A generalization of Ramsey theory for linear forests

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Abstract

Chung and Liu defined the *d-chromatic Ramsey numbers* as a generalization of Ramsey numbers by replacing a weaker condition. Let $1 < d < c$ and let $t = \binom{c}{d}$. Assume $A_1, A_2, \ldots, A_t$ are all d-subsets of a set containing c distinct colors. Let $G_1, G_2, \ldots, G_t$ be graphs. The d-chromatic Ramsey number denoted by $r_d^c(G_1, G_2, \ldots, G_t)$ is defined as the least number p such that, if the edges of the complete graph K_p are colored in any fashion with c colors, then for some i, the subgraph whose edges are colored by colors in A_i contains a G_i. In this paper, we determine $r_d^c(G_1, G_2, \ldots, G_t)$ where G_i is a linear forest (disjoint union of paths) and $d = c - 1 \leq 3$.

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1 Introduction

In this paper all graphs will be undirected, finite, and have no loops or multiple edges. If G is a graph, V will denote its vertex set and E its edge set. The number of vertices of G is denoted by $|G|$. As usual K_n will denote the complete graph on n vertices. By P_i (respectively, C_i) we will mean a path (respectively, cycle) with i vertices. Moreover, $P_{i(c_1,c_2,c_3)}$ and $C_{i(c_1,c_2,c_3)}$ respectively denote a path and a cycle with i vertices whose edges are colored in c_1, c_2, or c_3. A graph L is a *linear forest* if it is the disjoint union of nontrivial paths. For linear forest L, the definition of $L_{(c_1,c_2,c_3)}$ is similar. It is assumed throughout the paper that $2 \leq i \leq j \leq k \leq l$.

Let $G_1, G_2, \ldots, G_c$ be graphs. The *Ramsey number*, $r(G_1, G_2, \ldots, G_c)$ is defined to be the least number p such that if the edges of the complete graph K_p are colored in any fashion with c colors, then for some i the spanning subgraph whose edges are colored with the ith color contains a G_i. Gerencsér and Gyárfás in [4] found the value of $r(P_i, P_j)$. They proved that for $i \leq j$, $r(P_i, P_j) = j + [i/2] - 1$. For given linear forests L_1 and L_2, Faudree and Schelp [3] showed that if the number of odd components of L_i is j_i, $1 \leq i \leq 2$, then

$$r(L_1, L_2) = \max\{|L_1| + (|L_2| - j_2)/2 - 1, |L_2| + (|L_1| - j_1)/2 - 1\}.$$

More information about the Ramsey numbers of other graphs can be found in the survey [8].

Chung and Liu [2] defined the *d-chromatic Ramsey numbers* as a generalization of Ramsey numbers by replacing a weaker condition. Let $1 < d < c$ and let $t = \binom{c}{d}$. Assume $A_1, A_2, \ldots, A_t$ are all d-subsets of a set containing c distinct colors. Let $G_1, G_2, \ldots, G_t$ be graphs. The d-chromatic Ramsey number denoted by $r_d^c(G_1, G_2, \ldots, G_t)$ is defined as the least number p such that, if the edges of the complete graph K_p are colored in any fashion with c colors, then for some i, the subgraph whose edges are colored by colors in A_i contains a G_i. They also determined the values of $r_2^3(K_i, K_j, K_l)$ and $r_2^3(K_{1,i}, K_{1,j}, K_{1,l})$, [1, 2].

Note that for graphs G_1 and G_2, $r(G_1, G_2) = r_1^2(G_1, G_2)$. Moreover, for graphs G_1, G_2, and G_3 with $|G_1| \leq |G_2| \leq |G_3|$ it is shown [2] that $r_2^3(G_1, G_2, G_3) \leq r(G_1, G_2)$ and the equality holds if $|G_3| \geq r(G_1, G_2)$.

In [7] Meenakshi and Sundararaghavan found the value of $r_2^3(P_i, P_j, P_k)$. In fact, they proved the following theorem.

Theorem 1.1 *The value of* $r_2^3(P_i, P_j, P_k)$ *is equal to* $[\frac{4k+2j+i-2}{6}]$ *if* $k < r(P_i, P_j)$ *and is equal to* $r(P_i, P_j)$*, otherwise.*

Harborth and Möller in [5] called a special case of d-chromatic Ramsey numbers, *weakened Ramsey numbers*. In their notation, $R_{s,t}(G)$ is the minimum p such that any coloring of the edges of K_p with t colors contains a copy of G with at most s colors. In [5] the value of $R_{t-1,t}(K_n)$ is identified. The values of $R_{t-1,t}(K_{1,n})$ and $R_{t-2,t}(K_{1,n})$ are determined in [6].

In this paper we are mainly concerned with the numbers $r_{t-1}^t(G_1, G_2, \ldots, G_t)$, the smallest p such that if a complete graph K_p is colored with t colors, then there is a copy of G_i that avoids the color i for some i. We shall determine these numbers when $t = 3$ or 4, and the graphs G_i are linear forests.

2 Main results

Using Theorem 1.1, we are able to determine the exact value of $r_2^3(L_1, L_2, L_3)$ as follows.

Theorem 2.1 *Let* L_1*,* L_2*, and* L_3 *be linear forests with* $|L_1| \leq |L_2| \leq |L_3|$*. Then* $r_2^3(L_1, L_2, L_3) = r_2^3(P_{|L_1|}, P_{|L_2|}, P_{|L_3|})$ *if* $|L_3| < r_1^2(L_1, L_2) \leq r_1^2(P_{|L_1|}, P_{|L_2|})$ *and* $r_2^3(L_1, L_2, L_3) = r_1^2(L_1, L_2)$*, otherwise.*

As a special case, we have the following corollary for *stripes*, i.e. disjoint copies of a P_2.

Corollary 2.2 *For integers n_1, n_2, and n_3 with $n_1 \leq n_2 \leq n_3$, we have*

$$r_2^3(n_1P_2, n_2P_2, n_3P_2) = r_2^3(P_{2n_1}, P_{2n_2}, P_{2n_3}).$$

We now try to determine the value of $r_3^4(L_1, L_2, L_3, L_4)$. For this we shall find the value of $r_3^4(P_i, P_j, P_k, P_l)$.

Theorem 2.3 *We have $r_3^4(P_i, P_j, P_k, P_l) \leq r_2^3(P_i, P_j, P_k)$ and the equality holds if $l \geq r_2^3(P_i, P_j, P_k)$.*

Theorem 2.4 *If $l < r_2^3(P_i, P_j, P_k)$, then $r_3^4(P_i, P_j, P_k, P_l) \leq [\frac{8l+4k+2j+i-2}{14}]$.*

Theorem 2.5 *Let L_i, $1 \leq i \leq 4$, be linear forests with $|L_1| \leq |L_2| \leq |L_3| \leq |L_4|$. Then $r_3^4(L_1, L_2, L_3, L_4) > [\frac{8|L_4|+4|L_3|+2|L_2|+|L_1|-2}{14}] - 1$.*

Proof. If $|L_4| \geq r_2^3(L_1, L_2, L_3)$, then $r_3^4(L_1, L_2, L_3, L_4) = r_2^3(L_1, L_2, L_3)$. Thus we may assume that $|L_4| < r_2^3(L_1, L_2, L_3) \leq r_2^3(P_{|L_1|}, P_{|L_2|}, P_{|L_3|})$. Let $s = [\frac{8|L_4|+4|L_3|+2|L_2|+|L_1|-2}{14}]$, $x_1 = \lceil \frac{4|L_4|+2|L_3|+|L_2|-|L_1|-4s-1}{3} \rceil$, $x_2 = \lceil \frac{2|L_4|+|L_3|-|L_2|+|L_1|-2s-2}{3} \rceil$, $x_3 = 2s - |L_3| - |L_4|$, and $x_4 = s - |L_4|$. By $|L_4| < r_2^3(P_{|L_1|}, P_{|L_2|}, P_{|L_3|})$ and the definition of s and x_i's, it is straightforward to check that

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = s - 1, \\ x_1 + x_2 + x_3 = |L_4| - 1, \\ x_1 + x_2 + 2x_4 = |L_3| - 1, \\ x_1 + 2x_3 + 2x_4 \leq |L_2| - 1, \\ 2x_2 + 2x_3 + 2x_4 + 1 \leq |L_1| - 1, \\ x_i \geq 0, 1 \leq i \leq 4. \end{cases} \tag{1}$$

Now partition the vertices of K_{s-1} into four sets X_i, $1 \leq i \leq 4$ with $|X_i| = x_i$. Paint with 1 all edges which are incident with two vertices of X_1. For $i = 2, 3, 4$, paint with i the edges having two vertices in X_i or one vertex in X_i and one vertex in X_j where $j < i$. The conditions in 1 guarantee that K_{s-1} does not contain $L_{1(2,3,4)}$, $L_{2(1,3,4)}$, $L_{3(1,2,4)}$, and $L_{4(1,2,3)}$. ⊣

We can now summarize Theorems 2.3, 2.4, and 2.5 as follows.

Theorem 2.6 *The value of $r_3^4(P_i, P_j, P_k, P_l)$ is equal to $[\frac{8l+4k+2j+i-2}{14}]$ if $l < r_2^3(P_i, P_j, P_k)$ and is equal to $r_2^3(P_i, P_j, P_k)$, otherwise.*

Corollary 2.7 *Let L_i, $1 \leq i \leq 4$, be linear forests with $|L_1| \leq |L_2| \leq |L_3| \leq |L_4|$. Then $r_3^4(L_1, L_2, L_3, L_4) = r_3^4(P_{|L_1|}, P_{|L_2|}, P_{|L_3|}, P_{|L_4|})$ if $|L_4| < r_2^3(L_1, L_2, L_3) \leq r_2^3(P_{|L_1|}, P_{|L_2|}, P_{|L_2|})$ and $r_3^4(L_1, L_2, L_3, L_4) = r_2^3(L_1, L_2, L_3)$, otherwise.*

3 Concluding remark

Theorem 2.5 gives a lower bound for $r_3^4(P_i, P_j, P_k, P_l)$. The method applied in Theorem 2.5 can be easily generalized to the case of t colors. Also in Theorem 2.4, the argument for the case $i < j$ is applicable in the more general case. However in the case $i = j$ for large t too much cases should be considered and this may be hard to verify. It is our conjecture that for each $t \geq 3$, and for $n_1, n_2, \ldots, n_t$ with $n_1 \leq n_2 \leq \cdots \leq n_t$,

$$r_{t-1}^t(P_{n_1}, P_{n_2}, \ldots, P_{n_t}) = \left\lceil \frac{\sum_{i=0}^{t-1} 2^i n_{i+1} - 2}{\sum_{i=1}^{t-1} 2^i} \right\rceil,$$

where $n_t < r_{t-2}^{t-1}(P_{n_1}, P_{n_2}, \ldots, P_{n_{t-1}})$. It is an interesting result to find a proof for the upper bound that does not consider different cases.

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