COMBINATORICS SECTION

ERDŐS CENTENNIAL MEETING
BUDAPEST, HUNGARY

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Saturation Numbers for Graphs

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Definition

Given a fixed graph $H$, a graph $G$ is $H$-Saturated if it contains no copy of $H$, but $G + e$ contains a copy of $H$ for any edge $e \not\in G$. 
**Definition**

\[ ex(n, F) = \max\{|E(G)| : |V(G)| = n \text{ and } G \text{ is } F\text{-saturated}\}. \]
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\[ \text{Ex}(n, F) := \{G : |V(G)| = n, |E(G)| = \text{ex}(n, F), \text{ and } G \text{ is } F\text{-saturated}\} . \]
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\[ \text{sat}(n, F) = \min \{|E(G)| : |V(G)| = n \text{ and } G \text{ is } F\text{-saturated} \}. \]
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\[ \text{sat}(n, F) = \min \{|E(G)| : |V(G)| = n \text{ and } G \text{ is } F\text{-saturated}\}. \]

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\[ \text{Sat}(n, F) = \{ G : |V(G)| = n, |E(G)| = \text{sat}(n, F), \text{ and } G \text{ is } F\text{-saturated}\}. \]
**Weakly Saturated Graphs**

**Definition**

\[ wsat(n, F) = \min \{|E(G)| : |V(G)| = n, G \text{ does not have } F \text{ as a subgraph, but edges in } G \text{ can be ordered such that the addition of each edge results in a new copy of } F \} \]
**Definition**

$wsat(n, F) = \min \{|E(G)| : |V(G)| = n, G \text{ does not have } F \text{ as a subgraph, but edges in } \overline{G} \text{ can be ordered such that the addition of each edge results in a new copy of } F\}$.

**Definition**

$WSat(n, F) = \{G : |V(G)| = n, |E(G)| = wsat(n, F), \text{ and } G \text{ is } F\text{-weakly saturated}\}$.
Example

The complete bipartite graph $K_{n/2, n/2}$ is a $K_3$-saturated of order $n$ that has $n^2/4$ edges.
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Example

The star $K_{1, n-1}$ is a $K_3$-saturated of order $n$ that has $n - 1$ edges.
COMPLETE BIPARTITE – STAR GRAPHS

$K_{n/2, n/2}$
COMPLETE BIPARTITE – STAR GRAPHS

\[ K_{n/2, n/2} \]

\[ K_{1, n-1} \]
\textit{wsat}(n, K_3)
$\text{wsat}(n, K_3)$

$P_n$

$K_{1,n-1}$
Theorem

\[ \text{ex}(n, P_3) = \lfloor n/2 \rfloor. \]
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Theorem

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\[ \text{sat}(n, P_3) = \lfloor n/2 \rfloor. \]

Theorem

\[ \text{wsat}(n, P_3) = 1. \]
Theorem

For $t \geq 2$,

$$\text{ex}(n, tP_2) = (t - 1)n - t(t - 1)/2.$$
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For \( t \geq 2 \),

\[
\text{ex}(n, tP_2) = (t - 1)n - t(t - 1)/2.
\]

Example
The extremal graph is \( K_{t-1} + \overline{K}_{n-t+1} \).
Theorem

For $t \geq 2$,

$$sat(n, tP_2) = 3t - 3.$$
**Theorem**

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sat(n, tP_2) = 3t - 3.
\]

**Example**

*The extremal graph is \((t - 1)K_3 \cup \overline{K}_{n-3t+3}.*
SATURATION NUMBERS FOR MATCHINGS

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Saturation Numbers for Graphs

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Saturation Numbers for Matchings

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Theorem

For \( t \geq 2 \),

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wsat(n, tP_2) = t - 1.
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\[(t - 1)K_2 \cup \overline{K}_{n-2t+2}\]
Theorem (Turán(1954)) If $n > t$ and divisible by $t - 1$, then

$$ex(n, K_t) = \frac{(t - 2)n^2}{2(t - 1)}.$$
**Theorem**  
*(Turán(1954))* If \( n > t \) and divisible by \( t - 1 \), then

\[
\text{ex}(n, K_t) = \frac{(t - 2)n^2}{2(t - 1)}.
\]

**Theorem**  
*(Erdős, Hajnal, Moon(1964))* For \( n \geq t \)

\[
\text{sat}(n, K_t) = (t - 2)(n - 1) - \binom{t - 2}{2}.
\]
Theorem (Turán (1954)) If $n > t$ and divisible by $t - 1$, then

$$\text{ex}(n, K_t) = \frac{(t - 2)n^2}{2(t - 1)}.$$ 

Theorem (Erdős, Hajnal, Moon (1964)) For $n \geq t$

$$\text{sat}(n, K_t) = (t - 2)(n - 1) - \binom{t - 2}{2}.$$ 

Theorem (Lovász (1977)) For $n \geq t$

$$\text{wsat}(n, K_t) = (t - 2)(n - 1) - \binom{t - 2}{2}.$$
Example

The extremal graph is $K_n - (t - 1)K_{n/(t-1)}$. 

\[ \begin{align*} 
& K_{n/(t-1)} \quad \cdots \quad K_{n/(t-1)} \\
& 1 \quad \cdots \quad t-1
\end{align*} \]
Example

The extremal graph is $K_n - (t - 1)K_{n/(t-1)}$.

Example

The minimal saturated graph is $K_{t-2} + \overline{K}_{n-t+2}$.
Theorem
(Erdős, Simonovits (1972)) If $n$ is sufficiently large and $F$ is a graph with chromatic number $\chi(F) = p$, then

$$\text{ex}(n, F) = \frac{(p - 2)n^2}{2(p - 1)} + o(n^2).$$
Theorem (Erdős, Simonovits (1972)) If $n$ is sufficiently large and $F$ is a graph with chromatic number $\chi(F) = p$, then

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Theorem (Erdős, Simonovits (1972)) For $n$ sufficiently and $F$ is a graph with chromatic number $\chi(F) = p$, then

$$K_n - (p - 1)\overline{K}_{n/(p-1)} \approx Ex(n, F).$$
Theorem
(Kászonyi, Tuza (1986)) For a given graph $F$ of order $t$ and independence number $\alpha = \alpha(F)$, let $d = d(F)$ be the minimum degree of any vertex of $F - S$ relative to a maximum independent set $S$. Then,

$$sat(n, F) \leq (t - \alpha - 1)n + \lfloor (d - 1)(n - t + \alpha + 1)/2 \rfloor - \binom{t - \alpha}{2},$$

and $K_{t-\alpha-1} + H_d$, contains a saturated graph, where $H_d$ is a $(d - 1)$-regular graph of order $n - t + \alpha + 1$. 

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Saturation Numbers for Graphs
Theorem
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$$\text{sat}(n, F) \leq (t - \alpha - 1)n + \lceil(d - 1)(n - t + \alpha + 1)/2 \rceil - \binom{t - \alpha}{2},$$

and $K_{t - \alpha - 1} + H_d$, contains a saturated graph, where $H_d$ is a $(d - 1)$-regular graph of order $n - t + \alpha + 1$.

Corollary
(Kászonyi, Tuza (1986)) For each graph $F$ there is a constant $c = c(F)$ such that

$$\text{sat}(n, F) < cn.$$
$H_d$ is a $(d - 1)$-regular graph of order $n - t + \alpha + 1$. 

$K_{t-\alpha-1}$

$H_d$
**Theorem**

Let $F$ be a graph with $p$ vertices, $q$ edges, and minimal degree $\delta$. Then, for any $n \geq p$

$$q - 1 + (\delta - 1)(n - p)/2 \leq \text{wsat}(n, F) \leq (p - 1)(p - 2)/2 + (\delta - 1)(n - p + 1).$$
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**Example**

The extremal graph is $K_{\delta - 1} + (K_{p - \delta} \cup \overline{K}_{n - p + 1})$. 

![Diagram of the extremal graph](image)
Theorem

Let $F$ be a graph with $p$ vertices, $q$ edges, and minimal degree $\delta$. Then, for any $n \geq p$

$$q - 1 + (\delta - 1)(n - p)/2 \leq wsat(n, F) \leq (p - 1)(p - 2)/2 + (\delta - 1)(n - p + 1).$$
Theorem
Let $F$ be a graph with $p$ vertices, $q$ edges, and minimal degree $\delta$. Then, for any $n \geq p$

\[ q-1+(\delta-1)(n-p)/2 \leq wsat(n, F) \leq (p-1)(p-2)/2+(\delta-1)(n-p+1). \]

Theorem
(F. Gould, Jacobson) If $F$ is a graph with $p$ vertices and minimal degree $\delta$, then,

\[ \frac{\delta n}{2} - \frac{n}{\delta + 1} \leq wsat(n, F) \leq (\delta - 1)n + (p - 1)(p - 2\delta)/2 \]

for any $n$ sufficiently large.
Theorem
(F. Gould, Jacobson) Let $F$ be a graph with $p$ vertices, $q$ edges, and minimal degree $\delta$. Then, for any $n \geq p$

$$\frac{(\delta(F) - 1)n}{2} + c_1 \leq w_{sat}(n, F) \leq (\delta(F) - 1)n + c_2.$$
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(F. Gould, Jacobson) Let $F$ be a graph with $p$ vertices, $q$ edges, and minimal degree $\delta$. Then, for any $n \geq p$

$\frac{(\delta(F) - 1)n}{2} + c_1 \leq wsat(n, F) \leq (\delta(F) - 1)n + c_2.$

Theorem
$wsat(n, F) = (\delta(F) - 1)n + c_2$ for $\delta(F) = 1$ or 2.
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(F. Gould, Jacobson) Let $F$ be a graph with $p$ vertices, $q$ edges, and minimal degree $\delta$. Then, for any $n \geq p$

$$\frac{(\delta(F) - 1)n}{2} + c_1 \leq \text{wsat}(n, F) \leq (\delta(F) - 1)n + c_2.$$ 

Theorem
$\text{wsat}(n, F) = (\delta(F) - 1)n + c_2$ for $\delta(F) = 1$ or 2.

Question
Is $\text{wsat}(n, F) = (\delta(F) - 1)n + c_2$ for $\delta(F) \geq 3$? NO
Let $\mathcal{F}$ be a family of graphs. Then, $ex(n, \mathcal{F})$ satisfies:
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Let $\mathcal{F}$ be a family of graphs. Then, $ex(n, \mathcal{F})$ satisfies:

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$sat(n, \mathcal{G})$ and $wsat(n, \mathcal{F})$ does not satisfy any of these properties.
Theorem
J. Faudree, R. Faudree, R. Gould, M. Jacobson Given any positive integer $C$, any tree $T$ is a subtree of a tree $T' = T'(T, C)$ such that for $n$ sufficiently large

$$sat(T', n) \geq Cn.$$ 

Any tree $T'$ is a subtree of a tree $T'' = T''(T', C)$ such that for $n$ sufficiently large

$$sat(T'', n) < n.$$
Theorem

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$$\text{sat}(T'', n) < n.$$ 

Theorem

There are sequences of trees $T(1) \subset T(2) \subset \cdots T(m)$ such that for any positive integer $C$ and $n$ sufficiently large

$$\text{sat}(T(i), n) < n, \text{ for } i \text{ odd and } \text{sat}(T(i), n) > Cn, \text{ for } i \text{ even.}$$
Theorem

J. Faudree, R. Faudree, R. Gould, M. Jacobson  
For $t \geq 2$ and $n \geq t + 1$,

\[ sat(n, K_{1,t} + e) = n - 1, \]

and  \( Sat(n, K_{1,t} + e) = \{K_{1,n-1}\}. \)
Theorem

J. Faudree, R. Faudree, R. Gould, M. Jacobson For $t \geq 2$ and $n \geq t + 1$,

$$sat(n, K_{1,t} + e) = n - 1,$$

and $Sat(n, K_{1,t} + e) = \{K_{1,n-1}\}$.

Theorem

For $t \geq 2$ and $n \geq t + 1$,

$$sat(n, \{K_{1,t} + e, K_{1,t}\}) = sat(n, K_{1,t}) = (t - 1)n/2 - \frac{1}{2}[t^2/4].$$
Theorem
(Kászonyi, Tuza (1986)) For \( t \geq 2 \),

\[
\text{sat}(2k - 1, P_4) = k + 1; \quad \text{Sat}(2k - 1, P_4) = K_3 \cup (k - 2)K_2.
\]

and

\[
\text{sat}(2k, P_4) = k; \quad \text{Sat}(2k, P_4) = kK_2.
\]
(a) $G_p$ graph obtained from $K_p$ by adding a pendant edge.
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\[ K_p \quad G_p \]

(b) $wsat(n, G_p) = \binom{p}{2}$ for all $n \geq p + 1$.

c) $wsat(n, K_p) = \binom{p-2}{2} + (p - 2)(n - p + 2)$ for $n \geq p$. 
(a) $S_p$ star with $p$ vertices, $S_p^*$ graph obtained from $S_p$ by adding an edge.
(a) $S_p$ star with $p$ vertices, $S^*_p$ graph obtained from $S_p$ by adding an edge.

(b) $\text{wsat}(n, S^*_p) = p - 1$.

(c) $\text{wsat}(n, \{S_p, S^*_p\}) = \binom{p-1}{2}$. 

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Saturation Numbers for Graphs
BAD BEHAVIOR FOR (3) - WEAK SATURATION

$H_1$  $H_2$  $H_3$  $H_4$
BAD BEHAVIOR FOR (3) - WEAK SATURATION

\[ wsat(6, H_1) = 7. \]

(b) \[ wsat(7, H_1) = 6 \]
BAD BEHAVIOR FOR (3) - WEAK SATURATION

(a) \( wsat(6, H_1) = 7 \).

(b) \( wsat(7, H_1) = 6 \)

(a) \( wsat(6, 2K_3) = 10 \) and \( H_2 \in WSat(6, K_3) \).

(b) \( wsat(7, 2K_3) = 8 \) and \( H_3 \in WSat(6, K_3) \).

(c) \( wsat(8, 2K_3) = 8 \) and \( H_4 \in WSat(6, K_3) \).
Theorem

Let $F$ be a graph with $p$ vertices, $q$ edges, and minimal degree $\delta$. Then,

$$\text{wsat}(n, F) \leq \text{wsat}(p, F) + (\delta - 1)(n - p)$$

for any $n \geq p$. 
**Theorem**

Let $F$ be a graph with $p$ vertices, $q$ edges, and minimal degree $\delta$. Then,

$$\text{wsat}(n, F) \leq \text{wsat}(p, F) + (\delta - 1)(n - p)$$

for any $n \geq p$.

Let $F'_p \in \text{WSat}(p, F)$.
**Theorem**

Let $F$ be a graph with $p$ vertices, $q$ edges, and minimum degree $\delta = \delta(F)$. Let $q' = \text{wsat}(p, F)$. Then, for $n$ divisible by $p$,

$$\text{wsat}(n, F) \leq \frac{n}{p} q' + \left(\frac{n}{p} - 1\right) \binom{\delta}{2}.$$
Theorem
Let $F$ be a graph with $p$ vertices, $q$ edges, and minimum degree $\delta = \delta(F)$. Let $q' = \text{wsat}(p, F)$. Then, for $n$ divisible by $p$,

$$\text{wsat}(n, F) \leq \frac{n}{p} q' + \left(\frac{n}{p} - 1\right) \binom{\delta}{2}.$$ 

Let $F'_p \in \text{WSat}(p, F)$. 

\[ F'_p \quad B_\delta \quad F'_p \quad B_\delta \quad F'_p \quad B_\delta \quad \ldots \quad B_\delta \quad F'_p \]

\[ B_5 = \begin{array}{c}
\end{array} \]
Definition
A graph $G$ of order $p$ with $q$ edges is self weakly saturated if $\text{wsat}(p, G) = q - 1$. 
**Definition**

A graph $G$ of order $p$ with $q$ edges is **self weakly saturated** if $\text{wsat}(p, G) = q - 1$.

**Definition**

If a self weakly saturated graph $G$ has a vertex $v$ (called the **root**) and an ordering of the edges of $\overline{G}$ such that as each edge is added it is possible to choose a new copy of $G$ in which $v$ is always the same vertex, then the graph $G$ is a **rooted self weakly saturated** graph.
EXAMPLES
EXAMPLES

(1) \( P_n \) is rooted self weakly saturated

\[ P_4 \]

Figure: \( P_4 \) is a rooted self weakly saturated graph with vertex 1 as a root.
EXAMPLES

(1) $P_n$ is rooted self weakly saturated

Figure: $P_4$ is a rooted self weakly saturated graph with vertex 1 as a root.

(2) $K_n - F_p$, when $F_p$ is a forest with $p < n$ is self weakly saturated.
SELF WEAKLY SATURATED GRAPHS

EXAMPLES

(1) $P_n$ is rooted self weakly saturated

(2) $K_n - F_p$, when $F_p$ is a forest with $p < n$ is self weakly saturated.

(3) $C_n \cup e$ where $e$ is a 2-chord of $C_n$ is self weakly saturated.

Figure: $P_4$ is a rooted self weakly saturated graph with vertex 1 as a root.
Theorem
Let $F$ be a graph of order $p$ with $q$ edges containing a cut-vertex $v$ such that one of the components of $F - v$ along with $v$ forms a rooted self saturated tree, $T_m$ with root $v$. Then, for $n \geq 2p - m$,

$$\text{wsat}(n, F) = q - 1.$$
**Theorem**

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$$wsat(n, F) = q - 1.$$
Theorem

For any given pair of positive integers $p$ and $q$ with $p - 1 \leq q \leq \binom{p}{2}$, there is a connected self weakly saturated graph with $p$ vertices and $q$ edges.
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Question
Given a graph $G$ with $p$ vertices and $q$ edges, what conditions on the graphical parameters of $G$ would imply that $G$ is self weakly saturated.
Definition

A graph $H$ is vertex symmetric, if for any pair of vertices $u$ and $v$ in $H$, there is an automorphism $\theta$ of $G$ such that $\theta(u) = v$. 

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Definition
Given a vertex symmetric $H$, the graph $G = H \cup H$ will denote the graph obtained from $H \cup H$ by adding an edge between the two copies of $H$. 
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Definition
Given a vertex symmetric $H$, the graph $G = H - H$ will denote the graph obtained from $H \cup H$ by adding an edge between the two copies of $H$.

Theorem
If $H$ is a vertex symmetric graph of order $p$ with $\delta(H) = \delta$, then

$$\text{wsat}(2p, H - H) = \delta p.$$
Question

Is there a finite set of graphical parameters that will determine the saturation number (weak saturation number) of a graph, or at least determine the order of magnitude?
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Question

For a fixed graphs $H$, for which integers $m$ with

$$sat(n, H) \leq m \leq ex(H, n)$$

such that there is a $H$-saturated graph with $m$ edges (Edge Saturation Spectrum)
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Is there a finite set of graphical parameters that will determine the saturation number (weak saturation number) of a graph, or at least determine the order of magnitude?

Question

For a fixed graphs H, for which integers m with

$$\text{sat}(n, H) \leq m \leq \text{ex}(H, n)$$

such that there is a H-saturated graph with m edges (Edge Saturation Spectrum)

Question

Is there a universal lower bound for the weak saturation number in terms of the minimum degree of a graph? In particular is $$\text{ws}(n, G) \geq (\delta n)/2 + c$$ for some constant c.
THANKS
THANKS

QUESTIONS?