Strongly regular and pseudo geometric graphs

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Abstract

Very often, strongly regular graphs appear associated with partial geometries. The point graph of a partial geometry is the graph whose vertices are the points of the geometry and adjacency is defined by collinearity. It is well known that the point graph associated to a partial geometry is a strongly regular graph and, in this case, the strongly regular graph is named *geometric*. When the parameters of a strongly regular graph, Γ, satisfy the relations of a geometric graph, then Γ is named a *pseudo geometric* graph. Moreover, it is known that not every pseudo geometric graph is geometric. In this work, we characterize strongly regular graphs that are pseudo geometric and we analyze when the complement of a pseudo geometric graph is also pseudo geometric.

Keywords: Strongly regular graph, partial geometry, pseudo geometric graph.

1 Strongly regular graphs

A *strongly regular graph* with parameters (n, k, λ, μ) is a graph on n vertices which is regular of degree k, any two adjacent vertices have exactly λ common neighbours and two non–adjacent vertices have exactly μ common neighbours.

We recall that antipodal strongly regular graphs are characterized by satisfying $\mu = k$, or equivalently $\lambda = 2k - n$, which in particular implies that $2k \geq n$. In addition, any bipartite strongly regular graph is antipodal and it is characterized by satisfying $\mu = k$ and $n = 2k$.

It is well-known that distance–regular graphs with degree $k \geq 3$ other than bipartite and antipodal are primitive, see for instance [2, Th. 4.2.1].

We recall that if Γ is a primitive strongly regular graph with parameters (n, k, λ, μ), then its complement graph, $\overline{\Gamma}$, is also a primitive strongly regular graph with parameters $(n, n-k-1, n-2-2k+\mu, n-2k+\lambda)$, which in particular implies that $\mu \geq 2(k+1) - n$. Note that the complement of an antipodal strongly regular graph Γ is the disjoint union of m copies of a complete graph K_r for some positive integers m and r. Therefore, Γ is the complete multipartite graph $K_{m\times r}$. On the other hand, strongly regular graphs with the same parameters as their complement are called *conference graphs* and then, their parameters are $(4m+1, 2m, m-1, m)$ where $m \geq 1$. Moreover, it is known that such a graph exits iff $4m+1$ is the sum of two squares, see [4].

2 Partial geometries

Many strongly regular graphs appears associated with the so–called partial geometries. A *partial geometry with parameters* $s, t, \alpha \geq 1$, $pg(s, t, \alpha)$, is an incidence structure of points and lines such that every line has $s+1$ points, every point is on $t+1$ lines, two distinct lines meet in at most one point and given a line and a point not in it, there are exactly α lines through the point which meet the line. The number of points and lines in $pg(s, t, \alpha)$ are $n = \dfrac{1}{\alpha}(s+1)(st+\alpha)$ and $\ell = \dfrac{1}{\alpha}(t+1)(st+\alpha)$, respectively. Therefore, the parameters of a partial geometry satisfy the inequalities $1 \leq \alpha \leq \min\{t+1, s+1\}$ and moreover, α divides both $st(t+1)$ and $st(s+1)$. We refer the

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reader to the surveys [1,3,4,5] for the main features.

The *point graph of* $pg(s, t, \alpha)$ has the points as vertices and two vertices are adjacent iff they are collinear. Therefore, it is a regular graph with degree $k = s(t+1)$. Moreover, when $\alpha = s+1$, the partial geometry is called *Linear space* and its point graph is the complete graph K_n. When $\alpha \leq s$, the point graph is a strongly regular graph with parameters

$$(n, s(t+1), s-1+t(\alpha-1), \alpha(t+1)). \tag{1}$$

A strongly regular graph is called *pseudo geometric graph* if its parameters are of the former form, where $1 \leq \alpha \leq \min\{s, t+1\}$ and α divides $st(s+1)$.

A pseudo geometric graph is geometric iff there is a collection L of $(s+1)$–cliques with the property that each edge lies in just one clique of L. If some edge lies in no $(s+1)$–clique then the graph is clearly not geometric, but if there are too many cliques, it is often not clear whether a suitable collection can be selected.

Next we study the point graphs associated with the most well–known families of partial geometries.

(i) *Dual Linear Spaces:* In this case $\alpha = t+1 \leq s$. When, $t = 1$ and $s = m-2$ the corresponding pseudo geometric graph is the so–called *triangular graph* T_m whose parameters are $\left(\binom{m}{2}, 2(m-2), m-2, 4\right)$. Notice, that T_m is also the line graph of the complete graph K_m.

(ii) *Transversal Designs:* In this case $\alpha = s \leq t+1$ and hence the corresponding pseudo geometric graph is the *complete multipartite graph* $K_{(s+1)\times(t+1)}$ whose parameters are $((s+1)(t+1), s(t+1), (s-1)(t+1), s(t+1))$.

(iii) *Dual Transversal Designs:* In this case $\alpha = t \leq s$, $t > 1$ and hence the corresponding pseudo geometric graph is the *Pseudo–Latin square graph* $PL_r(m)$ whose parameters are $(m^2, r(m-1), r^2-3r+m, r(r-1))$, where $r = t+1$ and $m = s+1$.

(iv) *Generalized quadrangles:* In this case $\alpha = 1$, $s > 1$ and hence the parameters of the pseudo geometric graph are $((s+1)(st+1), s(t+1), s-1, t+1)$. Note that when $t = 1$ these graphs are the so–called *Hamming graph* $H(2, s+1)$ *or Lattice*. Observe that the complement of $H(2, s+1)$ is the pseudo–latin square graph $PL_s(s+1)$.

(v) *Proper pseudo geometric:* In this case $1 < \alpha < \min\{t, s\}$. Within this type it is worth to mention the *Kneser graphs* $K(m, 2)$*,* where $m \geq 6$ is even, in which case $s = \dfrac{m}{2} - 1$, $t = m-4$ and $\alpha = \dfrac{m}{2} - 2$. For arbitrary $m \geq 5$, the *Kneser graph* $K(m, 2)$ is the graph whose vertices represent the 2–subsets of $\{1,\dots,m\}$, and where two vertices are connected if and only if they correspond to disjoint subsets. The parameters of the Kneser graph $K(m,2)$ are $\left(\binom{m}{2},\binom{m-2}{2},\binom{m-4}{2},\binom{m-3}{2}\right)$, that coincide with the parameters of the complement of T_m. In addition, $K(m,2)$ for m odd is an example of strongly regular graph that is not a pseudo geometric graph, which also implies that the complement of a pseudo geometric graph is not necessarily a pseudo geometric graph, see below.

3 The characterization

The relation between triangular and Kneser graphs raises the question of when the complement of a pseudo geometric graph is also pseudo geometric. To do this we take into account that for any strongly regular graph, different from a conference graph, there exist $h\geq 1$ and $\beta\in\mathbb{Z}$ such that

$$(\mu-\lambda)^2+4(k-\mu)=h^2 \quad\text{and}\quad (n-1)(\mu-\lambda-h)=2(k+h\beta). \tag{2}$$

As a by–product of the first equality, we get that $\mu-\lambda$ and h have the same parity and hence $h+\mu-\lambda$ is an even integer. So for any strongly regular graph different from a conference graph, we can consider the integer $\nu=\dfrac{1}{2}(h+\mu-\lambda)\geq 1$. In addition, the second equality in (2) implies that $(n-1)\nu=k+h(\beta+n-1)$ and hence that $n-1$ divides $k+h\beta$.

Theorem 3.1 *A strongly regular graph Γ with parameters (n,k,λ,μ) is a pseudo geometric graph iff one of the following conditions hold:*

(i) *Γ is a conference graph with $n=\ell^2(p^2+q^2)^2$, where ℓ,p are odd integers, q is even and $\gcd(p,q)=1$. Moreover, μ is even and Γ is a dual transversal design whose corresponding pseudo–latin square graph is $PL_{\frac{1+\sqrt{n}}{2}}(\sqrt{n})$.*

(ii) *Γ is not a conference graph, ν divides both μ and k and moreover $\mu\leq\nu^2$. When Γ is antipodal, this property holds iff $k\leq(n-k)^2$.*

Proof. The graph Γ is pseudo geometric iff there exist integers $s,t,\alpha\geq 1$ with $1\leq\alpha\leq\min\{s,t+1\}$ and such that

$$k=s(t+1),\quad \mu=\alpha(t+1)\quad\text{and}\quad \lambda=s-1+t(\alpha-1),$$

and hence $t=\dfrac{\mu}{\alpha}-1$ and $s=\dfrac{k\alpha}{\mu}$, which implies $\alpha^2(k-\mu)-\alpha\mu(\lambda-\mu)-\mu^2=0$.

(i) If Γ is a conference graph $k = 2\mu$, $\mu - \lambda = 1$ and hence

$$\alpha = \frac{1}{2}\Big[-1 + \sqrt{1+4\mu}\Big].$$

Therefore, α is a positive integer iff $1 + 4\mu = h^2$ with h odd. In this case, $\alpha = \dfrac{h-1}{2}$, $s = h - 1$ and $t = \dfrac{2\mu}{h-1} - 1 = \dfrac{h-1}{2}$. As a by–product we obtain that μ must be even and that Γ is a dual transversal design, since $\alpha = t$ and $t \geq 2$. Moreover, Γ exists iff h^2 is the sum of two squares which implies that $h = \ell(p^2 + q^2)$ where $\gcd(p, q) = 1$.
(ii) If Γ is not a conference graph but $\mu = k$; *i.e.* Γ is antipodal, then $\lambda = 2k - n$, $\alpha = s = \dfrac{k}{n-k}$ and $t = n - k - 1$. On the other hand, $\nu = \mu - \lambda = h = n - k$ and the second equality in (2) implies that $k = -h\beta$; that is, ν divides k. Therefore Γ is pseudo geometric iff $\alpha \leq t + 1$; that is iff $k \leq (n - k)^2$.

On the other hand, if Γ is neither a conference graph nor an antipodal graph, then $k > \mu$, $h > |\mu - \lambda|$ and

$$\alpha = \frac{\mu(\lambda-\mu+h)}{2(k-\mu)} = \frac{2\mu(\lambda-\mu+h)}{4(k-\mu)} = \frac{2\mu(\lambda-\mu+h)}{(h+\mu-\lambda)(h+\lambda-\mu)} = \frac{\mu}{\nu},$$

which implies that $t = \nu - 1$ and $s = \dfrac{k}{\nu}$. Therefore, for Γ be a pseudo geometric graph these values must be integers and in consequence ν must be a divisor of both k and μ. Finally, $\alpha < s$, since $\mu < k$ and hence, Γ is pseudo geometric iff $\alpha \leq t + 1$; that is $\mu \leq \nu^2$. $\square$

The reasoning in part (ii) of the above Theorem, shows that for any antipodal strongly regular graph Γ, $n - k$ divides k and moreover there exist integers $t, s \geq 1$ such that the parameters of Γ are $((s+1)(t+1), s(t+1), (s-1)(t+1), s(t+1))$ and, in particular, Γ is bipartite iff $s = 1$ an hence the bipartite strongly regular graphs are $K_{2\times(t+1)}$, $t \geq 1$, and all of them are pseudo geometric. Finally, note that the graph $K_{(s+1)\times 2}$ is also known as *Cocktail party graph* that is pseudo geometric iff $s \leq 2$.

In the next result we study when the complement of a pseudo geometric graph is also pseudo geometric. We can assume that $\alpha \leq s-1$, since otherwise Γ is antipodal.

Corollary 3.2 *If Γ is a pseudo geometric graph with parameters $(n, s(t+1), s-1+t(\alpha-1), \alpha(t+1))$, $t, \alpha \geq 1$, $s \geq 2$ and $1 \leq \alpha \leq \min\{s-1, t+1\}$, then $\bar{\Gamma}$ is also a pseudo geometric graph iff α divides st and $(s-\alpha)(t-\alpha) \leq \alpha$.*

Moreover, the geometrical parameters of $\bar{\Gamma}$ *are* $\left(\dfrac{st}{\alpha}, s-\alpha, \dfrac{t(s-\alpha)}{\alpha}\right)$.

Let us study when some well–known families of strongly regular graphs, other than antipodal, are pseudo geometric graphs.

Corollary 3.3 *The following results hold:*

(i) The Kneser graph $K(m,2)$*,* $m \geq 4$*, is pseudo geometric iff* m *is even.*

(ii) The Negative–Latin square $NL_r(m)$*, whose parameters are*

$$(m^2, r(m+1), r(r+3)-m, r(r+1)),$$

where $1 \leq r < m \leq r(r+3)$ *is pseudo geometric iff* $m-r$ *divides both* $r(r+1)$ *and* $r(m+1)$ *and moreover* $r(r+1) \leq (m-r)^2$.

(iii) If Γ *is a symmetric balanced incomplete block design there exist integers* $t, s \geq 1$ *such that* $t+2 \leq s$ *and the parameters of* Γ *are*

$$(n, s(t+1), (s-t-1)(t+1), (s-t-1)(t+1)).$$

Therefore, Γ *is pseudo geometric iff* $t+2 \leq s \leq 2(t+1)$.

(iv) The graphs $NL_r(2r)$*,* $r \geq 1$*, are the unique negative–latin squares that are symmetric balanced incomplete block designs and they are not pseudo geometric.*

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