Asymmetric exclusion: a way to anomalous scaling

Joint work with Timo Seppäläinen

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An easy example

The totally asymmetric simple exclusion process

Exotic scaling

Proof
An easy Markov process with “usual” behavior

A particle jumps one step to the right with iid. Exp(1) waiting times. Its position at time $t$ is $S(t)$, counting the number of steps.
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0 1 2 3 4 5 6 7

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![Diagram of particle movement](image)
An easy example

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![Diagram showing a particle at position 1 on a line with positions 0 to 7 labeled.]
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![Diagram of particle position over time](image-url)
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TASEP Exotics Proof

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Continuous time Markov jump process with rate 1.
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A particle jumps one step to the right with iid. $\text{Exp}(1)$ waiting times. Its position at time $t$ is $S(t)$, counting the number of steps.

$\implies$ Continuous time Markov jump process with rate 1.

$\text{LLN: } \lim_{t \to \infty} \frac{S(t)}{t} = 1 \quad \text{a.s.}$
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**CLT:** $\lim_{t \to \infty} \frac{S(t) - t}{\sqrt{t}} \sim \mathcal{N}(0, 1)$. 
The totally asymmetric simple exclusion process

Bernoulli($\varrho$) product distribution.
The totally asymmetric simple exclusion process

Particles step to the right with rate 1, unless the destination site is occupied.
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Bernoulli(\(q\)) product distribution.

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The Bernoulli(\(\varrho\)) distribution is stationary (and non-reversible) for all \(0 \leq \varrho \leq 1\).

These are the important (\(=\) ergodic) stationary distributions.
The totally asymmetric simple exclusion process

An observer starts from the origin, and moves with velocity $V$. 
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An observer starts from the origin, and moves with velocity \( V \).

The quantity of our interest is:

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J_V(t) = \# \{ \text{particles that pass the observer by time } t \} - \# \{ \text{particles the observer passes by time } t \}.
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Again, counting the number of steps of a given type.
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Again, counting the number of steps of a given type.

$J_V(t) = \text{net flux of particles}$
The second class particle

Stochastic coupling: evolution as close as possible

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Its position at time $t$: $Q(t)$.

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Its position at time $t$: $Q(t)$.

Its velocity: $\lim_{t \to \infty} \frac{E_{Q(t)}}{t} = 1 - 2\varrho$

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\( J_V(t) = \text{net flux of particles} \)
Exotic scaling

On the characteristics $V = 1 - 2\rho$:

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On the characteristics $V = 1 - 2\varrho$:

Theorem (B. - Seppäläinen)

$$0 < \liminf_{t \to \infty} \frac{\text{Var}(J_{1-2\varrho}(t))}{t^{2/3}} \leq \limsup_{t \to \infty} \frac{\text{Var}(J_{1-2\varrho}(t))}{t^{2/3}} < \infty.$$ 

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1. An algebraic miracle

Miracle: exact identities.

Theorem (B. - Seppäläinen; ideas also from B. Tóth, H. Spohn, and M. Prähöfer)

\[
\mathbb{E} Q(t) = (1 - 2\varrho)t,
\]

\[
\text{Var}(J_{1-2\varrho}(t)) = c \cdot \mathbb{E}|Q(t) - \mathbb{E}Q(t)| = c \cdot \mathbb{E}|\tilde{Q}(t)|.
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Proof by combinatorial tricks, partial summations, covariances, independence.
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\[\text{Var}(J_{1-2\varphi}(t)) = c \cdot \mathbb{E} |Q(t) - \mathbb{E} Q(t)| = c \cdot \mathbb{E} |\tilde{Q}(t)|.\]

Proof by combinatorial tricks, partial summations, covariances, independence.

\[\mathbb{E} Q(t) = (1 - 2\varphi)t\]
\[\text{Var}(J_{1-2\varphi}(t)) = c \cdot \mathbb{E} |\tilde{Q}(t)|\]
2. Many second class particles

\[ EQ(t) = (1 - 2\varrho)t \quad \text{Var}(J_{1-2\varrho}(t)) = c \cdot E|\tilde{Q}(t)| \]
2. Many second class particles

Coupling three processes:

\[ E[Q(t)] = (1 - 2\varrho)t \]

\[ \text{Var}(J_{1-2\varrho}(t)) = c \cdot E[\tilde{Q}(t)] \]
2. Many second class particles

Coupling three processes:

\[ \begin{array}{ccccccccc}
& & & & & & & & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
-3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & i \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array} \]

Bernoulli(\(q\))  
Bernoulli(\(\lambda\))

Push \(Q\) abnormally to the right: \(\tilde{Q}(t) \geq u\)  
\(\Rightarrow\) abnormally many second class particles pass

\[
E(Q(t)) = (1 - 2q)t \\
Var(J_{1-2\varnothing}(t)) = c \cdot E|\tilde{Q}(t)|
\]
2. Many second class particles

Coupling three processes:

- Bernoulli($\rho$)
- Bernoulli($\lambda$)

Push $Q$ abnormally to the right: $\tilde{Q}(t) \geq u$

$\Rightarrow$ abnormally many second class particles pass
$\Rightarrow$ abnormally large difference between $J$ and $J^\lambda$

$\mathbb{E}Q(t) = (1 - 2\rho)t$

$\text{Var}(J_{1-2\rho}(t)) = c \cdot \mathbb{E}|\tilde{Q}(t)|$
2. Many second class particles

Coupling three processes:

Push $Q$ abnormally to the right: $\tilde{Q}(t) \geq u$
⇒ abnormally many second class particles pass
⇒ abnormally large difference between $J$ and $J^\lambda$
⇒ via Chebyshev’s inequality:

$$P\{\tilde{Q}(t) \geq u\} \leq c \cdot \frac{t^2}{u^4} \cdot \text{Var}(J_{1-2\varrho}(t))$$

after optimising in $\lambda$.

$$E_Q(t) = (1-2\varrho)t$$
$$\text{Var}(J_{1-2\varrho}(t)) = c \cdot E|\tilde{Q}(t)|$$
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$$P\{\tilde{Q}(t) \geq u\} \leq c \cdot \frac{t^2}{u^4} \cdot \text{Var}(J_{1-2\varrho}(t)) \quad \text{Var}(J_{1-2\varrho}(t)) = c \cdot E|\tilde{Q}(t)|$$
2. Many second class particles

Repeat to the left:

\[ P\{|\tilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot \text{Var}(J_{1-2\varrho}(t)). \]

\[ P\{\tilde{Q}(t) \geq u\} \leq c \cdot \frac{t^2}{u^4} \cdot \text{Var}(J_{1-2\varrho}(t)) \]

\[ \text{Var}(J_{1-2\varrho}(t)) = c \cdot E|\tilde{Q}(t)| \]
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Repeat to the left:

\[ \mathbb{P}\{|\tilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot \text{Var}(J_{1-2\varnothing}(t)). \]

Recall the miracle:

\[ \mathbb{P}\{|\tilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot \mathbb{E}|\tilde{Q}(t)| = : c \cdot \frac{t^2}{u^4} \cdot \mathbb{E}. \]
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Repeat to the left:

\[ P\{ |\widetilde{Q}(t)| > u \} \leq c \cdot \frac{t^2}{u^4} \cdot \text{Var}(J_{1-2\ell}(t)). \]

Recall the miracle:

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Innocent as it looks... but already implies the 2/3 scaling.

\[ P\{ \widetilde{Q}(t) \geq u \} \leq c \cdot \frac{t^2}{u^4} \cdot \text{Var}(J_{1-2\ell}(t)) \quad \text{Var}(J_{1-2\ell}(t)) = c \cdot \mathbb{E}|\widetilde{Q}(t)| \]
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Innocent as it looks... but already implies the 2/3 scaling.

\[ P\{|\tilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E \quad \text{Var}(J_{1-2\varrho}(t)) = c \cdot E|\tilde{Q}(t)| \]
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\[ E = E|\tilde{Q}(t)| = \int_0^\infty \mathbb{P}\{\left|\tilde{Q}(t)\right| > u\} \, du \]
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\[ P\{|\tilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E \quad \text{Var}(J_{1-2\eta}(t)) = c \cdot E|\tilde{Q}(t)| \]
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\[
\leq E \int_{1/2}^\infty P\{|\tilde{Q}(t)| > vE\} \, dv + \frac{1}{2} E
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\[ \leq E \int_0^{1/2} P\{|\tilde{Q}(t)| > vE\} \, dv + \frac{1}{2} E \]
\[ \leq c \cdot \frac{t^2}{E^2} + \frac{1}{2} E, \]

that is: \( E^3 \leq c \cdot t^2. \)
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\[ \leq c \cdot \frac{t^2}{E^2} + \frac{1}{2}E, \]

that is: \( E^3 \leq c \cdot t^2. \)

\[ \text{Var}(J_{1-2\varrho}(t)) \stackrel{\text{Miracle}}{=} \text{const.} \cdot E \leq c \cdot t^{2/3}. \]

\[ P\{|\tilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E \quad \text{Var}(J_{1-2\varrho}(t)) = c \cdot E|\tilde{Q}(t)| \]
Lower bound

In the upper bound, the relevant orders were

\[ u \equiv \text{(deviation of } Q(t) \text{)} \sim t^{2/3}, \quad \rho - \lambda \sim t^{-1/3}. \]
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works with similar arguments: compare models of which the densities differ by \( t^{-1/3} \), and use connections between \( Q(t) \), the green second class particles, and heights.

Lower bounds tend to be more messy.
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The lower bound

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works with similar arguments: compare models of which the densities differ by \( t^{-1/3} \), and use connections between \( Q(t) \), the green second class particles, and heights.

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Thank you.