On Graceful Digraphs [1]

S. M. Hegde [2]

Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka Surathkal
Mangalore, India

Shivarajkumar [3]

Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka Surathkal
Mangalore, India

Abstract

In 1994, Du and Sun conjectured that, for any positive even n and any integer m, the digraph $n.\overrightarrow{C_m}$ is graceful. In this paper we prove the conjecture. Also, in 1985, Bloom and Hsu mentioned that the question remains open for the non generalized graceful labelings. For example, the following question is currently unanswered: How many distinct graceful labelings does a designated graceful digraph have? In this paper, we give an upper bound for the distinct graceful labeling of unidirectional cycle $\overrightarrow{C_n}$.

Keywords: System of simultaneous congruences, Partitions.

---

[1] The work reported in this paper is a part of the work done under the project No.SR/S4/MS-425/2007 funded by the Department of Science and Technology (DST) Government of India for which we are thankful.

[2] Email: `smhedge@nitk.ac.in`

[3] Email: `shivarajmacs@gmail.com`

1 Introduction

The concept of graceful labeling of undirected graph was extended to a digraph by Bloom and Hsu [1] as follows:
A digraph D with e edges is labeled by assigning a distinct integer value $\theta(v)$ from $\{0, 1, ..., e\}$ to each vertex v. The vertex values, in turn, induce a value $\theta(u, v)$ on each edge (u, v) where $\theta(u, v) = \theta(v) - \theta(u) mod(e + 1)$. If the edge values are all distinct and nonzero, then the labeling is called a *graceful labeling* of a digraph.

Bloom and Hsu [1] established some connection between graceful digraphs and latin squares, Abelian groups, Galois field, and neofields. Also they have specified the relation between graceful unicycles and complete mappings by establishing the relation of each to a particular class of permutations.
The following Theorem is given in [1].

Theorem 1.1 *Let* $D = \bigcup_{i=1}^{t} \overrightarrow{C_{k_i}}$, *the union of* t *disjoint identical unicycles on* n *vertices.* D *is graceful if (a)* $t = 1$ *and* n *is even; of if (b)* $t = 2$; *or if (c)* $n = 2$ *or* $n = 6$. *Moreover,* D *is not graceful if* tn *is odd.*

2 Preliminaries

In this section, we study the graceful labeling of digraphs using the *system of simultaneous congruences.*

Suppose D is a graceful digraph with m vertices and n edges. Let $a_1, a_2, ..., a_m$ be the labels of the vertices of D. As D is graceful, corresponding to each $k = 1, 2, ..., n$, we obtain a nonzero element b_k of Z_{n+1} given by: $a_i - a_j \equiv b_k(mod\ n + 1)$, for some $i, j \in \{1, 2, ..., n\}$.

For example, consider the unidirectional cycle $\overrightarrow{C_n}$. The following system of simultaneous congruences represents a graceful labeling of unidirectional cycle $\overrightarrow{C_n}$,

$$\begin{aligned} a_2 - a_1 &\equiv b_1(mod\ n + 1) \\ a_3 - a_2 &\equiv b_2(mod\ n + 1) \\ &\vdots \\ a_1 - a_n &\equiv b_n(mod\ n + 1). \end{aligned}$$

As the left hand sides of the above congruences add up to zero, the right hand sides also add up to zero. Hence, the above system of congruences is solvable if and only if

$$b_1 + b_2 + ... + b_n \equiv 0(mod\ n+1).$$

Since the sum of all labels on the edges of $\overrightarrow{C_n}$ is $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, which is not zero under $(mod\ n+1)$ when n is odd, $\overrightarrow{C_n}$ is graceful only if n is even. As $\overrightarrow{C_n}$ is graceful, the vertex labels $a_i{}'s$ should be distinct. From the above system of congruences, it is easy to see that the $a_i{}'s$ are distinct if and only if the edge labels of any unidirectional path of $\overrightarrow{C_n}$ do not add up to zero under $(mod\ n+1)$.

From the above observations, the following theorem is immediate.

Theorem 2.1 *In the unidirectional cycle $\overrightarrow{C_n}$, if the orientation of an acyclic unipath is changed then the resulting directed cycle is not graceful.*

3 Gracefulness of the Digraph $n.\overrightarrow{C_m}$

In the year 1994, Du and Sun [2] put forward a conjecture as following:

Conjecture 3.1 *For any positive even n and any integer m, the digraph $n.\overrightarrow{C_m}$ is graceful.*

Here $n.\overrightarrow{C_m}$ denotes the digraph obtained from n copies of the directed cycle $\overrightarrow{C_m}$ which have one common vertex.
In this section, we prove the existence of graceful labeling for $n.\overrightarrow{C_m}$.

It follows from section 2 that, it is necessary that the sum of the labels on the edges each unidirectional cycle $\overrightarrow{C_m}$ in $n.\overrightarrow{C_m}$ is congruent to zero under $(mod\ nm+1)$.
The following theorem is proved in [3].

Theorem 3.2 *Suppose n is odd and $k \mid (n-1)$, where $k > 1$. Then the nonzero residues $(mod\ n+1)$ can be partitioned into $(n-1)/k$ sets of cardinality k, so that the sum of the elements of each set is $\equiv 0(mod\ n)$.*

From the Theorems 1.1 and 3.2, it is easy to see the following theorem.

Theorem 3.3 *For any positive even n and any integer m, the digraph $n.\overrightarrow{C_m}$ is graceful.*

4 Distinct Graceful labelings of unidirectional cycles.

In this section we discuss the distinct graceful labelings of unidirectional cycle $\overrightarrow{C_n}$.

A graceful digraph D does not have a unique graceful labeling, since adding a constant modulo $(e+1)$ to all of the vertex labels of a digraph preserves the edge labels and therefore generates a new graceful labeling of D.

Bloom and Hsu [1] observed the following property for the given digraph D with vertex labeling $\theta(v)$. Suppose we replace the pair of edges (u, v) and (x, y) that are labeled respectively by values k and $-k = e+1-k \ mod\ (e+1)$ to form a digraph D'. Then in D', $\theta(v, u) = -k$ and $\theta(y, x) = k$ and hence the set of vertex labels remains unchanged by this exchange. Also they gave the following definition:

Definition 4.1 Digraphs D_1 and D_2 with common underlying graph G are said to be *similar* (or *edge-pair similar*) if there is an identical vertex labeling which is graceful for both.

Definition 4.2 Let $uv = b_i$ and $xy = b_j$ be any two edges of the unidirectional cycle $\overrightarrow{C_n}$, then the distance between b_i and b_j is the length of shortest path between v and x.

It is easy to see that a maximum distance between any pair of edges in a unidirectional cycle $\overrightarrow{C_n}$ is $\frac{n}{2} - 1$. Hence we get a maximum of $(\frac{n}{2} - 1)$ distinct graceful directed cycles by reversing the direction of a pair of edges. Let $(\frac{n}{2} - 1)$ be denoted by a_1.

Similarly, we get $\sum\limits_{i_1=1}^{(n/2-1)} i_1$, $\sum\limits_{i_2=2}^{(n/2-1)} \sum\limits_{i_1=1}^{(n/2-i_2)} i_1$ and $\sum\limits_{i_3=3}^{(n/2-1)} \sum\limits_{i_2=i_3}^{(n/2-1)} \sum\limits_{i_1=1}^{(n/2-i_2)} i_1$

distinct graceful directed cycles after reversing the direction of two, three and four pairs of edges respectively. Note that in a unidirectional cycle $\overrightarrow{C_n}$, replacement of $\lfloor \frac{n}{4} \rfloor + j$ $(1 \leq j \leq \lfloor \frac{n}{4} \rfloor)$ pairs of edges is the same as replacing the remaining $\lfloor \frac{n}{4} \rfloor - j$ pairs of edges.

Therefore continuing with the above procedure we get,

$$\sum_{i_{\lfloor \frac{n}{4} \rfloor}=\lfloor \frac{n}{4} \rfloor}^{(n/2-1)} \sum_{i_{\lfloor \frac{n}{4} \rfloor -1}=i_{\lfloor \frac{n}{4} \rfloor}}^{(n/2-1)} \cdots \sum_{i_1=1}^{(n/2-i_2)} i_1 \text{ number of distinct graceful directed cycles.}$$

Let $a_2 = \sum\limits_{i_1=1}^{(n/2-1)} i_1$, $a_3 = \sum\limits_{i_2=2}^{(n/2-1)} \sum\limits_{i_1=1}^{(n/2-i_2)} i_1$ and $a_4 = \sum\limits_{i_3=3}^{(n/2-1)} \sum\limits_{i_2=i_3}^{(n/2-1)} \sum\limits_{i_1=1}^{(n/2-i_2)} i_1$.

Similarly let $a_{d+4} = \sum_{i_{d+4}=d+4}^{(n/2-1)} \sum_{i_{d-3}=i_{d+4}}^{(n/2-1)} ... \sum_{i_1=1}^{(n/2-i_2)} i_1$ for all $d = \{1, 2, ..., \lfloor \frac{n}{4} \rfloor - 4\}$.

On simplification, a_{d+4} can be written as:

$$a_{d+4} = \sum_{d=1}^{\lfloor \frac{n}{4}-4 \rfloor} \sum_{m=1}^{\frac{n}{2}-d-3} P_d(m). \sum_{i_2=m+d+2}^{\frac{n}{2}-1} \sum_{i_1=1}^{\frac{n}{2}-i_2} i_1, \text{ for all } d = \{1, 2, ..., \lfloor \tfrac{n}{4} \rfloor - 4\},$$

where $P_d(m) = \frac{1}{d!} \prod_{k=0}^{d-1} (m + k)$ denotes the diagonal elements of $Pascal's$ $triangle$ [4].

The above discussion leads to the following theorem.

Theorem 4.3 *If a unidirectional cycle $\overrightarrow{C_n}$ is graceful with the graceful vertex labeling $\theta(V)$, then the unidirectional cycle $\overrightarrow{C_n}$ has no more than $\sum_{i=1}^{\lfloor n/4 \rfloor} a_i$ distinct graceful directed cycles, each cycle having identically labeled vertices in the common underlying undirected cycle.*

References

[1] Bloom, G. S.,and Hsu, *On graceful directed graphs*, Siam. J. Alg. Discr. Methods **6** (1985), 519–536.

[2] Du, Z., Sun H., $n.\overrightarrow{C_{2p}}$ *are graceful*, Journal of beijing University of Posts and Telecommunications, **17**, (1994), 85–88.

[3] Richard J. Friedlander, Basil Gordan, Peter Tannenbaum, *Partition of groups and complete mappings*, Pacific Journal of Mathematics, **92**, (1981), 283–293.

[4] Fowler, David , *The binomial coefficient function*, The American Mathematical Monthly, 103, **1**, 1–17.

[5] Bloom, G. S.,and Hsu, *On graceful digraphs and a problem in network addressing*, Congressus Numerantium **35**, (1982), 91–103.

[6] Gallian, J. A., *A dynamic survey of graph labeling*, Electronic journal of combinatorics, ♯DS6, **16**, (2009), 1–219.