Combinatorics of coloured hard-dimers

Maria Simonetta Bernabei and Horst Thaler
Department of Mathematics, University of Camerino
E-mail: simona.bernabei@unicam.it, horst.thaler@unicam.it

Outline

The aim is to get information on the numbers of hard-dimer configurations. We define a coloured hard-dimer configuration (CHDC) to be a finite sequence σ_n, of n blue and red vertices, together with coloured dimers on σ_n which must not intersect. Here a coloured dimer is an edge connecting two nearest vertices of the same colour, see Figure 1. The dimer's colour is given by the colour of its boundary vertices.

Fig. 1. A coloured hard-dimer configuration (n=12) with 2 blue dimers, 1 red dimer and 3 inner vertices.

1) On the one hand we are interested in the asymptotic behaviour of CHDCs when for given n all σ_n's are considered. For this it is useful to consider the variables $n_b(D), n_r(D)$ and $n_{br}(D)$, which count for each CHDC D, the number of blue dimers, red dimers and inner vertices, respectively. If the space of CHDCs is equipped with a uniform probability measure, the variables just defined become random variables (r.v.s) whose joint generating function can be written down explicitly. We obtain an explicit expression for the joint mass function of the r.v.s $(n_b + n_r, n_b + n_r + \gamma_b + \gamma_r)$, where γ_b, γ_r indicate the numbers of blue and red vertices which are not occupied by dimers.

2) On the other hand we treat the enumeration problem for given but arbitrary σ_n. More precisely, suppose we are given a triple of nonnegative integers (i, j, l), where i, j count the numbers of blue, red dimers and l the number of inner vertices. We would like to know how many CHDCs of a given type (i, j, l) there are on σ_n.

Central Limit Theorem

The combinatorial generating function of the variables n_b, n_r, n_{br} is defined by

$$\tilde{F}_n = \sum_{\sigma_n} F_n(u, v, w),$$

where

$$F_n(u, v, w) = \sum_{\text{CHDCs on } \sigma_n} u^{n_b(D)} v^{n_r(D)} u^{n_{br}(D)}, \quad u, v, w \in \mathbb{R}.$$

Elementary combinatorial considerations give the following expression

$$\tilde{F}_n(u, v, w) =$$

$$2^n \left(1 + \sum_{t=1}^{N} \sum_{s=1}^{[\frac{t}{2}]} \binom{n-t+s}{s} \binom{t-s-1}{s-1} \left(\frac{u+v}{4}\right)^s \left(\frac{w}{2}\right)^{t-2s}\right), \quad (1)$$

where $t = n - \gamma_b(D) - \gamma_r(D),\ s = n_b(D) + n_r(D)$.

Upon performing the change of variable $k = n - t + s$, one finds by (1) and an appropriate normalization, the corresponding probability mass function

$$\tilde{P}_n(s, k) = P_n(n_b + n_r = s; n_b + n_r + \gamma_b + \gamma_r = k)$$

with

$$\tilde{P}_n(s, k) = \begin{cases} \left(\frac{2}{3}\right)^{n-1}, & \text{for } s = 0 \\ \binom{k}{s}\binom{n-k-1}{s-1}\left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right)^{n-k}, & \text{otherwise.} \end{cases} \quad (2)$$

Here, P_n is the uniform probability measure on the set of CHDCs of length n.

Summing over s in (2) shows that $n_b + n_r + \gamma_b + \gamma_r$ is a binomial r.v. Although the distribution of $n_b + n_r$ is not of common type its mean and variance can be calculated using conditional expectations.

Since (2) involves binomial coefficients it is possible, like in De Moivre Laplace's theorem, to show the following central limit theorem

$$\lim_{n\to\infty} \sum_{\substack{s:\, x\in(a,b)\\ k:\, y\in(a',b')}} \tilde{P}_n(s, k) = \frac{1}{2\pi\sqrt{\frac{2}{3}}} \int_a^b \int_{a'}^{b'} e^{-\frac{3}{4}\left(z^2 + z'^2 + \frac{2\sqrt{3}}{3} z z'\right)} dz dz',$$

where

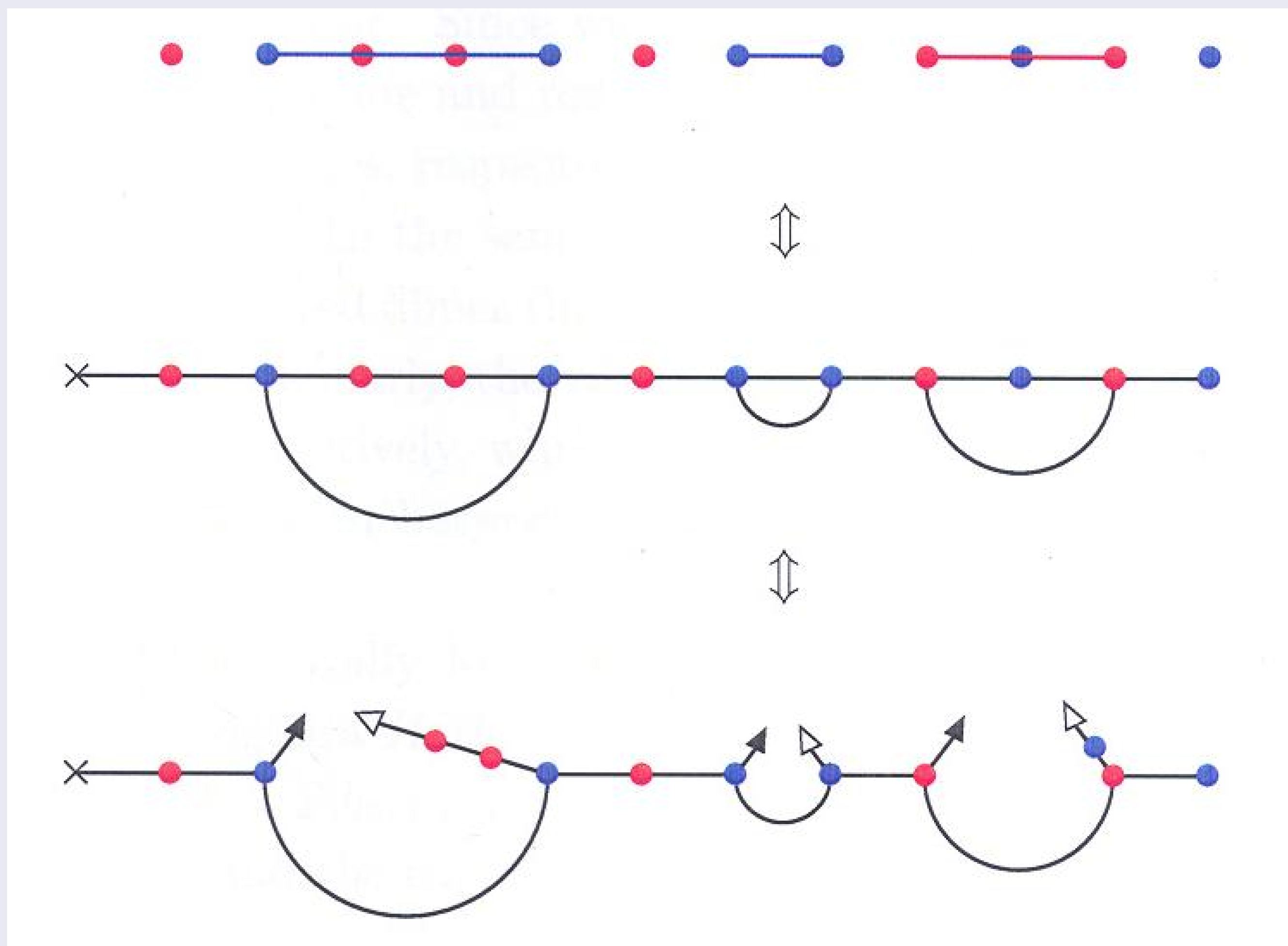
$$x = \frac{s - (2/9)n}{\sqrt{6n}/9}, \quad y = \frac{k - (2/3)n}{\sqrt{2n}/3}.$$

References

1. M. S. Bernabei, H. Thaler: Central limit theorem for coloured hard-dimers, Journal of Probability and Statistics, Volume 2010, Article ID 781681, 13 pages (2010).
2. M. S. Bernabei, H. Thaler: A noncommutative enumeration problem, (arXiv:1004.1738)

Problem 2

This problem can be naturally treated by means of noncommutative formal power series. The basic idea, as indicated in the figure below, is to establish bijections between CHDCs and a certain class of connected graphs, which in turn are identified with a particular class of rooted trees involving bud(black)-leaf(white) pairs.



For example, the tree above can be encoded into the monomial

$$b_3^2 r_3 y^3 r * b * r * r * b * r * b * b * r * b * r * b,$$

where r, b are treated as noncommutative variables. The commutative variables b_3 and r_3 are assigned to those trivalent blue and red vertices, respectively, which possess a bud leg pair as subtree. Finally, a commutative y is assigned to every bivalent vertex sitting in between a three vertex and a leaf. In this manner each CHDC on σ_n characterized by (i, j, l) is counted once.
It becomes thus natural to consider the set of trees as elements of the noncommutative algebra $K\langle\langle b, r\rangle\rangle$, where $K = \mathbb{Z}[b_3, r_3, y]$. The concrete polynomials S_b, S_r, standing for the sets of trees starting with blue or red vertices, can be found to satisfy a linear system of equations, which can be solved explicitly.

Moreover we could show that $S = S_b + S_r = \sum_x c_x x$ is a recognizable series in the sense that there is a representation in terms of matrices $B, R \in K^{3\times 3}$ and tuples λ, γ such that

$$c_x = \lambda^\top \cdot \mu(x) \cdot \gamma.$$

Owing to the latter result and using arguments from ergodic theory we have verified that the number of CHDCs grows exponentially and that asymptotically this growth rate is almost surely the same.