The Metric Dimension of the Composition Product of Graphs

Mohsen Jannesari [1]

Department of Mathematical Sciences
Isfahan University of Technology
Isfahan, Iran

Behnaz Omoomi [2]

Department of Mathematical Sciences
Isfahan University of Technology
Isfahan, Iran

Abstract

For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the ordered k-vector $r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the (metric) representation of v with respect to W, where $d(x, y)$ is the distance between the vertices x and y. The set W is called a resolving set for G if distinct vertices of G have distinct representations with respect to W. The minimum cardinality of a resolving set for G is its metric dimension. In this paper, we study the metric dimension of the composition product of graphs G and H, $G[H]$. First, we introduce a new parameter which is called adjacency metric dimension of a graph. Then, we obtain the metric dimension of $G[H]$ in terms of the order of G and the adjacency metric dimension of H.

Keywords: Composition Product, Resolving set, Metric dimension, Basis, Adjacency metric dimension.

1 Introduction

Throughout this paper, $G = (V, E)$ is a finite simple graph. We use $\overline{G}$ for the complement of graph G. The distance between two vertices u and v, denoted

[1] Email: m.jannesari@math.iut.ac.ir
[2] Email: bomoomi@cc.iut.ac.ir

by $d_G(u,v)$, is the length of a shortest path between u and v in G. Also, $N_G(v)$ is the set of all neighbors of vertex v in G. We write these simply $d(u,v)$ and $N(v)$, when no confusion can arise. The notations $u \sim v$ and $u \nsim v$ denote the adjacency and non-adjacency relation between u and v, respectively.

For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r(v|W) := (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$$

is called the (*metric*) *representation* of v with respect to W. The set W is called a *resolving set* for G if distinct vertices have different representations. A resolving set W for G with minimum cardinality is called a *basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\beta(G)$. The concept of (metric) representation is introduced by Slater [6] (see [3]).

Two distinct vertices u, v are said *twins* if $N(v)\backslash\{u\} = N(u)\backslash\{v\}$. It is called that $u \equiv v$ if and only if $u = v$ or u, v are twins. In [4], it is proved that "$\equiv$" is an equivalent relation. The equivalence class of vertex v is denoted by v^*. Hernando et al. [4] proved that v^* is a clique or an independent set in G. As in [4], we say v^* is of type (1), (K), or (N) if v^* is a class of size 1, a clique of size at least 2, or an independent set of size at least 2. We denote the number of equivalence classes of G with respect to "$\equiv$" by $\iota(G)$. We mean by $\iota_{_K}(G)$ and $\iota_{_N}(G)$, the number of classes of type (K) and type (N) in G, respectively. We also use $a(G)$ and $b(G)$ for the number of all vertices in G which have at least an adjacent twin and a non-adjacent twin vertex in G, respectively. On the other way, $a(G)$ is the number of all vertices in the classes of type (K) and $b(G)$ is the number of all vertices in the classes of type (N). Clearly, $\iota(G) = n(G) - a(G) - b(G) + \iota_{_N}(G) + \iota_{_K}(G)$.

Lemma 1.1 [1,2]

(i) If $n \notin \{3,6\}$, then $\beta(C_n \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor$,

(ii) If $n \notin \{1,2,3,6\}$, then $\beta(P_n \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor$.

The metric dimension of cartesian product of graphs is studied by Caseres et al. in [2]. They obtained the metric dimension of cartesian product of graphs G and H, $G \square H$, where $G, H \in \{P_n, C_n, K_n\}$.

The *composition product* of graphs G and H, denoted by $G[H]$, is a graph with vertex set $V(G) \times V(H) := \{(v,u) \mid v \in V(G), u \in V(H)\}$, where two vertices (v,u) and (v',u') are adjacent whenever, $v \sim v'$, or $v = v'$ and $u \sim u'$. When the order of G is at least 2, it is easy to see that $G[H]$ is a connected graph if and only if G is a connected graph.

This paper is aimed to investigate the metric dimension of composition product of graphs. The main goal of Section 2 is introducing a new parameter, which we call it adjacency metric dimension. In Section 3, we prove some relations to determine the metric dimension of composition product of graphs, $G[H]$, in terms of the order of G and the adjacency metric dimension of H. As a corollary of our main theorems, we obtain the exact value of the metric dimension of $G[H]$, where $G = C_n (n \geq 5)$ or $G = P_n (n \geq 4)$, and $H \in \{P_m, C_m, \overline{P}_m, \overline{C}_m, K_{m_1,\ldots,m_t}, \overline{K}_{m_1,\ldots,m_t}\}$.

2 Adjacency Resolving Sets

S. Khuller et al. [5] have considered the application of the metric dimension of a connected graph in robot navigation. In that sense, a robot moves from node to node of a graph space. If the robot knows its distances to a sufficiently large set of landmarks, its position on the graph is uniquely determined. This suggests the problem of finding the fewest number of landmarks needed, and where should be located, so that the distances to the landmarks uniquely determine the robot's position on the graph. The solution of this problem is the metric dimension and a basis of the graph.

Now let there exist a large number of landmarks, but the cost of computing distance is much for the robot. In this case, robot can determine its position on the graph only by knowing landmarks which are adjacent to it. Here, the problem of finding the fewest number of landmarks needed, and where should be located, so that the adjacency and non-adjacency to the landmarks uniquely determine the robot's position on the graph is a different problem. The answer to this problem is one of the motivations of introducing *adjacency resolving sets* in graphs.

Definition 2.1 Let G be a graph and $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(G)$. For each vertex $v \in V(G)$ the adjacency representation of v with respect to W is k-vector

$$r_2(v|W) := (a_G(v, w_1), a_G(v, w_2), \ldots, a_G(v, w_k)),$$

where

$$a_G(v, w_i) = \begin{cases} 0 \text{ if } v = w_i, \\ 1 \text{ if } v \sim w_i, \\ 2 \text{ if } v \nsim w_i. \end{cases}$$

If all distinct vertices of G have distinct adjacency representations, W is called an adjacency resolving set for G. The minimum cardinality of an adjacency resolving set is called adjacency metric dimension of G, denoted by $\beta_2(G)$. An adjacency resolving set of cardinality $\beta_2(G)$ is called an adjacency basis of G.

By the definition, if G is a connected graph with diameter 2, then $\beta(G) = \beta_2(G)$. The converse is false; it can be seen that $\beta_2(C_6) = 2 = \beta(C_6)$ while, $diam(C_6) = 3$.

In the following, we obtain some useful results on the adjacency metric dimension of graphs.

Proposition 2.2 *For every connected graph G, $\beta(G) \leq \beta_2(G)$.*

Proposition 2.3 *For every graph G, $\beta_2(G) = \beta_2(\overline{G})$.*

Let G be a graph of order n. It is easy to see that, $1 \leq \beta_2(G) \leq n - 1$. In the following proposition, we characterize all graphs G with $\beta_2(G) = 1$ and all graphs G of order n and $\beta_2(G) = n - 1$.

Proposition 2.4 *If G is a graph of order n, then* *(i)* $\beta_2(G) = 1$ *if and only if* $G \in \{P_1, P_2, P_3, \overline{P}_2, \overline{P}_3\}$.

(ii) $\beta_2(G) = n - 1$ *if and only if* $G = K_n$ *or* $G = \overline{K}_n$.

Proposition 2.5 *For every graph* G, $\beta(G \vee K_1) - 1 \leq \beta_2(G) \leq \beta(G \vee K_1)$. *Moreover,* $\beta_2(G) = \beta(G \vee K_1)$ *if and only if* G *has an adjacency basis for which no vertex has adjacency representation entirely* 1 *with respect to it.*

Proposition 2.6 *If* $n \geq 4$, *then* $\beta_2(C_n) = \beta_2(P_n) = \lfloor \frac{2n+2}{5} \rfloor$.

Proposition 2.7 *If* $K_{m_1,m_2,\ldots,m_t}$ *is the complete* t*-partite graph, then*

$$\beta_2(K_{m_1,m_2,\ldots,m_t}) = \beta(K_{m_1,m_2,\ldots,m_t}) = \begin{cases} m - r - 1 & \text{if } r \neq t, \\ m - r & \text{if } r = t, \end{cases}$$

where $m_1, m_2, \ldots, m_r$ *are at least* 2, $m_{r+1} = \cdots = m_t = 1$, *and* $\sum_{i=1}^{t} m_i = m$.

3 Composition Product of Graphs

Throughout this section, G is a connected graph of order n, and H is an arbitrary graph of order m.

Lemma 3.1 *Let* G *be a connected graph of order* n *and* H *be an arbitrary graph. Then,* $\beta(G[H]) \geq n\beta_2(H)$.

Theorem 3.2 *Let* G *be a connected graph of order* n *and* H *be an arbitrary graph. If there exist two adjacency bases* W_1 *and* W_2 *of* H *such that, there is no vertex with adjacency representation entirely* 1 *with respect to* W_1 *and no vertex with adjacency representation entirely* 2 *with respect to* W_2, *then* $\beta(G[H]) = \beta(G[\overline{H}]) = n\beta_2(H)$.

In the following three theorems, we obtain $\beta(G[H])$, when H does not satisfy the assumption of Theorem 3.2.

Theorem 3.3 *Let* G *be a connected graph of order* n *and* H *be an arbitrary graph. If for each adjacency basis* W *of* H *there exist vertices with adjacency representations entirely* 1 *and entirely* 2 *with respect to* W, *then* $\beta(G[H]) = \beta(G[\overline{H}]) = n(\beta_2(H) + 1) - \iota(G)$.

Theorem 3.4 *Let* G *be a connected graph of order* n *and* H *be an arbitrary graph. If* H *has the following properties*

(i) for each adjacency basis of H *there exist a vertex with adjacency representation entirely* 1,

(ii) there exist an adjacency basis W *of* H *such that there is no vertex with adjacency representation entirely* 2 *with respect to* W,

then $\beta(G[H]) = n\beta_2(H) + a(G) - \iota_K(G)$.

Theorem 3.5 *Let* G *be a connected graph of order* n *and* H *be an arbitrary graph. If* H *has the following properties*

(i) for each adjacency basis of H *there exist a vertex with adjacency representation entirely* 2,

(ii) there exist an adjacency basis W of H such that there is no vertex with adjacency representation entirely 1 with respect to W,

then $\beta(G[H]) = n\beta_2(H) + b(G) - \iota_{_N}(G)$.

Corollary 3.6 *If G has no pair of twin vertices, then* $\beta(G[H]) = n\beta_2(H)$.

Corollary 3.7 *Let $G = P_n$, $n \geq 4$ or $G = C_n$, $n \geq 5$. Then,* $\beta(G[P_m]) = \beta(G[C_m]) = \beta(G[\overline{P}_m]) = \beta(G[\overline{C}_m]) = n\lfloor \frac{2m+2}{5} \rfloor$. *And,*

$$\beta(G[\overline{K}_{m_1,m_2,\ldots,m_t}]) = \beta(G[K_{m_1,m_2,\ldots,m_t}]) = \begin{cases} n(m-r-1) & \text{if } r \neq t, \\ n(m-r) & \text{if } r = t, \end{cases}$$

where $m_1, m_2, \ldots, m_r$ are at least 2, $m_{r+1} = \cdots = m_t = 1$, and $\sum_{i=1}^{t} m_i = m$.

Corollary 3.8 *Let $H = K_{m_1,m_2,\ldots,m_t}$, where $m_1, m_2, \ldots, m_r$ are at least 2, $m_{r+1} = \cdots = m_t = 1$, and $\sum_{i=1}^{t} m_i = m$. Then,*

$$\beta(K_n[H]) = \begin{cases} n(m-r)-1 & \text{if } r \neq t, \\ n(m-r) & \text{if } r = t. \end{cases}$$

References

[1] Buczkowski, P. S., G. Chartrand, C. Poisson and P. Zhang *On k-dimensional graphs and their bases*, Periodica Mathematica Hungarica **46(1)** (2003), 9–15.

[2] Caceres, J., C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas and C. Seara, *On the metric dimension of some families of graphs*, Electronic Notes in Discrete Mathematics **22**, (2005), 129–133.

[3] Harary, F., and R. A. Melter, *On the metric dimension of a graph*, Ars Combinatorics **2** (1976), 191–195.

[4] Hernando, C., M. Mora, I. M. Pelayo, C. Seara and D. R. Wood, *Extemal graph theory for metric dimension and diameter*, The Electronic Journal of Combinatorics (2010), #R30.

[5] Khuller, S., B. Raghavachari and A. Rosenfeld, *Landmarks in graphs*, Discrete Applied Mathematics **70(3)** (1996), 217–229.

[6] Slater, P. J., *Leaves of trees*, Congressus Numerantium **14** (1975), 549–559.