Universal Graphs with Forbidden Subgraphs

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Budapest
1. Origins

2. Universality and Homogeneity

3. Universality without Homogeneity
1 Origins

2 Universality and Homogeneity

3 Universality without Homogeneity
Origins

[RADO64]: Universal Graphs

- Universal (countable) graphs exist
- Universal locally finite graphs do not exist (de Bruijn)

[KomPach91] (survey): WHEN do universal graphs exist?
Origins

[RADO64]: Universal Graphs
[KomPach91] (survey): WHEN do universal graphs exist?

[ERDŐS-RÉNYI63]: Automorphisms

- $\text{Aut}(\Gamma) = 1$ for $\Gamma$ random finite
- $\text{Aut}(\Gamma)$ rich for $\Gamma$ random infinite

Thus there is a striking contrast . . . : while „almost all" finite graphs are asymmetric, „almost all" infinite graphs are symmetric.
[RADO64]: Universal Graphs

[KomPach91] (survey): WHEN do universal graphs exist?

[ERDŐS-RÉNYI63]: Automorphisms

Thus there is a striking contrast . . . : while „almost all" finite graphs are asymmetric, „almost all" infinite graphs are symmetric.

[KPT05] \( \text{Aut} \Gamma \) has fixed points \( \iff \) Structural Ramsey

[Pes98] \( \text{Aut}(\mathbb{Q}) \) has fixed points \( \iff \) Ramsey
We follow Rado’s line (or Komjáth/Pach’s interpretation of it) . . .
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Homogeneity

Definition (Homogeneity)

\[ A \simeq B \iff A \sim B \text{ (conjugate under } \text{Aut}(\Gamma)) \]
Homogeneity

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\[ A \cong B \iff A \sim B \]

Consequences

- Universality (modulo finite substructures)
- Uniqueness (modulo finite substructures)
- Oligomorphomic (finitely many orbits on \( n \)-tuples)

As observed by Urysohn in 1924.
Homogeneity

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As observed by Urysohn in 1924 . . .
Urysohn 1924 (Letter)

“... [a] condition of homogeneity: the latter being, that it is possible to map the whole space onto itself ... so as to carry an arbitrary finite set $M$ into an equally arbitrary set $M_1$, congruent to the set $M$.”

Ref: [Hušek08]
Urysohn 1924 (Letter)

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\( \bar{U} \): universal complete separable metric space
\( \bar{U}_Q \): universal rational-valued metric space

- \( \bar{U}_Q \) is a universal graph (edges: \( d(u, v) = 1 \))
  cf. Moss, Cameron . . .
Theorem (Lachlan/Woodrow 1980)

The homogeneous graphs are (up to complementation)

- $C_5$, $K_3 \otimes K_3$ (9 vertices)
- $m \cdot K_n$ ($m, n \leq \infty$)
- Generic $K_n$-free [Henson71]

However, a structural Ramsey theorem requires an order ...
Classification Theorem with Order

Theorem (Cherlin 2013)

The homogeneous ordered graphs are

- Generic linear extensions of homogeneous partial orders with edge relation “comparability” (cf. [Schmerl79])
- Generically ordered homogeneous graphs (cf. [LachWood80])
- Generically ordered homogeneous tournaments with edges “a → b ⇐⇒ a < b” (cf. [Lachlan84])
- Homogeneous permutations (cf. [Cameron03])
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A Decision Problem

Survey: [KomjathPach91]
Narrowing the focus:
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Problem

\[ C : \text{finite set of finite, connected, forbidden subgraphs} \]

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Variant

Forbid induced subgraphs
A Decision Problem

Survey: [KomjathPach91]
Narrowing the focus:

**Problem**

C: finite set of finite, connected, forbidden subgraphs
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**Variant**

Forbid induced subgraphs

- More general
- **Undecidable** via Wang’s domino problem
- for the brave . . .
What so special about SUBGRAPHS?

- Sample Theorems
- Conjectures
- Underlying Theory [CheSheShi97]
### Sample Theorems

<table>
<thead>
<tr>
<th>Who,When</th>
<th>What</th>
<th>Which</th>
</tr>
</thead>
<tbody>
<tr>
<td>KMP88</td>
<td>Forbid a long path</td>
<td>$\exists$</td>
</tr>
<tr>
<td></td>
<td>No short odd cycles</td>
<td>&quot;</td>
</tr>
<tr>
<td>ChShe07</td>
<td>Tree</td>
<td>path or near-path</td>
</tr>
<tr>
<td>ChShi96</td>
<td>Set of cycles</td>
<td>short odd cycles</td>
</tr>
<tr>
<td>ChSheShi97</td>
<td>Hom-closed set</td>
<td>$\exists$</td>
</tr>
<tr>
<td>FürKom97</td>
<td>2-connected</td>
<td>complete</td>
</tr>
<tr>
<td>Kom99,ChTal07</td>
<td>2 blocks</td>
<td>$(K_m, K_n)$: $\min(m, n) \leq 5$ not $(5, 5)$!</td>
</tr>
</tbody>
</table>
Conjectures (1 Constraint)

Conjectures on existence of universal $C$-free graphs

1 (Solidity) Blocks of $C$ should be complete
2 (Block-Path) After pruning trees, $C$ should become a block-path
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Theorem (ChShe, in prep)

If the constraint $C$ is a block path, and a universal $C$-free graph exists then $C$ has complete blocks.

Corollary

$(2) \implies (1)$
Methods

- Pruning
- Algebraic Closure (+ Füredi-Komjáth method)
Non-Definition — $a$ is $C$-algebraic over $X$ if forbidding $C$ bounds the number of vertices like $a$. 
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- Obviously
- Or by transitivity
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Example

Let \( C \) contain a star (i.e., we bound the vertex degrees). Then
- Obviously algebraic means \textit{neighbor}
- Algebraic means \textit{in the connected component}
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Example (cont.)
- Forbidding \( C_4 \) makes a common neighbor unique. This can be iterated.
- Forbidding \( C \), 2-connected but not complete, with \( a, b \) non-adjacent, makes \( \bar{a} \) unique over \( C \setminus \{a, b\} \), where \( \bar{a} \) results by setting \( a = b \).
Oligomorphic Universality

Theorem

Let $C$ be a finite set of finite connected forbidden subgraphs with all blocks complete. Then the following are equivalent.

- There is a universal $C$-free graph with oligomorphic automorphism group;
- The algebraic closure of a vertex is always finite.

The halting problem for the relation obviously algebraic in
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Example

If $C$ contains a star, decidable:

- Algebraic closure = connected component
- Oligomorphic iff some path forbidden
The first method of pruning:

- For a tree, remove its leaves.
- Generally, remove a minimal block-leaf (or more generally, a “corner”)

**Lemma**

*If* $C$ *prunes to* $C'$, *then a universal* $C$-*free graph will contain a universal* $C'$-*free graph. So we may argue inductively.*
Pruning

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Applications: from trees to near-paths (by treating the minimal case).
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Applications: from trees to near-paths (by treating the minimal case).
— And probably . . .
A tentative Result

Theorem (CheShe, in progress)

Let $C$ be a block-path with $\ell \geq 6$ blocks, all complete, of sizes $m_i = |B_i| \geq 3$ all $i$, and allowing a universal $C$-free graph. Then up to reversal the sequence $(m_i)$ is one of: $(4, 4, 3^*)$, $(3, m, 3^*)$, $(m, 3^*)$

Is the end in sight? Not yet —
A Problem for Graph Theorists

Problem

$C: K_n$ plus $n$ paths, 1 at each vertex. 
Is there a universal $C$-free graph?
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Problem

$C: K_n \text{ plus } n \text{ paths, 1 at each vertex.}$

Is there a universal $C$-free graph?

Is this a problem for graph theorists?

Think about $acl_C$ . . .

Menger’s theorem?
Problem

T the theory of existentially complete C-free graphs. When is T stable (etc.)?
Variations

Problem

*T the theory of existentially complete \( C \)-free graphs. When is \( T \) stable (etc.)?

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*Which finitely determined permutation pattern classes allow a universal permutation?
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Problem

$T$ the theory of existentially complete $\mathcal{C}$-free graphs. When is $T$ stable (etc.)?

Problem

Which finitely determined permutation pattern classes allow a universal permutation?

*(Permutation: A structure with two orderings.)*
Variations

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Problem

Which finitely determined permutation pattern classes allow a universal permutation?

(Permutation: A structure with two orderings.)

Subquestion When there is an oligomorphic universal permutation, are the orderings involved dense?