Multiplicative Functions and Small Divisors

by

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* A report of joint work of \{ k. Alladi
P. Erdős
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and later work of
K. Soundararajan
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P. Suryamohan
& R. Munshi
Let $S \subseteq \mathbb{Z}^{+}$ and

$$g(n) = \prod_{p|n, \ p=\text{prime}} g(p) \quad (\text{strongly mult.}), \quad g \geq 1$$

**Problem:** Obtain upper bound for

$$S(g; x) = \sum_{n \leq x, \ n \in S} g(n)$$

**Idea:** Write

$$g(n) = \sum_{d|n} h(d)$$

Then $h$ is multiplicative, $0 \leq h$,

$$h(p) = g(p) - 1, \quad h(p^a) = 0, \quad \text{for } a \geq 2.$$ 

Thus

$$S(g; x) = \sum_{n \leq x, \ n \in S} \sum_{d|n} h(d) \cdot \ln$$

$$= \sum_{d \leq x} h(d) \sum_{n \in S, \ n \equiv 0 \pmod{d}} 1 = \sum_{d|S_d(x)} h(d) \left| S_d(x) \right|$$

where

$$S_d(x) = \{ m \in S \mid m \equiv 0 \pmod{d}, \ m \leq x \}$$
Suppose we have the estimate
\[
\left| S_d(x) \right| \leq \frac{1}{\log x} \sum_{d \mid x} \frac{h(d)}{d}, \ \text{\( \omega \)-mult. (1)}
\]
then we would get (with \( x = \left| S_d(x) \right| \))
\[
S(g; x) \leq x \sum_{d \leq x} \frac{h(d) \omega(d)}{d} \leq x \prod_{p \leq x} \left( 1 + \frac{\omega(p) \log p}{p} \right)
\]
It turns out that (1) usually holds only for \( d \leq x^\beta \), with some \( \beta < 1 \).

For example
a) if \( S = \{ p + a \mid p \leq x \} \), shifted primes,
then \( \beta = \frac{1}{2} \) (Brun-Titchmarsh)
b) if \( S = \{ P(n) \mid P(n) \leq x \} \), \( P(t) \in \mathbb{Z}^+ [t] \),
then \( \beta = \frac{1}{\deg P} \)

So what we require are inequalities like
\[
\sum_{d \leq n} h(d) \ll \sum_{d \leq n^\beta} h(d), \quad (2)
\]
with \( 0 < \beta < 1 \).
To lead us to such estimates, we formulate

Meta theorem: "Large divisors of an integer are more composite than the smaller ones."

We will characterize this meta theorem in various ways.

On the basis of the meta theorem, I made

Weak conjecture (Alladi, 1983 Asilomar Conf.)

For each integer \( k \geq 2 \), there exists \( c_k \)

such that if \( 0 \leq h(p) \leq c_k \), then

\[
\sum_{d \mid n} h(d) \ll_k \sum_{d \leq n^{1/k}} h(d) \dfrac{d}{\ln n}, \quad d \leq n^{1/k}
\]

An example:

Let \( r \) be large & \( p_1 \sim p_2 \sim \ldots \sim p_r \) primes.

Let \( n = p_1 p_2 \ldots p_r \). Then \( d \mid n \) satisfies

\[ d \leq n^{1/k} \iff \nu(d) \leq \frac{r}{k} \]
Suppose $h(p) = c$, $\forall$ $p$. Then

$$\sum' h(d) = (1+c)^r$$

and

$$\sum' h(d) \approx \sum' \binom{r}{k} c^l$$

d $\leq n^{\frac{1}{k}}$, $l \leq \frac{c}{k}$

The peak of $\binom{r}{k} c^l$ occurs when $l \sim \frac{rc}{1+c}$.

So $\frac{rc}{1+c} > \frac{c}{k} \iff c > \frac{1}{k-1}$,

Then the sum in (3) is $o((1+c)^r)$. This example shows that $c_k \leq \frac{1}{k-1}$.

Strong Conjecture: (Alladi, 1983)

$$c_k = \frac{1}{k-1}, \text{ for } k = 2, 3, \ldots$$

Remark: For purpose of applications to Probabilistic Number Theory that I had, I needed only inequalities like (2) for $0 \leq h(p) \leq \delta$, with some $\delta > 0$, and so the truth of the weak conjecture was sufficient.
A mapping for sets and divisors

If \( n \neq 0 \), then trivially we have that half the divisors of \( n \) are \(< \sqrt{n} \) and half are \( > \sqrt{n} \), as seen by the correspondence

\[
d/n \leftrightarrow \frac{n}{d} \mid n.
\]

A more interesting (deeper) mapping

Conjecture (Alladi, 1982)

There exists a mapping \( m \) from the set \( S^< \) of divisors of \( n \) which are \(< \sqrt{n} \) to the set \( S^> \) of divisors of \( n \) which are \( > \sqrt{n} \) such that

(i) \( m \) is a bijection

(ii) If \( d' = m(d) \) for \( d \in S^< \) and \( d' \in S^> \), then \( m(d) = d' \equiv 0 \pmod{d} \).

Remark (i) The conjecture, if true, would immediately imply
Lemma 1: Let \( n \) be sq. free and \( h \) mult. such that \( 0 \leq h \leq 1 \). Then
\[
\sum_{d \in S^<_{\sqrt{n}}} h(d) \leq 2 \sum_{d \in S^<_{\sqrt{n}}} h(d), \quad d < \sqrt{n}
\]

Proof: Write
\[
\sum_{d \in S^<_{\sqrt{n}}} h(d) = \sum_{d \in S^<_{\sqrt{n}}} h(d) + \sum_{d \in S^>_{{\sqrt{n} \atop \sqrt{n}}}} h(d')
\]

The second sum is less than the first because
\[ h(d') = h(dd'') \leq h(d) \]

(ii) One does not need the conj. To prove Lemma 2, which follows from a monotonicity principle namely

Lemma 2: If \( n \) is sq. free & \( h, h' \) mult. such that
\[ 0 \leq h' \leq h \]

then
\[
\frac{\sum_{d \in S^<_{\sqrt{n}}} h'(d)}{\sum_{d \in S^<_{\sqrt{n}}} h'(d)} \leq \frac{\sum_{d \in S^<_{\sqrt{n}}} h(d)}{\sum_{d \in S^<_{\sqrt{n}}} h(d)}
\]

and the fact that the conj. holds when \( h = 1 \).
Lemma 1 was precisely what I needed to establish the Erdős–Kac–Kubilius thm. on the set of shifted primes (KA, Pac. J. Math., 1983). I informed Erdős about my conjecture and said that I could not prove it. He replied that if we strengthen the conjecture, it could be proved by induction on the number of prime factors:

**Theorem (Erdős, 1982)** — private communication.

Let \( n \) be sq. free, and \( t \in [1, \sqrt{n}] \). Then there exists a mapping \( m_t = m_{t,n} \) from the set \( S^<_t \) of the divisors of \( n \) which are \( < t \) to the set \( S^>_t \) of the divisors of \( n \) that are \( > n/t \) such that

(i) \( m_t \) is a bijection

(ii) If \( d' = m_t(d) \), \( d \in S^<_t \), \( d' \in S^>_t \), then \( m_t(d) = d' \equiv 0 \) (mod \( d \)).
At this stage Vaaler noticed that this could be formulated more generally in terms of sets and measures:

**Theorem'**: Let $S$ be a finite set and $\lambda$ a finite measure on the set of all subsets of $S$. For each $t \geq 0$, define

$$A(t, S) = \{ E \subseteq S | \lambda(E) \leq t \}$$

Then there is a permutation

$$\pi_{t, S} : A(t, S) \rightarrow A(t, S)$$

such that

$$\pi_{t, S}(E) \cap E = \emptyset, \forall E \in A(t, S).$$


**Cor**: Let $S, \lambda$ as above. Define

$$B(t, S) = \{ E \subseteq S | \lambda(S) - t \leq \lambda(E) \}$$

Then there is a bijection

$$\sigma_{t, S} : A(t, S) \rightarrow B(t, S)$$

s.t. $E \subseteq \sigma_{t, S}(E), \forall E \in A(t, S)$. 
In that same paper, we prove

**Theorem 1:**

Let \( h \) be sub-multiplicative and satisfy

\[
0 \leq h(p) \leq c < \frac{1}{k-1}.
\]

Then

\[
\sum' h(d) \leq \left\{1 - \frac{k c}{1+c}\right\}^{-1} \sum' h(d) \quad \text{dln, } d \leq n^{1/k}
\]

(This settles the weak conjecture with any \( c < \frac{1}{k-1} \))

**Proof:** Begin with the decomposition

\[
\sum' h(d) = \sum h(d) + \sum h(pd) \quad \text{dln, } \text{dln/p, dln/p}
\]

for any \( p|n, \ p = \text{prime} \). Consequently

\[
h(p) \sum' h(d) = h(p) \sum h(d) + h(p) \sum' h(pd) \quad \text{dln/p, dln/p}
\]

\[
\geq \sum' h(pd) \quad \text{dln/p, (1+h(p)) \sum' h(pd) \quad dln/p}
\]

Thus

\[
\sum' h(pd) \leq \frac{h(p)}{1+h(p)} \sum' h(d). \quad (4)
\]
Next write

\[ \sum h(d) \geq \sum h(d) \frac{\log(n^{\frac{1}{k}}/d)}{\ln n}, \quad d \leq n^{\frac{1}{k}} \] \hspace{\textwidth}

\[ = \sum h(d) - \frac{k}{\ln n} \sum h(d) \log d \tag{5} \]

Note that (4) implies

\[ \sum h(d) \log d = \sum h(d) \sum \log p = \sum \log p \sum h(p)_{\text{odd}} \] \hspace{\textwidth}

\[ \leq \left( \sum h(d) \right) \left( \sum \frac{h(p) \log p}{\ln p + h(p)} \right) \leq \frac{c \log n \sum h(d)}{1 + c} \tag{6} \]

Theorem 1 follows from (5) & (6).

\[
\text{Theorem 2: (Alladi-Erdős-Vaaler, J.N.T. 1989)}
\]

If \( k \geq 2 \) is an integer and \( 0 \leq h(p) \leq \frac{1}{k-1} \), where \( h \) is \( k \)-strongly multiplicative, then for all squarefree \( n \)

\[ \sum h(d) \leq (2k + o(1)) \sum h(d) \] \hspace{\textwidth}

\[ \frac{\ln n}{\ln n}, \quad d \leq n^{\frac{1}{k}}, \] \hspace{\textwidth}

where \( o(1) \to 0 \) as \( n \to \infty \).

(This settles the strong conjecture.)
To prove this we use a powerful theorem on hypergraphs due to Baranyai (1973):

**Baranyai's Thm:** Let \( k, m \) be positive integers.

Let \( S \) be a set with \( km \) elements. Then the \( \binom{km}{m} \) subsets of \( S \) having \( m \) elements each, can be grouped \( k \) at a time such that in every such group, the \( k \) subsets of size \( m \) generate a partition of \( S \).

**Proof of Thm 2:** In view of the monotonicity, it suffices to prove Thm 2 in the case \( h(\rho)=c = \frac{1}{k-1} \).

Let \( v(n) = km + l, \quad 0 \leq l \leq k-1 \).

For \( j \leq m \) consider a divisor \( N \) of \( n \) with \( v(N) = k(m-j) \). By Baranyai's Thm, the divisors \( d \) of \( N \) having \( v(d)=m-j \), can be grouped \( k \) at a time such that they are mutually coprime and the product is \( N \). In every such group, one of these \( k \) divisors is \( \leq n^{1/k} \).
So there are
\[ \geq \frac{1}{k} \binom{k(m-j)}{m-j} \]

divisors of \( N \) which are \( \leq n^{\frac{1}{k}} \).

The number of ways of choosing such \( N \) is
\[ \binom{km+l}{k(m-j)} \]

Every such divisor \( d \) can be the divisor of at most
\[ \binom{km+l-m+j}{(k-1)(m-j)} \]
such numbers \( N \). Thus we have at least
\[ \frac{1}{k} \binom{k(m-j)}{m-j} \binom{km+l}{k(m-j)} \]

\[ \geq \frac{1}{k} \sum_{j=0}^{k-1} \binom{km+l}{m-j} \binom{1}{k-1} \]

divisors of \( n \) with \( v(d) = m-j \) \& \( d \leq n^{\frac{1}{k}} \). Thus
\[ \sum_{d \leq n^{\frac{1}{k}}} h(d) \geq \frac{1}{k} \sum_{j=0}^{k-1} \binom{km+l}{m-j} \binom{1}{k-1} \]

\[ \approx \frac{1}{k \cdot 2} (1+c)^{\nu(n)} = \frac{1}{2k} \sum_{d \leq n^{\frac{1}{k}}} h(d) \]

and this proves Theorem 2.
**Question:** Can the implicit "constant" $2k + o(1)$ in Thm 2 be replaced effectively by an expression (possibly depending on $n$) so that the inequality is valid for all $n$?

Theorem 1 has an expression valid for all $n$, but $\left\{1 - \frac{k \varepsilon}{1+c}\right\}^{-1} = \infty$ when $c = \frac{1}{k-1}$.

An answer to the above question is given by Theorem 3: (Alladi- Erdös-Vaaler, JNT 1989)

Let $k \geq 2$, let $h$ be multiplicative, and satisfy $0 \leq h(p) \leq \frac{1}{k-1}$. Then for all sq. free $n$:

$$\sum\frac{h(d)}{d \ln n} \leq \frac{k \nu(n)}{k-1} \sum\frac{h(d)}{d \ln n, d \leq n^{1/k}}$$

To prove Thm 3, we use the more general monotonicity given by Lemma M: Let $n$ be sq. free & $0 < \alpha < 1$. For fixed $\alpha$ and $n$ the quantity

$$R_{\alpha, n}(h) = \left(\sum\frac{h(d)}{d \ln n, d \leq n^{\alpha}}\right)/\sum\frac{h(d)}{d \ln n}$$

decreases as $n$ increases.
Proof of Lemma M: The lemma is trivial when \( v(n) = 1 \). So let \( v(n) \geq 2 \).

Define

\[ \chi_\alpha(x) = \begin{cases} 1, & \text{if } x \leq \alpha \\ 0, & \text{if } x > \alpha. \end{cases} \]

Then

\[ R_{\alpha, n}(h) = \sum_{\substack{d | n, \; q | n \implies q \text{ prime} \quad d \cdots \quad \pi(1 + h(d)) \quad d \cdots \quad \pi(1 + h(q)) \quad q \cdots \quad \pi(1 + h(q))}} \]

\[ = \sum_{p | n, \; q \neq p} \left\{ \chi_\alpha \left( \frac{\log d}{\log n} \right) \frac{h(d)}{1 + h(p)} + \chi_\alpha \left( \frac{\log p + \log d}{\log n} \right) \frac{h(pd)}{1 + h(p)} \right\} \times \]

\[ \frac{1}{\pi(1 + h(q))}, \quad q \mid n, \; q \neq p \]

\[ = \sum_{p | n, \; q \neq p} \left\{ \chi_\alpha \left( \frac{\log d}{\log n} \left( 1 - \frac{h(p)}{1 + h(p)} \right) + \chi_\alpha \left( \frac{\log p + \log d}{\log n} \right) \frac{h(pd)}{1 + h(p)} \right\} \times \]

\[ \frac{h(d)}{\pi(1 + h(q))}, \quad q \mid n, \; q \neq p \]

for some \( p | n \). Note that

\[ \chi_\alpha \left( \frac{\log d}{\log n} \right) \geq \chi_\alpha \left( \frac{\log p + \log d}{\log n} \right) \]
and so $R_{x, n}(h)$ decreases by increasing $h(p)$, and by not changing $h(q)$ for $q 
eq p$.

Then by increasing $h(q)$ in succession for other primes $q$, we get Lemma M.

Proof of Theorem 3: let $F(x, c, n)$ denote the value of $R_{x, n}(h)$ when $h(p) = c$, $q = p$.

To get a lower bound for $F$ we consider bounding $\chi_{\alpha}(x)$ from below, where $x = \frac{\log 2}{\log n}$.

$\chi_{\alpha}(x)$ in blue

$1 - \frac{x}{\alpha}$ in green

One possibility is to take $y = 1 - \frac{x}{\alpha}$. This was the choice in the proof of Thm 1.

So we now seek a quadratic polynomial.

To this end, let

$-\frac{1}{\alpha} \leq t \leq \frac{1}{\alpha}$
Then

\[ x(x) \equiv f(x) = tx^2 - (at + \frac{1}{\alpha})x + 1 \]

With this choice

\[ F(d, c, n) \geq \frac{1}{H(n)} \sum_d \left\{ \frac{t \log^2 d}{\log^2 n} \log \frac{d}{\log n} - (at + \frac{1}{\alpha}) \log \frac{d}{\log n} \right\} + h(d) \]

where

\[ H(n) = \sum_d h(d) \]

Note that

\[ \frac{1}{\log n} \sum_d h(d) \log d = \sum_d \frac{h(d)}{\log n} \sum_p \log p \]

\[ = \sum_p \frac{\log p}{\log n} \sum_d h(pd) = \frac{H(n)}{\log n} \sum_p \frac{h(p) \log p}{1 + h(p)} = \frac{c + H(n)}{1 + c} \]

(9)

Similarly, it can be shown that

\[ \frac{1}{\log^2 n} \sum_d h(d) \log^2 d \]

\[ = H(n) \left\{ \left(\frac{c}{1+c}\right)^2 + \frac{c}{(1+c)^2} \log^2 n \sum_p \log^2 p \right\} \]

(10)

By writing

\[ \log^2 d = \left(\sum_{p|d} \log p\right)^2 \]

and expanding the square
Thus (9) & (10) yield

\[ F(\alpha, c, \gamma, n) \geq f\left( \frac{\gamma}{\alpha + \gamma} \right)^2 + \frac{\epsilon c}{(\alpha + \gamma)^2} \sum_{p} \log^2 p. \] (11)

By the Cauchy-Schwarz inequality

\[ 1 = \sum_{p \leq n} \frac{\log p}{\log n} \leq \sqrt{\nu(n)} \left( \sum_{p \leq n} \frac{\log^2 p}{\log^2 n} \right)^{1/2} \]

and so

\[ \sum_{p \leq n} \frac{\log^2 p}{\log^2 n} \geq \frac{1}{\nu(n)}. \]

Therefore

\[ F(\alpha, c, \gamma, n) \geq f\left( \frac{\gamma}{\alpha + \gamma} \right)^2 + \frac{\epsilon c}{(\alpha + \gamma)^2} \cdot \nu(n). \]

Obviously we wish to make \( t \) maximal.

Since \( \alpha = \frac{1}{k} \) in Thm. 2, we take \( t = k, c = \frac{t}{\nu(n)} \).

With these choices it turns out that

\[ f\left( \frac{c}{\alpha + \gamma} \right) = 0 \quad (!) \]

Thus

\[ F\left( \frac{1}{k}, \frac{1}{k-1}, \gamma, n \right) \geq \frac{k-1}{k \cdot \nu(n)}. \]

and this proves Theorem 2.
Remarks:

(i) One would expect to get better bounds by increasing the degree of the minorizing polynomial. This would involve expressions

\[ \frac{1}{\log n} \sum_{j=3}^{\infty} \log^j p \]

These would be complicated for large \( m \) and might yield worse bounds unless the cancellation among the higher moments are calculated properly.

(ii) In Thm 3., we do not require \( k \) to be an integer.
Improvements and simplification by later authors

(1) What we needed to prove Theorem 2 was inequality (7) which we deduced as a consequence of Baranyai's theorem. Suryamohan (Glasgow Math. J., 2004) showed that (7) could be deduced directly by a simple counting argument without appeal to Baranyai's theorem. We show his proof now:

Proposition: Let \( k, \ell \geq 1 \) be integers and let \( N = p_1 p_2 \ldots p_{k \ell} \) where \( p_1 < p_2 < \ldots < p_{k \ell} \) are primes. Then the number of \( d | N \) with \( d \leq N^{\frac{1}{k \ell}} \) and having exactly \( \ell \) prime divisors is

\[
\geq \frac{1}{k \ell} \binom{k \ell}{\ell}.
\]

Note: The proposition is our inequality (7) with \( \ell \) in place of \( m-j \).
Proof (Suryamohan)

Let \( S_{ke} = \{1, 2, 3, \ldots, kl\} \) and \( \pi = (\sigma_1, \sigma_2, \ldots, \sigma_{ke}) \) be any permutation of \( S_{ke} \). We set
\[
\xi_{\pi} = \{A_1, A_2, \ldots, A_k\},
\]
where
\[
A_j = (\sigma_i - (i-1)e+1, \sigma_i - (i-1)e+2, \ldots, \sigma_i) .
\]
For each set \( B \) with \( |B| = l \), let
\[
\delta_{\pi}(B) = \begin{cases} 
1, & \text{if } B \in \xi_{\pi} \\
0, & \text{otherwise}
\end{cases}
\]
For each \( A \subseteq S_{ke} \), let \( d_A \) be the associated divisor of \( N \) given by the product \( \prod_{i \in A} \pi_i \).

Let
\[
\xi_1 = \left\{ A \subseteq S_{ke} \mid |A| = l, \ d_A \leq N^{\frac{1}{2}} \right\}
\]
\[
\xi_2 = \left\{ B \subseteq S_{ke} \mid |B| = l, \ d_B > N^{\frac{1}{2}} \right\}
\]
Clearly if \( C \subseteq S_{ke}, \ |C| = l \), then \( C \) belongs exactly to one of \( \xi_1 \) or \( \xi_2 \). Thus
\[
|\xi_1| + |\xi_2| = \binom{ke}{l} \tag{12}
\]
Note that
\[ \prod_{i=1}^{k} d_i = N \]
Thus \( \exists \) some \( i \) such that \( d_{A_i} \leq N^{1/k} \). Thus
\[ |\mathcal{S}_2 \cap \mathcal{S}_\pi| \leq (k-1) |\mathcal{S}_1 \cap \mathcal{S}_\pi|, \quad \forall \pi \]
Consequently
\[ \sum_{B \in \mathcal{S}_2} \delta_{\pi}(B) \leq (k-1) \sum_{A \in \mathcal{S}_1} \sum_{\pi} \delta_{\pi}(A). \quad (13) \]
If we sum the expressions in (13) over all \( \pi \), we get
\[ \sum_{B \in \mathcal{S}_2} \sum_{\pi} \delta_{\pi}(B) \leq (k-1) \sum_{A \in \mathcal{S}_1} \sum_{\pi} \delta_{\pi}(A) \quad (14) \]
It is now crucial to note that
\[ \sum_{\pi} \delta_{\pi}(C) = k \times \mathcal{L}!(kl-k)! \quad \forall C \text{ with } |C| = l \] 
(15)
is independent of \( C \). Thus (14) & (15) imply
\[ |\mathcal{S}_2| \leq (k-1) |\mathcal{S}_1|, \]
which together with (12) yields
\[ k |\mathcal{S}_1| \geq (k^l) \]
which is the assertion of the proposition.
Just as our Theorem 1 holds for all sub-multiplicative functions satisfying suitable bounds, S. Srinivasan (Glasgow Math. J., 1994) showed that our Theorem 2 holds for sub-multiplicative functions as well.

K. Soundararajan (J. N. T., 1992) achieved several improvements of our results in JNT, 1989 paper.

First Improvement: In Theorem 2, with $h$ multiplicative, $0 \leq h(p) \leq \frac{1}{k-1}$, for $k \geq 2$, he showed

$$\sum_{d|m} h(d) \leq (k + o(1)) \sum_{d|m} h(d)$$

where $o(1) \to 0$ as $v(n) \to \infty$.

Remark: Thus Soundararajan cut the implicit constant in Thm. 2 by half. This is crucial because when $k = 2$ it corresponds better with the bound

$$\sum_{d|m} h(d) \leq 2 \sum_{d|m} h(d)$$

for $0 \leq h \leq 1$. 
Second Improvement: In Theorem 3 for \( k \geq 2 \), and \( h \) multiplicative satisfying \( 0 \leq h(p) \leq \frac{1}{k-1} \), he showed that

\[
\sum h(d) \ll \sqrt{\nu(n)} \sum_{n \leq \sqrt{n}} h(d), \quad \text{for sq. free,}
\]

\[
d \ln \]

where the implicit constant is absolute.

Third Improvement: He extended Theorem 2 to rational values \( k \) as follows: Let \( k \geq 2 \) be rational, \( h \) mult., and \( 0 \leq h(p) \leq \frac{1}{k-1} \). Then

\[
\sum h(d) \leq (\gamma_k + o(1)) \sum h(d)
\]

\[
d \ln \]

\[
d \ln, \quad d \leq n^{\frac{1}{k}},
\]

for sq. free \( n \), where

\[
\gamma_k = 1 + a_0 + a_1 + \ldots + a_r,
\]

with \( k-1 = [a_0, a_1, \ldots, a_r] \) being the continued fraction of \( k-1 \). Here also \( o(1) \to 0 \) as \( \nu(n) \to \infty \).

Remark: If \( k \geq 2 \) is an integer, then \( r = 0 \), and \( a_0 = k-1 \). Thus \( 1 + a_0 = k \) which is what one has in the First Improvement of Thm. 2.
For \( t \geq 0 \), let \( F_t \) denote the set of multiplicative functions \( F : \mathbb{Z}^+ \to [0, \infty) \) such that \( F(p) \geq t \) for all primes \( p \).

Let \( G_t \) denote the set of multiplicative functions \( G : \mathbb{Z}^+ \to [0, \infty) \) such that \( 0 \leq G(p) \leq t \), \( \forall p \).

For square-free \( n \), put
\[
q(t, n) = \inf \left\{ \left( \sum_{d \mid n} F(d) \frac{d \ln d}{\ln n} \right) \middle| F \in F_t \right\}
\]
and
\[
b(t, n) = \sup \left\{ \left( \sum_{d \mid n} G(d) \frac{t/(t+1)}{\ln n} \right) \middle| G \in G_t \right\}
\]
Also, let
\[
A(t) = \inf \left\{ q(t, m) \mid m \text{ sq. free} \right\}
\]
\[
B(t) = \sup \left\{ b(t, m) \mid m \text{ sq. free} \right\}
\]

For \( F \in F_t \), we have by definition
\[
\sum_{d \mid n, \ d \geq n^{t/(t+1)}} F(d) \frac{d \ln d}{\ln n} \geq A(t) \sum_{d \mid n} F(d) \frac{d \ln d}{\ln n}
\]
Our Theorem 2 is equivalent to

\[ A(k) \approx \frac{1}{2k+2 + o(1)} \]

for integers \( k \geq 1 \).

(Note: Soundararajan has replaced \( k \) in our Theorem 2 by \( k+1 \)).

He establishes three results:

**Theorem S₁**: For all \( t \geq 0 \)

\[ A(t+1) \geq \frac{A(t)}{A(t)+1} \]

In particular,

\[ A(k) \geq \frac{1}{k+1} \quad \forall \ k \geq 0, \ k \in \mathbb{Z}. \]

**Theorem S₂**: For all \( t \geq 0 \),

\[ B(t+1) \leq \frac{1}{2-B(t)} \]

In particular,

\[ B(k) = \frac{k}{k+1} \quad \forall \ k \in \mathbb{Z}^+. \]

**Theorem S₃**: For all \( t \geq 0 \),

\[ A\left(\frac{1}{t}\right) + B(t) = 1. \]
Using Theorem 3, he extends our Theorem (more specifically the assertion \( A(k) > \frac{1}{k+1} \) in his Theorem 5_1) to rational \( k \) as follows:

**Theorem 5_4**: Let \( k > 0 \) be rational and \( k = [a_0, a_1, \ldots, a_r] \) its continued fraction expansion. Then

\[
A(k) \geq \frac{1}{1 + a_0 + a_1 + \ldots + a_r}
\]

and

\[
B(k) \leq \frac{a_0 + a_1 + \ldots + a_r}{1 + a_0 + a_1 + \ldots + a_r}
\]

**Note**: Even if we write

\[
k = [a_0, a_1, \ldots, a_{r-1}, a_r-1, 1]
\]

we have

\[
1 + a_0 + \ldots + a_{r-1} + a_r - 1 + 1 = 1 + a_0 + \ldots + a_r
\]

and so the above inequalities for \( A(k) \) and \( B(k) \) do not change.
Ritabrata Munshi (Ramanujan J., 2011) considered the problem of obtaining bounds of the type

\[ \Omega(n) \ll \sum_{d \leq n^{\delta}} \tau(d)^{1/2}, \]

where \( \tau(n) \) is the divisor function. He was motivated by applications of such inequalities in analytic number theory.

Landreau (Bull. LMS, 1989) had shown

\[ \Omega(n) \leq k/k-1 \sum_{d \leq \sqrt[1/k]{n}} \tau(d)^k \cdot \frac{1}{\ln d}, \]

(related to earlier work of Wolke (JLMS 1972))

For certain small \( k \), Friedlander & Iwaniec improved on Landreau by showing

\[ \Omega(n) \leq 9 \sum_{d \leq n^{1/3}} \tau(d) \cdot \frac{1}{\ln d}, \]

\[ \Omega(n) \leq 256 \sum_{d \leq n^{1/4}} \tau(d)^{\log_2 9} \cdot \frac{1}{\ln d}, \]

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\[ \Omega(n) \leq 256 \sum_{d \leq n^{1/4}} \tau(d)^{\log_2 9} \cdot \frac{1}{\ln d}, \]
Munshi (2011) improves these results as follows.

For \( \delta \in (0, \frac{1}{2}) \), define

\[
\beta(\delta) = \frac{\log \delta}{\log 2} + \frac{1}{\delta} \left\{ 1 + (1-\delta) \frac{\log (1-\delta)}{\log 2} \right\}
\]

Then \( \beta(\delta) \) is a strictly decreasing function of \( \delta \).

Moreover

\[
\mathcal{E}(n) \ll \delta, \beta \sum_{\delta, \beta} \tau(d)^{\beta} \text{dln}, \ d \leq n^\delta
\]

**Remark:** Munshi observes that (16) is optimal by taking \( n = p_1 p_2 \ldots p_r \), with \( p_1, p_2, \ldots, p_r \) large primes, as we did. Then

\[
\sum_{\delta, \beta} \tau(d)^{\beta} = \left( \frac{r}{r/k} \right)^{\beta r/k} \text{dln}, \ d \leq n^{1/k}
\]

This leads to the requirement

\[
\frac{\beta r}{k} - \frac{r}{k} \log_2 \left( \frac{1}{k} \right) - (r - \frac{r}{k}) \log_2 \left( 1 - \frac{1}{k} \right) > r
\]

\( \iff \beta > \beta(\delta). \)
For $S \leq \mathbb{Z}^+$, define

$$S_d(x) = \sum_{\substack{n \leq x, \ n \in S \\ n \equiv 0 \ (\text{mod} \ d)}} 1$$

Write

$$S_d(x) = \frac{x \omega(d)}{d} + R_d(x), \quad x = S_d(x),$$

where $\omega(d)$ is multiplicative.

**Assumptions:**

(i) $|R_d(x)| \ll \frac{x \omega(d)}{d}, \quad 1 \leq d \leq x^{\beta}$, for some $\beta < 1$

(ii) **Bombieri Type condition**

$$\sum_{d \leq x^{\beta} / \log x} |R_d(x)| \ll \frac{x}{\log x}$$

Next, let $f$ be a strongly additive function:

$$f(n) = \sum_{p \mid n} f(p)$$

We will focus on real valued $f$ and even $f > 0$. 
Consider

\[ A(x) = \sum_{p \leq x} \frac{f(p) \omega(p)}{p} \]  
(mean of \( f(n), n \in S \))

and

\[ B(x) = \sum_{p \leq x} \frac{|f^2(p) \omega(p)|}{p} \]  
(variance of \( f(n), n \in S \))

**Problem**: Obtain bounds for the moments

\[ \sum_{n \leq x, n \in S} |f(n) - A(x)|^2 \]

**Elliott** (Canadian J. Math. 1980) has considered and solved this problem elegantly when \( S = \mathbb{Z}^+ \).

**Reductions**: Use \( |a + b|^2 \leq |a|^2 + |b|^2 \)

So we assume \( f \geq 0 \) because, if \( f \) is real, we may write

\[ f = f^+ - f^- \]

where

\[ f^+(p) = \max(0, f(p)) \]

\[ f^-(p) = \min(0, f(p)) \]
Consider the distribution function

\[ F_x(x) = \frac{1}{X} \sum_{n \leq x, n \in S} \frac{1}{f(n) - A(x)} < \lambda \sqrt{B(x)} \]

Then

\[ \frac{1}{B(x)^{\ell/2}} \sum_{n \leq S, n \leq x} (f(n) - A(x))^\ell = \int \lambda^\ell dF_x(\lambda) \] (1)

Next consider the bilateral Laplace transform

\[ T_u(x) = \int e^{u\lambda} dF_x(\lambda) \]

If \( |T_u(x)| \ll 1 \) for \( |u| \leq R \) for some \( R > 0 \), then we can say the expression in (1) is bounded for each \( \ell \).

Note that

\[ T_u(x) = \frac{1}{X} \sum_{n \leq x, n \in S} e^{u(f(n) - A(x))/\sqrt{B(x)}} \]

\[ = \frac{1}{X} \sum_{n \leq x, n \in S} g(n) \]

where

\[ g(n) = e^{uf(n)/\sqrt{B(x)}} \] is strongly multiplicative.
Case 1: \( u \leq 0 \). Here \( 0 \leq g \leq 1 \).

I used sieve methods (Springer Lecture Notes, # 1122, (1984)) to show

\[
S(g, x) = \sum_{n \leq x, n \in S} g(n) \ll x \prod_{p \leq x} \left( 1 - \frac{(1-g(p))\omega(p)}{p} \right)
\]

uniformly for \( 0 \leq g \leq 1 \).

Case 2: \( u \geq 0 \). Here \( g \geq 1 \).

Assumption: \( \max_{p \leq x} f(p) \ll \sqrt{B(x)} \)

With \( R \) chosen sufficiently small, we can make

\[
1 \leq g(p) \leq 1 + \frac{1}{2(k-1)}
\]

\( \Rightarrow \) \( 0 \leq h(p) = g(p) - 1 \leq \frac{1}{2(k-1)} \).

Thus we will have

\[
g(n) = \sum_{d \mid n} h(d) \ll k \sum_{d \leq n^{1/k}} h(d) \frac{d \ln n}{\ln d}
\]

This will lead to the estimate

\[
S(g, x) \ll x \prod_{p \leq x} \left( 1 + \frac{h(p)\omega(p)}{p} \right)
\]
All these estimates lead to

**Theorem A**: If $f$ is strongly additive and

$$0 \leq |f(p)| \ll \sqrt{B(x)},$$

then

$$\sum |f(n) - A(x)|^l \ll x^{l/2} B(x)^{l/2}, \quad \text{for } l > 0, \ n \leq x, \ n \in S$$

This extends Elliott's result for $\mathbb{Z}$ to more general sets $S$.

**Remark**: We have indicated only bounds for moments here. The bilateral Laplace Transform approach along with sieve methods yields asymptotic estimates for the moments of additive functions in special sets of integers $S$, and this in turn leads to Erdős-kac-Kubilius type theorems. (See KA, Springer Lecture Notes, #1122 (1984), or #1375 (1989)).
Open Problems

(New) Weak Conjecture: The implicit constant in Theorem 2 is absolute.

(New) Strong Conjecture: For $k = 2, 3, \ldots$, the implicit constant in Theorem 2 is

$$\left(1 + \frac{1}{k-1}\right)^{k-1}.$$