Power domination in some classes of graphs

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Abstract

The problem of monitoring an electric power system by placing as few phase measurement units (PMUs) in the system as possible is closely related to the well-known domination problem in graphs. The power domination number $\gamma_p(G)$ is the minimum cardinality of a power dominating set of G. In this paper, we investigate the power domination problem in Mycielskian and generalized Mycielskian of graphs. We also find the power domination number of direct product of some classes of graphs.

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1 Introduction

A dominating set of a graph $G = (V, E)$ is a set of vertices S such that every vertex (node) in $V \backslash S$ has at least one neighbor in S. The problem of finding a dominating set of minimum cardinality is an important problem that has been extensively studied. The minimum cardinality of a dominating set of G is its domination number, denoted by $\gamma(G)$. Our focus is on a variation called the power dominating set (PDS) problem.

The power domination problem arose in the context of monitoring electric power networks. A power network contains a set of nodes and a set of edges connecting the nodes. It also contains a set of generators, which supply power, and a set of loads, where the power is directed to. In order to monitor a power network we need to measure all the state variables of the network by placing measurement devices. A Phase Measurement Unit (PMU) is a measurement device placed on a node that has the ability to measure the voltage of the node and current phase of the edges connected to the node and to give warnings of system-wide failures. The goal is to install the minimum number of PMUs such that the whole system is monitored.

This problem has been formulated as a graph domination problem by Haynes et al. in [5]. However, this type of domination is different from the standard domination type problem, since the domination rules can be iterated. The propagation rules are derived from the Ohm's and Kirchoff's laws for an electric circuit.

Now, let the graph $G = (V, E)$ represent an electric power system, where a vertex represents an electrical node and an edge represents a transmission line joining two electrical nodes. In the following, we use the notations as in [3]. For a vertex v of G, let $N(v)$ denote the open neighborhood of v, and for a subset S of $V(G)$, let $N(S) = \bigcup_{v \in S} N(v) - S$. The closed neighborhood $N[S]$ of a subset S is the set $N[S] = N(S) \bigcup S$.

For a connected graph G and a subset $S \subseteq V(G)$, let $M(S)$ denote the *set monitored by S*. It is defined recursively as follows:
1.*Domination step*
$M(S) \leftarrow S \bigcup N(S)$
2.*Propagation step*
As long as there exists $v \in M(S)$ such that $N(v) \bigcap (V(G) - M(S)) = \{w\}$

set $M(S) \leftarrow M(S) \bigcup \{w\}$.
A set S is called a power dominating set of G if $M(S) = V(G)$ and the power domination number of G, $\gamma_p(G)$, is the minimum cardinality of a power dominating set of G. For any graph G, $1 \leq \gamma_p(G) \leq \gamma(G)$.

The problem of deciding if a graph G has a power dominating set of cardinality k has been shown to be NP-complete even for bipartite graphs, chordal graphs [5] or even split graphs [8]. On the other hand, the problem has efficient solutions on trees [5], as well as on interval graphs [8] . Some upper bounds for $\gamma_p(G)$ are discussed in [11] and [12].

The power domination number and minimal power dominating sets for grid graphs were obtained by Dorfling and Henning in [4]. In [3], Dorbec et al. determined $\gamma_p(G)$ for all direct product of paths except for the odd component of the direct product of two odd paths. They also computed the same for the strong product of paths and the lexicographic product of any two graphs. Later, Barrera and Ferrero obtained upper bounds for $\gamma_p(G)$ when G is a cylinder, a torus, or a generalized Petersen graph, and identified many cases where their bounds coincide with the power domination number [2].

In this paper, we investigate the power domination problem in Mycielskian of graphs. We also show that the power domination number of generalized Mycielskian of even path is one. We compute the power domination number of the direct product of some classes of graphs.

All the graphs considered here are undirected, finite and simple. For all basic concepts and notations not mentioned in this paper, we refer [10].

Definition 1.1 For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with vertex set $V \bigcup V' \bigcup z$, where $V' = \{u' : u \in V\}$, and edge set $E \bigcup \{uv' : uv \in E\} \bigcup \{v'z : y' \in V'\}$. The vertex u' is called the twin of the vertex u (and u the twin of u') and the vertex z is called the root of $\mu(G)$.

Definition 1.2 Let G be a graph with vertex set $V^0 = \{v_1^0, v_2^0 \dots v_n^0\}$ and edge set E^0. Given an integer $m \geq 1$ the $m-$Mycielskian of G denoted by $\mu_m(G)$, is the graph with vertex set $V^0 \bigcup V^1 \bigcup V^2 \bigcup \dots \bigcup V^0 \bigcup \{u\}$, where $V^i = \{v_j^i : v_j^0 \in V^0\}$ is the ith distinct copy of V^0 for $i = 1, 2, \dots m$ and edge set $E^0 \cup \left(\bigcup_{m-1}^{i=0} \{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0\} \right) \cup \{v_j^m u : v_j^m \in V^m\}$.

Some interesting investigations on Mycielskian and generalized Mycielskian of graphs are in [6], [1] and [9].

Definition 1.3 The *direct product* of graphs G and H, denoted by $G \times H$, have the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \times H$ if they are adjacent in both the coordinates.

Definition 1.4 For any vertex $g \in G$, the subgraph of $G \times H$ induced by $\{g\} \times V(H)$ is called the $H - fiber$ at g and denoted by ${}^gH$. For any vertex $h \in H$, the subgraph of $G \times H$ induced by $V(G) \times \{g\}$ is called the $G - fiber$ at h and denoted by G^h.

The direct product is associative and commutative. The fibers induce edgeless graphs. For more information on the direct product we refer the reader to [7].

2 Power domination in the Mycielskian of a graph

In this section we shall investigate the power domination number of the Mycielskian of a graph. It is proved in [6] that, $\gamma(\mu(G)) = \gamma(G) + 1$. But we have that, $\gamma_p(\mu(G)) \leq \gamma_p(G) + 1$. The upper bound is attained for $G \cong C_n; n \geq 4$. We shall obtain a sufficient condition which improves the bound for $\gamma_p(\mu(G))$, using the following concepts introduced in [8].

Let H be an induced subgraph of G. The *out-degree* of $v \in H$ is the number of vertices in $G \backslash H$ adjacent to v and the edge vw connecting a vertex $v \in H$ and $w \notin H$ is called an outgoing edge. The set $S_0 \subseteq V(G)$ is referred to as the *kernel*. The set of vertices which are directly dominated by S_0 form the *first generation descendants* or *1-descendants*, denoted by S_1 and the subgraph induced by $S_0 \bigcup S_1$ is called the *derived kernel of first generation*. The *ith generation descendants* or *i-descendants*, S_i are those vertices which are monitored by propagation from S_{i-1}.

Theorem 2.1 *If G has a minimum power dominating set in which every vertex has a neighbor of outdegree one in S_1, then $\gamma_p(\mu(G)) \leq \gamma_p(G)$.*

Theorem 2.2 *Let $V(\mu(G)) = V \bigcup V' \bigcup \{z\}$ and $S'_0 \subseteq V'$ with $|S'_0| > 1$. Then S'_0 is a power dominating set of $\mu(G)$ if and only if G is a disjoint union of K_2s.*

Theorem 2.3 *Let G be a connected graph with $\gamma_p(\mu(G)) \leq \gamma_p(G)$ and $\gamma_p(\mu(G)) \neq 1$. Then $\gamma_p(\mu(G)) = \gamma_p(G)$.*

Corollary 2.4 *If G is a connected graph, then*
$\gamma_p(\mu(G)) \in \{1, \gamma_G, \gamma_p(G) + 1\}$.

Theorem 2.5 *If G has an universal vertex, then $\gamma_p(\mu(G)) = 1$.*

It follows that the power domination number of the Mycielskian of the complete graph, the wheel, the n-fan and the n-star is equal to one. If P_n is the path $\langle v_1 v_2 \ldots v_n \rangle$, then v_2 is a power dominating set for $\mu(P_n)$. Hence, $\gamma_p(\mu(P_n)) = 1$ and the condition of Theorem 2.5 is not necessary.

In the following theorem, we show that the difference between $\gamma_p(G)$ and $\gamma_p(\mu(G))$ can be arbitrarily large.

Theorem 2.6 *For every $n \geq 1$, there exists graphs with $\gamma_p(G) = n$ and $\gamma_p(\mu(G)) = 1$.*

3 Generalized Mycielskian of even paths

In this section we show that the power domination number of generalized Mycielskian of even path is one and hence they form a suitable structure for electrical networks.

Theorem 3.1 *For an even integer n, $\gamma_p(\mu_m(P_n)) = 1$.*

4 The direct product

Theorem 4.1 *$\gamma_p(K_m \times K_n) = 2$, for $m + n > 5$.*

Theorem 4.2 *Let $m \geq 3$, $n \geq 4$ and $G = K_m \times C_n$. Then*
$\gamma_p(G) = 2k$, if $n = 4k$
$\gamma_p(G) = 2k + 1$, if $n = 4k + 1$
$\gamma_p(G) = 2k + 2$, if $n = 4k + 2$ or $n = 4k + 3$.

Remark 4.3 When m is even, $C_m \times P_n$ is a disconnected graph with two isomorphic components. When m is odd $C_m \times P_n$ is a connected graph.

Theorem 4.4 *Let n be an even integer and $G = C_m \times P_n$. Then*
$\gamma_p(G) = 2\left\lceil \frac{n}{3} \right\rceil$*, if m is even.*
$\gamma_p(G) = \left\lceil \frac{n}{3} \right\rceil$*, if m is odd.*

Theorem 4.5 *If G and H are graphs with two universal vertices each, then* $\gamma_p(G \times H) \leq 2$.

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