On root vertices of rooted HISTs of planar triangulations

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Abstract

A spanning tree with no vertex of degree two of a graph is called a *homeomorphically irreducible spanning tree* (or *HIST*) of the graph. It has been proved that every planar triangulation with at least four vertices has a HIST [1]. Moreover, in [7], it has been proved that if a planar triangulation G has $2n$ ($n \geq 2$) vertices, then, for any vertex v of G, G has a HIST H such that the degree of v in H is at least three. We call such a HIST a *rooted HIST of G with the root v*. In this paper, we characterize v of G such that G has a rooted HIST with the root v.

Keywords: HIST, homeomorphically irreducible spanning tree, triangulation.

1 Introduction

Let G be a graph and let H be a subgraph of G. If H contains all vertices of G, then H is called a *spanning subgraph* of G. (If H is a tree, then it is called

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a *spanning tree* of G.) It is a fundamental problem deciding whether a graph has particular types of spanning subgraph in graph theory. For example, in the Hamiltonian path problem, we seek a spanning subgraph with all but two vertices of degree two. In this paper, we search "homeomorphically irreducible spanning trees", a class antithetical to Hamiltonian paths.

A graph is said to be *homeomorphically irreducible* if it has no vertices of degree two. Let T be a spanning tree of a graph G. If T has no vertex of degree two, then T is called a *homeomorphically irreducible spanning tree* (or *HIST*) of G. For example, the octahedron has a HIST with two vertices of degree three and four vertices of degree one. (See Figure 1.) Joffe has constructed infinite families of 4-regular, 3-connected planar graphs that have no HISTs [5]. Albertson, Berman, Hutchinson, and Thomassen have shown that it is an NP-complete problem deciding whether a graph contains a HIST [1]. So, we consider the problem restricted to "triangulations".

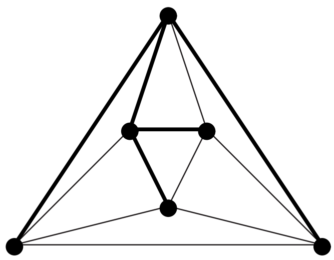

Fig. 1. A HIST on the octahedral graph.

A *triangulation* on a surface is a simple graph embedded in the surface such that each face is bounded by a 3-cycle, where a *k-cycle* means a cycle of length k. A *near triangulation* R is a graph on the plane with boundary cycle of length $m \geq 3$ such that each face of R is triangular except for the outer face. Hill conjectured that every planar triangulation other than the complete graph K_3 with three vertices has a HIST [4]. Malkevitch extended Hill's conjecture to near triangulations [6]. For their conjectures, Albertson, Berman, Hutchinson, and Thomassen have proved the following.

Theorem 1.1 [Albertson, Berman, Hutchinson, and Thomassen [1]] *Every near triangulation with at least four vertices has a HIST.*

Moreover, they extend Hill's conjecture to all triangulations on any surface, i.e., "*every triangulation on any surface with at least four vertices has a HIST*". For this conjecture, Davidow, Hutchinson and Huneke have proved that every toroidal triangulation has a HIST [2]. In [3], Fiedler, Huneke, Richter and Robertson have proved that every projective planar triangulation has a near triangulation as a spanning subgraph. By their result and Theorem 1.1, it has already been proved that every triangulation on the projective plane has a HIST. These results assert nothing whether the degree of a fixed vertex of a graph is at least three or not in a HIST of the graph. So, new notions called "rooted HISTs" and "rooted near 1-HISTs" have been proposed in [7].

Let G be a graph and let $|G|$ denote the number of vertices of G. Fix a vertex v of G such that the degree of v in G, denoted by $d_G(v)$, is at least three (resp., two). If G has a spanning tree T such that $d_T(v) \geq 3$ (resp., $d_T(v) \geq 2$) and that $d_T(u) \neq 2$ where $u \neq v$, then T is said to be a *rooted HIST* (resp., *rooted near* 1-*HIST*) of G with the root v. (Note that a rooted HIST is a HIST, but a rooted near 1-HIST might not be a HIST.) For any vertex v of G, where $d_G(v) \geq 3$ (resp., $d_G(v) \geq 2$), if G has a rooted HIST (resp., rooted near 1-HIST) with the root v, then we call that G satisfies the *rooted HIST property* (resp., *rooted near* 1-*HIST property*). For these properties of near triangulations, the following has already been proved.

Theorem 1.2 [Tsuchiya [7]] *Let R be a near triangulation.*

(i) If $|R| = 2n-1$ $(n \geq 2)$, then R satisfies the rooted near 1-*HIST property.*

(ii) If $|R| = 2n$, then R satisfies the rooted HIST property.

By the definitions of rooted HISTs and rooted near 1-HISTs, if a graph G has no vertex of degree at most two and satisfies the rooted HIST property, then G satisfies the rooted near 1-HIST property. Since every planar triangulation with at least four vertices has no vertex of degree at most two, Theorem 1.2 implies the following.

Corollary 1.3 *Every planar triangulation satisfies the rooted near* 1-*HIST property.*

Does planar triangulation satisfy the rooted HIST property? Unfortunately, the answer is no since there exist planar triangulations not satisfying the rooted HIST property. In this paper, we characterize a vertex v of a planar triangulation G such that G has a rooted HIST with the root v. Let $G - v$ denote a near triangulation obtained from G by removing v and edges incident to v. The following is our main result.

Theorem 1.4 *Let G be a planar triangulation and let v be a vertex of G. Then, G has a rooted HIST with the root v if and only if*

(i) $d_G(v) \neq 2$ and

(ii) $G - v$ is neither K_4 nor the octahedron when $d_G(v) = 3$.

Since Theorem 1.4 also characterizes planar triangulations satisfying the rooted HIST property, it implies the following.

Corollary 1.5 *A planar triangulation G satisfies the rooted HIST property if and only if G is not a triangulation shown in Figure 2*

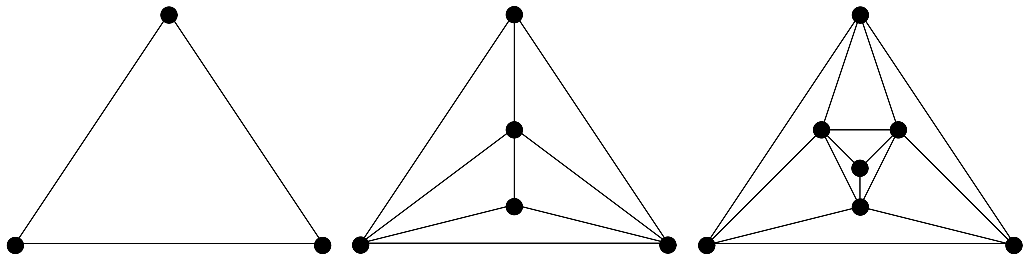

Fig. 2. Planar triangulations not satisfying the rooted HIST property.

2 Lemmas

In this section, we prove two lemmas in order to prove Theorem 1.4. Let G be a graph and let $S = \{v_1, v_2, \cdots, v_k\}$ be a subset of vertices of G. Let $G - S$ denote a graph obtained from G by removing each v_i and edges incident to v_i. If $G - S$ is not connected, then S is said to be a *k-separator* of G.

Let G be a planar triangulation and let v be a vertex of G. By Theorem 1.2, if $|G| = 2n$, then G has a rooted HIST with the root v. Therefore, we must consider the case when $|G| = 2n - 1$.

Lemma 2.1 *Let G be a planar triangulation with $2n-1$ vertices and let v be a vertex of G. If $d_G(v) \geq 4$, then G has a rooted HIST with the root v.*

Proof. Since $d_G(v) \geq 4$, we may suppose $|G| \geq 5$ (i.e., $n \geq 3$). Let u be a vertex of G adjacent to v. Note that $G' = G - u$ is a near triangulation. Since $|G| = 2n-1$ and $d_G(v) \geq 4$, $|G'| = 2n-2$ and $d'_G(v) \geq 3$. So, by Theorem 1.2, G' has a rooted HIST H' with the root v. Moreover, $H' \cup vu$ is a rooted HIST of G with the root v. □

Lemma 2.2 *Let G be a planar triangulation with $2n-1$ vertices and let v be a vertex of G such that $d_G(v) = 3$. If $G - v$ is neither K_4 nor the octahedron, then G has a rooted HIST with the root v.*

Proof (Sketch) Since $|G| = 2n - 1$ and $d_G(v) = 3$, we may suppose $|G| \geq 5$ (i.e., $n \geq 3$). Let x, y and z be vertices adjacent to v and let S denote a 3-separator of G whose vertices are x, y and z. (Note that the subgraph of

G induced by $\{v, x, y, z\}$ is K_4 since G is a triangulation.) Without loss of generality, we may suppose $d_G(x) \geq d_G(y) \geq d_G(z)$.

We consider two cases. One is the case when G has a 3-separator other than S, and the other is that when G has no 3-separator other than S. The former case, let $S' = \{p, q, r\}$ be a 3-separator of G where $p \notin \{x, y, z\}$. In this case, we prove that $G - p$ has a rooted HIST H with the root v and that we obtain a rooted HIST of G with the root v from H by adding a suitable edge adjacent to p. The latter case, we prove that $G - \{v, y, z\}$ has a rooted HIST H with the root x and that we obtain a rooted HIST of G with the root v from H by adding three edges vx, vy and vz. □

3 Proof of the main result

Proof of Theorem 1.4 Let G be a planar triangulation and let v be a vertex of G. If $d_G(v) = 2$, then G is K_3, and hence G has no rooted HIST with the root v. If $d_G(v) = 3$ and $G - v$ is K_4, then let u be a vertex of G such that $d_G(u) = 3$ where $u \neq v$. In this case, if G has a rooted HIST H with the root v, then each edge incident to v is contained in H. Observe that the degree of each vertex adjacent to v must be one in H. (Otherwise, H has a cycle.) So, $u \notin H$, a contradiction. Therefore, G has no rooted HIST with the root v. By similar arguments, if $d_G(v) = 3$ and $G - v$ is the octahedron, then G has no rooted HIST with the root v, and the necessity holds

So, we consider the sufficiency. By Theorem 1.2, if $|G| = 2n$, then G has a rooted HIST with the root v. Therefore, we may suppose $|G| = 2n - 1$. Since $d_G(v) \neq 2$ and $G - v$ is neither K_4 nor the octahedron when $d_G(v) = 3$, we consider two cases. One is the case when $d_G(v) \geq 4$, and the other is that when $d_G(v) = 3$ and $G - v$ is neither K_4 nor the octahedron. The former case, G has a rooted HIST with the root v by Lemma 2.1, and the latter case, G has a rooted HIST with the root v by Lemma 2.2. □

4 Conclusion

Theorem 1.4 characterizes a vertex v of a planar triangulation G such that G has a rooted HIST with the root v. Moreover, this result also characterizes planar triangulations satisfying the rooted HIST property (Corollary 1.5). When we consider the existence of a HIST of a triangulation on a surface, we often consider spanning subgraph of it which is a planar graph. For example, in [2], Davidow, Hutchinson and Huneke have proved that a triangulation G

on the torus has a triangulation G' on the annulus as a spanning subgraph, and that G' has a HIST. (Note that a triangulation on the annulus is a planar graph.) Therefore, our main result might be useful when we prove the conjecture that every triangulation on a surface with at least four vertices has a HIST [1].

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