On some classes of graphs having its geodetic number less than or equal to its Steiner number

Ismael G. Yero [1] and Juan A. Rodríguez-Velázquez [2]

Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain.

Abstract

A set of vertices S of a graph G is a geodetic set of G if every vertex $v \notin S$ lies on a shortest path between two vertices of S. A Steiner set of G is a set of vertices W of G such that every vertex of G belongs to the set of vertices of a connected subgraph of minimum size containing the vertices of W. In this work we show that if G is a graph of diameter two, then every Steiner set of G is also a geodetic set of G. Moreover, we also study some classes of graphs with diameter greater than two in which every Steiner set is a geodetic set.

Keywords: Geodetic sets, Steiner sets, corona graph.

1 Introduction

The Steiner distance of a set of vertices of a graph was introduced as a generalization of the distance between two vertices [3]. In this sense, Steiner sets in graphs could be understood as a generalization of geodetic sets in graphs. Nevertheless, its relationship is not exactly obvious. Some of the primary results in this topic were presented in [4], where the authors tried to show that

[1] Email: `ismael.gonzalez@urv.cat`
[2] Email: `juanalberto.rodriguez@urv.cat`

every Steiner set of a graph is also a geodetic set. Fortunately, the author of [9] showed by a counterexample that not every Steiner set of a graph is a geodetic set, and it was pointed out an open question related to characterizing those graphs satisfying that every Steiner set is geodetic or vice versa. Some relationships between Steiner sets and geodetic sets were obtained in [1,2,4,7,8,9]. For instance, [2] was dedicated to obtain some families of graphs in which every Steiner set is a geodetic set, but the problem of characterizing such a graphs remains open.

We begin by stating some terminology and notation. In this paper $G = (V, E)$ denotes a connected simple graph of order $n = |V|$. The diameter $D(G)$ of G is the maximum among all distances between any two vertices of G.

A shortest $u - v$ path is called $u - v$ *geodesic*. We define $I_G[u, v]$ to be the set of all vertices lying on some $u - v$ geodesic of G, and for a nonempty set $S \subseteq V$, $I_G[S] = \bigcup_{u,v \in S} I_G[u, v]$ ($I[S]$ for short). A set $S \subseteq V$ is a *geodetic set* of G if $I_G[S] = V$ and a geodetic set of minimum cardinality is called a *minimum geodetic set* [6]. The cardinality of a minimum geodetic set of G is called the *geodetic number* of G and it is denoted by $g(G)$. A vertex $v \in V$ is *geodominated* by a pair $x, y \in V$ if v lies on an $x - y$ geodesic of G.

For an integer $k \geq 2$, a vertex v of a graph G is *k-geodominated* by a pair x, y of vertices in G if $d(x, y) = k$ and v lies on an $x - y$ geodesic of G. A subset $S \subseteq V$ is a *k-geodetic set* if each vertex v in $\overline{S} = V - S$ is k-geodominated by some pair of vertices of S. The minimum cardinality of a k-geodetic set of G is its *k-geodetic number* $g_k(G)$. It is clear that $g(G) \leq g_k(G)$ for every k.

For a nonempty set W of vertices of a connected graph, the *Steiner distance* of W is the minimum size of a connected subgraph of G containing W [3]. Necessarily, such a subgraph is a tree and it is called a *Steiner tree* with respect to W or a Steiner W-tree, for short. For a set $W \subseteq V$, the set of all vertices of G lying on some Steiner W-tree is denoted by $S_G[W]$ (or by $S[W]$, if there is no ambiguity). If $S_G[W] = V$, then W is called a Steiner set of G. The *Steiner number* of a graph G, denoted by $s(G)$, is the minimum cardinality among the Steiner sets of G.

2 Results

We begin by proving that every Steiner set of a graph with diameter two is also a geodetic set.

Theorem 2.1 *If G is a graph of diameter two, then every Steiner set for G is a geodetic set for G.*

Corollary 2.2 *If G is a graph of diameter two, then $g(G) \leq s(G)$.*

Next we will present some results about geodetic sets and Steiner sets of corona product graphs in order to show that in every corona product graph it is satisfied that every Steiner set is also a geodetic set. Notice that, there are infinite number of corona product graphs with diameter greater than two.

Let G and H be two graphs and let n be the order of G. The corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and then joining by an edge, all the vertices from the i^{th}-copy of H with the i^{th}-vertex of G.

2.1 Geodetic number of corona product graphs

Proposition 2.3 *Let G be a connected graph of order n_1 and let H be a graph of order n_2. If $n_1 \geq 2$ or ($n_1 = 1$ and H is a non-complete graph), then*

$$n_1 g(H) \leq g(G \odot H) \leq n_1 n_2.$$

The upper bound is achieved if and only if H is isomorphic to a graph in which every connected component is isomorphic to a complete graph.

Moreover, if no connected component of H is isomorphic to a complete graph, then $g(G \odot H) \leq n_1(n_2 - 1)$.

Theorem 2.4 *Let G be a connected graph of order n and let H be a non-complete graph. Then, $g(G \odot H) = ng(K_1 \odot H)$.*

The geodetic number of wheel graphs and fan graphs were studied in [2] and [5].

Remark 2.5 [2] If $n \geq 4$, then $g(W_{1,n}) = \left\lceil \frac{n}{2} \right\rceil$.

Remark 2.6 [2,5] If $n \geq 3$, then $g(F_{1,n}) = \left\lceil \frac{n+1}{2} \right\rceil$.

As a particular cases of Theorem 2.4 and by using the above remarks we obtain the following results.

Corollary 2.7 *Let G be a connected graph of order n_1.*

(i) *If $n_2 \geq 4$, then $g(G \odot C_{n_2}) = n_1 g(W_{1,n_2}) = n_1 \left\lceil \frac{n_2}{2} \right\rceil$.*
(ii) *If $n_2 \geq 3$, then $g(G \odot P_{n_2}) = n_1 g(F_{1,n_2}) = n_1 \left\lceil \frac{n_2+1}{2} \right\rceil$.*

Now we are interested in those graphs in which $g(H) = g(K_1 \odot H)$.

Theorem 2.8 *For a connected graph H, the following statements are equivalent:*

- $g(H) = g(K_1 \odot H)$.
- $g(H) = g_2(H)$.

Theorem 2.9 *Let G be a connected graph of order n and let H be a connected non-complete graph. Then the following statements are equivalent:*

- $g(G \odot H) = ng(H)$.
- $g(H) = g_2(H)$.

Since for every graph H of diameter two we have $g(H) = g_2(H)$, Theorem 2.9 leads to the following result.

Corollary 2.10 *Let G be a connected graph of order n and let H be a graph. If $D(H) = 2$, then $g(G \odot H) = ng(H)$.*

Another consequence of Theorem 2.8 is the following result.

Corollary 2.11 *Let G and H be two connected graphs of order n_1 and n_2, respectively. Let N_k be the empty graph of order $k \geq 2$. Then $g(G \odot (H \odot N_k)) = n_1 n_2 k$.*

The following result improves the lower bound in Proposition 2.3 for those graphs whose geodetic number is different from its 2-geodetic number.

Theorem 2.12 *Let G be a connected graph of order n and let H be a non-complete graph. If $g(H) \neq g_2(H)$, then $g(G \odot H) \geq n\left(g(H) - 1\right)$.*

2.2 Steiner number of corona product graphs

Proposition 2.13 *Let G be a connected graph of order n_1 and let H be a graph of order n_2. If $n_1 \geq 2$ or ($n_1 = 1$ and H is a non-complete graph), then*

$$n_1 s(H) \leq s(G \odot H) \leq n_1 n_2.$$

The upper bound is achieved if and only if H is isomorphic to a graph in which every connected component is isomorphic to a complete graph.

Moreover, if no connected component of H is isomorphic to a complete graph, then $s(G \odot H) \leq n_1(n_2 - 1)$.

Note that an example of corona graph where $s(G \odot H) = n_1(n_2 - 1)$ is showed in Corollary 2.17.

Theorem 2.14 *Let G be a connected graph of order $n \geq 2$ and let H be any non complete graph. Then, $s(G \odot H) = ns(K_1 \odot H)$.*

Notice that the above theorem leads to the lower bound of Proposition 2.13.

The Steiner number of wheel graphs and fan graphs were studied in [2] and [5].

Remark 2.15 [2] If $n \geq 4$, then $s(W_{1,n}) = n - 2$.

Remark 2.16 [2,5] If $n \geq 3$, then $g(F_{1,n}) = n - 1$.

As a particular cases of Theorem 2.14 and by using the above remarks we obtain the following results.

Corollary 2.17 *Let G be a connected graph of order $n_1 \geq 2$. Then,*

(i) $s(G \odot N_{n_2}) = n_1 s(S_{1,n_2}) = n_1 n_2$.

(ii) *If $n_2 \geq 4$, then $s(G \odot C_{n_2}) = n_1 s(W_{1,n_2}) = n_1(n_2 - 2)$.*

(iii) *If $n_2 \geq 3$, then $s(G \odot P_{n_2}) = n_1 s(F_{1,n_2}) = n_1(n_2 - 1)$.*

Theorem 2.18 *Let H be a connected non complete graph. Then the following statements are equivalent:*

- $s(K_1 \odot H) = s(H)$.
- $D(H) = 2$.

By Theorem 2.14 and Theorem 2.18 we obtain the following result.

Theorem 2.19 *Let H be a connected non complete graph and let G be a graph of order n. Then the following statements are equivalent:*

- $s(G \odot H) = ns(H)$.
- $D(H) = 2$.

2.3 Relationships between the geodetic number and the Steiner number of corona graphs

Now, from Theorem 2.4, Theorem 2.14 and Corollary 2.2 we obtain the following interesting result in which we give an infinite number of graphs G satisfying that $g(G) \leq s(G)$.

Theorem 2.20 *Let G be a connected graph of order $n \geq 2$ and let H be any non complete graph. Then, $g(G \odot H) \leq s(G \odot H)$.*

As a consequence of almost all the previous results we obtain the following theorem on corona product graphs whose second factor has diameter two.

Theorem 2.21 *Let G be a connected graph and H be any graph of diameter two. Then the following statements are equivalent:*

- $s(G \odot H) = g(G \odot H)$.
- $s(H) = g(H)$.

Acknowledgements

This work was partly supported by the Spanish Ministry of Science and Education through the project MTM2011-25189.

References

[1] G. Chartrand, F. Harary, P. Zhang, Extremal problems in geodetic graph theory, *Congressus Numerantium* **130** (1998) 157–168.

[2] C. Hernando, T. Jiang, M. Mora, I. M. Pelayo, Carlos Seara, On the Steiner, geodetic and hull numbers of graphs, *Discrete Mathematics* **293** (1–3) (2005) 139–154.

[3] G. Chartrand, O.R. Oellermann, S. Tian, H. B. Zou, Steiner distance in graphs, *Časopis pro Pěstování Matematiky* **114** (1989) 399–410.

[4] G. Chartrand, P. Zhang, The Steiner number of a graph, *Discrete Mathematics* **242** (2002) 41–54.

[5] R. Eballe, S. Canoy, Jr., Steiner sets in the join and composition of graphs, *Congressus Numerantium* **170** (2004) 65–73.

[6] F. Harary, E. Loukakis, C. Tsouros, The geodetic number of a graph, *Mathematical and Computer Modelling* **17** (11) (1993) 89–95.

[7] L. D. Tong, Geodetic sets and Steiner sets in graphs, *Discrete Mathematics* **309** (2009) 4205–4207.

[8] O. R. Oellermann, M. L. Puertas, Steiner intervals and Steiner geodetic numbers in distance-hereditary graphs, *Discrete Mathematics* **307** (1) (2007) 88–96.

[9] I. M. Pelayo, Comment on "The Steiner number of a graph" by G. Chartrand and P. Zhang [*Discrete Mathematics* **242** (2002) 41–54], *Discrete Mathematics* **280** (2004) 259–263.