On the M–matrix inverse problem for symmetric singular tridiagonal matrices

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Abstract

A well–known property of an irreducible non–singular M–matrix is that its inverse is positive. However, when the matrix is an irreducible and singular M–matrix it is known that it has a generalized inverse which is non–negative, but this is not always true for any generalized inverse. We focus here in characterizing when the Moore–Penrose inverse of a symmetric, singular, irreducible and tridiagonal M–matrix is itself a M–matrix.

Keywords: M–matrix, Moore–Penrose inverse, Laplacian, tridiagonal matrix.

1 Statement of the Problem

The matrices that can be expressed as $L = kI - A$, where $k > 0$ and $A \geq 0$, appear in relation with systems of equations or eigenvalue problems in a broad variety of areas including finite difference methods for solving partial differential equations, input–output production and growth models in economics or Markov processes in probability and statistics. Of course, the combinatorial community can recognize within this type of matrices, the combinatorial Laplacian of a k–regular graph where A is the adjacency matrix.

If k is at least the spectral radio of A, then L is called an *M–matrix*. A well–known property of an irreducible non–singular M–matrix is that its inverse is non–negative, [3]. However, when the matrix is an irreducible and singular M–matrix it is known that it has a generalized inverse which is non–negative, but this is not always true for any generalized inverse. For instance, it may happens that the Moore–Penrose inverse has some negative entries. We focus here at characterizing when the Moore–Penrose inverse of a symmetric, singular, irreducible and tridiagonal M–matrix is itself a M–matrix. This problem has been widely studied for several types of matrices, see for instance [2,5,6,7].

Given $n \geq 2$, $\mathsf{c} = (c_1, \dots, c_{n-1}) \in (0, +\infty)^{n-1}$ and $\mathsf{d} = (d_1, \dots, d_n) \in [0, +\infty)^n$ we look for conditions under which the tridiagonal matrix

$$\mathsf{M} = \begin{bmatrix} d_1 & -c_1 & & & \\ -c_1 & d_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{n-2} & d_{n-1} & -c_{n-1} \\ & & & -c_{n-1} & d_n \end{bmatrix} \tag{1}$$

is a singular M–matrix. Moreover, when M satisfies this property we are also interested in characterizing when its Moore-Penrose inverse, $\mathsf{M}^\dagger$, is also a M-matrix. In particular, the matrix obtained by choosing $d_1 = c_1$, $d_n = c_{n-1}$ and $d_i = c_{i-1} + c_i$ for $i = 2, \dots, n-1$ is nothing but the combinatorial Laplacian of

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Γ, the weighted path on n vertices whose conductances are given by the values $c_1,\dots,c_{n-1}$. Therefore, M can be considered as a *perturbed Laplacian of* Γ in the sense of [1] and also as one of the so–called *discrete Schrödinger operators of* Γ, see for instance [4] and references therein for several physical interpretations. So, we ask which perturbed Laplacians or Schrödinger operators of Γ, are singular, positive semi–definite and such that their Moore–Penrose inverse are also M–matrices.

In the sequel, any $\mathsf{c}=(c_1,\dots,c_{n-1})\in(0,+\infty)^{n-1}$ and $\mathsf{w}=(w_1,\dots,w_n)\in(0,+\infty)^n$ such that $w_1^2+\ldots+w_n^2=1$ are called *conductance* and *weight*, respectively. The set of weights is denoted by $\Omega(n)$ and e is the vector whose entries are all equal to 1.

The authors proved in [2] that the matrix given in (1) is a singular M–matrix iff there exists $\mathsf{w}\in\Omega(n)$ such that

$$d_1=\frac{c_1\omega_2}{\omega_1},\quad d_n=\frac{c_{n-1}\omega_{n-1}}{\omega_n}\quad\text{and}\quad d_j=\frac{1}{\omega_j}(c_j\omega_{j+1}+c_{j-1}\omega_{j-1}) \tag{2}$$

for any $j=2,\dots,n-1$. Moreover the weight w is uniquely determined by d and c. For this reason we denote by $\mathsf{M}(\mathsf{c},\mathsf{w})$ the matrix given in (1) where the diagonal entries are determined by (2).

Given a conductance c, the set of weights such that $\mathsf{M}^\dagger(\mathsf{c},\mathsf{w})$ is a M–matrix is denoted by $\Omega(\mathsf{c})$, whereas given a weight w, the set of conductances such that $\mathsf{M}^\dagger(\mathsf{c},\mathsf{w})$ is a M–matrix is denoted by $\mathsf{C}(\mathsf{w})$. Therefore, $\mathsf{w}\in\Omega(\mathsf{c})$ iff $\mathsf{c}\in\mathsf{C}(\mathsf{w})$. We drop w in all the expressions when w is constant; that is when $\mathsf{w}=n^{-\frac{1}{2}}\mathsf{e}$.

2 Characterization of $\mathsf{M}^\dagger(\mathsf{c},\mathsf{w})$ as a M–matrix

Given a conductance c and a weight w, our analysis is based in the following expression of $\mathsf{M}^\dagger(\mathsf{c},\mathsf{w})=(g_{ij})$, where

$$g_{ji}=g_{ij}=w_iw_j\left[\sum_{k=1}^{i-1}\frac{\left(\sum\limits_{l=1}^{k}w_l^2\right)^2}{c_kw_kw_{k+1}}+\sum_{k=i}^{n-1}\frac{\left(\sum\limits_{l=k+1}^{n}w_l^2\right)^2}{c_kw_kw_{k+1}}-\sum_{k=i}^{j-1}\frac{\left(\sum\limits_{l=k+1}^{n}w_l^2\right)}{c_kw_kw_{k+1}}\right]$$

for any $1\le i\le j\le n$ that was obtained in [2, Corollary 5.2] and on the fact that the Moore–Penrose inverse of a symmetric and positive semi–definite matrix is itself symmetric and positive semi–definite.

Theorem 2.1 *Given a conductance* c *and a weight* w*, then* $\mathsf{M}^\dagger(\mathsf{c},\mathsf{w})$ *is a* M*–matrix iff* $g_{ii+1} \le 0$ *for any* $i = 1,\dots,n-1$*, that is; iff*

$$\frac{\Big(\sum\limits_{l=i+1}^{n} w_l^2\Big)\Big(\sum\limits_{l=1}^{i} w_l^2\Big)}{c_i w_i w_{i+1}} \ge \sum_{k=1}^{i-1} \frac{\Big(\sum\limits_{l=1}^{k} w_l^2\Big)^2}{c_k w_k w_{k+1}} + \sum_{k=i+1}^{n-1} \frac{\Big(\sum\limits_{l=k+1}^{n} w_l^2\Big)^2}{c_k w_k w_{k+1}}, \quad i = 1,\dots,n-1.$$

The above result determines a set of nonlinear inequalities involving the conductance and the weight that seems difficult to treat. In the literature, one can only find results for the constant weight. In fact, the conclusion of the above Theorem for w constant was given in [5, Lemma 3.1].

Corollary 2.2 *When the weight is constant, then* $\mathsf{M}^\dagger(\mathsf{c})$ *is a* M*–matrix iff* $n \le 4$ *and moreover either* $\frac{1}{2} \le \frac{c_1}{c_2} \le 2$ *when* $n = 3$ *or* $c_1 = c_3$ *and* $c_2 = 2c_1$ *when* $n = 4$*.*

The above result is implicitly contained in [2] and it was also obtained in [5] by using a different approach. When $n = 2$, then $\mathsf{M}^\dagger(\mathsf{c},\mathsf{w})$ is always a M–matrix. In fact, for any $c > 0$ and any $0 < x < 1$, if $\mathsf{w} = (x, \sqrt{1-x^2})$, we get

$$\mathsf{M}(\mathsf{c},\mathsf{w}) = c\begin{bmatrix} \frac{\sqrt{1-x^2}}{x} & -1 \\ -1 & \frac{x}{\sqrt{1-x^2}} \end{bmatrix}, \quad \mathsf{M}^\dagger(\mathsf{c},\mathsf{w}) = \frac{x(1-x^2)}{c}\begin{bmatrix} \sqrt{1-x^2} & -x \\ -x & \frac{x^2}{\sqrt{1-x^2}} \end{bmatrix}.$$

Corollary 2.3 *When* $n = 3$*,* $\mathsf{M}^\dagger(\mathsf{c},\mathsf{w})$ *is a* M*–matrix iff*

$$\frac{w_1^3}{w_3(1-w_3^2)} \le \frac{c_1}{c_2} \le \frac{w_1(1-w_1^2)}{w_3^3}.$$

Moreover, for any conductance c*, it is satisfied that*

$$\Omega(\mathsf{c}) = \left\{\Big(w_1, \sqrt{1-(1+t^2)w_1^2}, tw_1\Big) : 0 < t < \frac{c_2}{c_1},\ \ 0 < w_1 \le \sqrt{\frac{tc_1}{c_2 + t^3 c_1}}\right\}$$
$$\bigcup \left\{\Big(w_1, \sqrt{1-(1+t^2)w_1^2}, tw_1\Big) : \frac{c_2}{c_1} \le t,\ \ 0 < w_1 < \sqrt{\frac{1}{1+t^2}}\right\}.$$

The cases w constant and $n = 2, 3$ are the only ones in which we tackle directly the system of inequalities in Theorem 2.1. For $n \ge 4$ we will follow a different way that also works for $n = 2, 3$. For the sake of simplicity we only analyze here the case $n = 4$. So, given $\mathsf{w} \in \Omega(4)$ we consider the irreducible Z-matrix

$$\mathsf{A}(\mathsf{w}) = \begin{bmatrix} \frac{w_1(w_2^2+w_3^2+w_4^2)}{w_2} & -\frac{(w_3^2+w_4^2)^2}{w_2 w_3} & -\frac{w_4^3}{w_3} \\ -\frac{w_1^3}{w_2} & \frac{(w_1^2+w_2^2)(w_3^2+w_4^2)}{w_2 w_3} & -\frac{w_4^3}{w_3} \\ -\frac{w_1^3}{w_2} & -\frac{(w_1^2+w_2^2)^2}{w_2 w_3} & \frac{(w_1^2+w_2^2+w_3^2)w_4}{w_3} \end{bmatrix}$$

whose determinant is $D(\mathsf{w}) = \dfrac{w_1 w_4}{w_2^2 w_3^2}(w_2^2 w_3^2 - w_1^2 w_4^2)$. Observe that $\mathsf{A}(\mathsf{w})$ can be interpreted as the coefficient matrix of the inequalities system. Then, if $D(\mathsf{w}) \neq 0$ we get

$$\mathsf{A}^{-1}(\mathsf{w}) = \frac{w_2 w_3}{w_2^2 w_3^3 - w_1^2 w_4^2} \begin{bmatrix} \frac{w_3(w_1^2+w_2^2)}{w_1} & \frac{w_4^2(1-w_4^2)+w_3^4}{w_1 w_3} & \frac{w_4^2(w_3^2+w_4^2)}{w_1 w_3} \\ w_1^2 & w_2^2 + w_3^2 & w_4^2 \\ \frac{w_1(w_1^2+w_2^2)}{w_2 w_4} & \frac{w_1^2(1-w_1^2)+w_2^4}{w_2 w_4} & \frac{w_2(w_3^2+w_4^2)}{w_4} \end{bmatrix}.$$

If for a conductance c, we define $\mathsf{c}^{-1} = (c_1^{-1}, c_2^{-1}, c_3^{-1})^T$, then from Theorem 2.1, $\mathsf{M}^\dagger(\mathsf{c}, \mathsf{w})$ is a M–matrix iff $\mathsf{A}(\mathsf{w})\mathsf{c}^{-1} \geq \mathsf{0}$. Therefore, by applying well-known properties about Z-matrices, see [3], if $\mathsf{C}(\mathsf{w}) \neq \emptyset$ for a weight w, then $\mathsf{A}(\mathsf{w})$ is a M–matrix. Conversely when $\mathsf{A}(\mathsf{w})$ is a non singular M–matrix then $\mathsf{c} \in \mathsf{C}(\mathsf{w})$ iff $\mathsf{c}^{-1} = \mathsf{A}^{-1}(\mathsf{w})\mathsf{a}$, where $\mathsf{a} \geq 0$ is non null, since $\mathsf{A}^{-1}(\mathsf{w}) > \mathsf{0}$. So, our next aim is to characterize when $\mathsf{A}(\mathsf{w})$ is a M–matrix for a given $\mathsf{w} \in \Omega(4)$.

Theorem 2.4 *Given* $\mathsf{w} \in \Omega(4)$ *if* $\mathsf{c}(\mathsf{w}) = \left(\dfrac{\omega_1(\omega_1^2 + \omega_2^2)}{\omega_2(\omega_3^2 + \omega_4^2)}, \dfrac{(\omega_1^2 + \omega_2^2)}{\omega_2 \omega_3}, \dfrac{\omega_4}{\omega_3} \right)$, *then* $\mathsf{A}(\mathsf{w})\mathsf{c}^{-1}(\mathsf{w}) = D(\mathsf{w})\mathsf{e}$. *Therefore, the following properties hold:*

(i) $\mathsf{C}(\mathsf{w}) = \emptyset$ *iff* $w_1 w_4 > w_2 w_3$.

(ii) $\mathsf{A}(\mathsf{w})$ *is a singular M–matrix iff* $w_1 w_4 = w_2 w_3$. *In this case, we get that* $\mathsf{C}(\mathsf{w}) = \{t\mathsf{c}(\mathsf{w})\}_{t>0}$ *and moreover,* $\displaystyle\bigcup_{\substack{\mathsf{w}\in\Omega(4)\\ w_1 w_4 = w_2 w_3}} \mathsf{C}(\mathsf{w}) = \{\mathsf{c} : c_2^2 \geq 4c_1 c_3\}$.

(iii) $\mathsf{A}(\mathsf{w})$ *is an invertible M–matrix iff* $w_1 w_4 < w_2 w_3$ *and then* $\mathsf{c} \in \mathsf{C}(\mathsf{w})$ *iff*

there exists $a_1, a_2, a_3 \geq 0$ *such that* $a_1 + a_2 + a_3 > 0$ *and*

$$c_1 = \frac{w_1 w_3}{w_3^2(w_1^2+w_2^2)a_1 + [w_4^2(w_1^2+w_2^2)+w_3^2(w_3^2+w_4^2)]a_2 + w_4^2(w_3^2+w_4^2)a_3},$$

$$c_2 = \frac{1}{w_1^2 a_1 + (w_2^2+w_3^2)a_2 + w_4^2 a_3},$$

$$c_3 = \frac{w_2 w_4}{w_1^2(w_1^2+w_2^2)a_1 + [w_1^2(\omega_3^2+w_4^2)+w_2^2(w_1^2+w_2^2)]a_2 + w_2^2(w_3^2+w_4^2)a_3}.$$

In particular, $\{t\mathsf{c}(\mathsf{w})\}_{t>0} \subset \mathsf{C}(\mathsf{w})$.

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