exists a constant $\gamma_m > 0$ depending only on $m$ with the following property: If $\Delta \in \Omega_{\rho,n}$, then

$$|\Lambda_{P_m(\Delta)}(a) - 2m - 1| \leq \gamma_m \left( \frac{n}{\rho} + \left( \frac{m}{m+1} \right)^n \right).$$

**Corollary 4.2.** Let $(\rho_n)_{n=1}^{\infty}$ satisfy

$$\lim_{n \to \infty} \frac{n}{\rho_n} = 0.$$ 

Let $(\Delta_n)_{n=1}^{\infty}$ satisfy $\Delta_n \in \Omega_{\rho_n,n}$ for all $n \in \mathbb{N}$, then

$$\lim_{n \to \infty} \Lambda_{P_m(\Delta_n)}(a) = 2m + 1.$$ 

See also K. Höllig [JAT, 1981] for ”geometric knot sequences” $\Delta_n$. 

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I conclude my talk with the following

**Theorem 4.3.** For any $x \in [a, b]$, there exists a sequence $(\Delta_n)_{n=1}^{\infty}$ (depending on $x$) such that

$$\lim_{n \to \infty} \Lambda_{P_m}(\Delta_n)(x) = 2m + 1.$$