Sortabilities of properties in multi-dimensional parameter spaces

Huilan Chang*
Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan 30010

Abstract. The theory of sortability of partition property was introduced to prove the existence of an optimal partition satisfying the property for optimal partition problems over single-parameter space, and then extended to multi-dimensional parameter spaces. For each partition property of interest, almost all levels of sortabilities were obtained; however, the part-specific-sortabilities are hard to be determined for many properties. In this paper, we establish a rule to generate examples that reveal the non-part-specific-sortabilities of these properties for almost all cases. Such rule also has potential of generating more concise examples to support known results.

Keywords: Optimal partition; objective function; partition property; sortability;

1 Introduction

For a finite set Θ of distinct points in a multi-dimensional parameter space $\mathbb{R}^d$, a *partition* of Θ is of the form $\pi = (\pi_1, \cdots, \pi_p)$ where $\pi_1, \cdots, \pi_p$ are disjoint nonempty subsets of Θ whose union is Θ; moreover, p is referred to as the *size* of π, $\pi_1, \cdots, \pi_p$ the *parts* of π, and $(n_1, \cdots, n_p)$ the *shape* of π where $n_i := |\pi_i|$. An *optimal partition problem* deals with the selection of a partition from a given family Π of partitions of Θ so as to optimize a given objective function. In many applications, partitions in the candidate family Π have prescribed sizes which are

*This research is partially supported by NSC 99-2811-M-009-056.

constants in terms of n and even have prescribed shapes. Π is called a *shape-family* (*size-family*) if all partitions have a prescribed shape (size), and an *open-family* if no restriction is placed. The family Π usually contains exponentially (in n) many candidates and it is usually difficult to find an optimal partition analytically. In the literature, a common approach used to reduce the candidate partitions is to identify a property of optimal partitions. Thus if there are only polynomial number of partitions satisfying the property, we can find an optimal one in polynomial time by examining all partitions satisfying the property. For example, a partition is *cone-separable* if the cones spanned by any two parts only have the origin in common. Some objective functions were proved to have optimal partitions that are cone-separable [8] and the number of cone-separable partitions of size p over a set of n points in $\mathbb{R}^d$ is at most $O(n^{(d-1)\binom{p}{2}})$ which is polynomial in n [6].

Hwang *et al.* [9] proposed a strategy to prove the existence of a partition that is optimal over a family Π and satisfies a property Q. The main idea of the strategy is to show that for any optimal partition π over Π not satisfying Q, there is a finite sequence of transformations of partitions in Π starting at π such that the optimality is preserved and the transformations guarantee ending at a partition satisfying Q. The success of the strategy is decided by two sequential notions "invariance" which ensures transformations staying inside the family and "sortability" which concerns whether transformations end at a partition satisfying Q, complementing the first notion. We will introduce their formal descriptions together with a commonly used transformation – (*local*) k-*sorting* (introduced in [9]).

A k-*subpartition* of a partition π is a set of k parts of π. A Q-k-*sorting* sorts a k-subpartition K of π not satisfying Q into a partition K' such that K' satisfies Q. A k-sorting is further characterized by different constraints on K': A *size-sorting* is a sorting with $J(K) = J(K')$ where $J(K)$ denotes the index set of a subpartition K; a *shape-sorting* preserves not only the index set but also the shape; a *open-sorting* allows K' to be any partition satisfying Q.

A t-family Π, $t \in \{\text{size, shape, open}\}$, is Q-k-*invariant* if for every partition in Π not satisfying Q and a k-subpartition K not satisfying Q, there exists a Q-k-t-sorting of K which yields a partition also in Π. Chang *et al.* [4] introduced four levels of invariance families. Π is $(\textit{strong}, k, t)$-*invariant* if for every subpartition K not satisfying Q and every Q-k-sorting of K, π' is in Π; Π is $($*part-specific*$, k, t)$*-invariant* if K is specific but the sorting is arbitrary; Π is $($*sort-specific*$, k, t)$*-invariant* if K is arbitrary but the sorting is specific; Π is $($*weak*$, k, t)$*-invariant* if both K and the sorting are specific. A family Π with each partition satisfying Q is surely Q-(l, k, t)-invariant and is referred to as a *trivial* invariant family. For simplicity's sake, we say that a partition family satisfies Q if it contains a partition satisfying Q. Accordingly, sortability can be classified into four levels. If l is a member of {strong, weak} (or {part-specific, sort-specific}), then l^{-1} denote the other member of that pair. Then Q is (l, k, t)*-sortable* if and only if there exists a non-trivial Q-(l^{-1}, k, t)-invariant family and every such family satisfies Q. Therefore, the studies turn to concern the sortabilities of partition properties.

The sortability theory was first introduced to deal with the objects in one-dimensional parameter space [9], and the (l, k, t)-sortabilities for partition properties of interest were later completely determined [4]. Hwang *et al.* [6] first extended the sortability theory to multi-dimensional parameter space. The properties of interest (will be defined in Section 2) are acyclic, convex-separable (CvS), non-penetrating (NP), noncrossing (NC), cone-separable (CnS), sphere-separable (SS), and monopoly. Most of their sortabilities were obtained [6, 5] except: k-part-specific-sortabilities for NP, SS, CvS, and NC when $k \geq 2$ and CnS when $k = 2$. In this paper, we prove their non-sortabilities for $k \geq 3$ by generating invariant families from an identical rule; further, such rule can also be applied to generate a weak-2-invariant-family not satisfying NP which is much more concise than the known example in [5].

2 Main Result

For a finite set $\Omega \subset \mathbb{R}^d$, let *Cone*$(\Omega)$ denote the cone spanned by Ω with its vertex at the origin 0, and let *Conv*(Ω) denote the convex hull of Ω. A cone is *pointed* if for any nonzero point v, not both v and $-v$ are in the cone. Furthermore, Ω is said to *penetrate* another finite set $\Omega' \subset \mathbb{R}^d$ if $\Omega \cap \text{Conv}(\Omega') \neq \emptyset$; in this case we write $\Omega \rightarrow \Omega'$.

The following properties of a partition π are considered in the literature [1, 2, 3, 5, 6, 7].

Convex Separable (CvS):	For all i, j, Conv$(\pi_i) \cap$ Conv$(\pi_j) = \emptyset$.
Noncrossing (NC):	For all i, j, either Conv$(\pi_i) \cap$ Conv$(\pi_j) = \emptyset$ or $\pi_i \subset$ Conv(π_j) and Conv$(\pi_i) \cap \pi_j = \emptyset$ or vice versa.
Nonpenetrating (NP):	For all i, j, $\pi_i \not\to \pi_j$.
Cone Separable (CnS):	For all i, j, Cone$(\pi_i) \cap$ Cone$(\pi_j) = \{0\}$.
Sphere Separable (SS):	For all i, j, there exists a sphere $S \subset \mathbb{R}^d$ such that one part is within S and the other outside of S.

The implications among the partition properties were given in [6] as in Figure 1 where $Q \Rightarrow Q'$ means that if a partition satisfies Q, then it also satisfies Q'.

$$\begin{array}{ccccc} & & & & NP \\ & & & \nearrow\!\!\!\!\nearrow & \\ CnS & \Rightarrow & CvS & \Rightarrow & SS \\ & & & \searrow\!\!\!\!\searrow & \\ & & & & NC \end{array}$$

Figure 1: Implications among properties.

For a finite set $\Omega \subset \mathbb{R}^d$ and $\delta > 0$, let $\Omega(\delta)$ denote the set $\Omega \cup \{\delta v : v \in \Omega\}$. For the sake of simplicity, we define $\Omega(0) = \Omega$. For a sphere $S = \{x \in \mathbb{R}^d : \|x - v\| \leq R\}$, the boundary of S is given by bd$(S) = \{x \in \mathbb{R}^d : \|x - v\| = R\}$. Let $\mathcal{B}$ denote the unit sphere centering at 0.

Lemma 1. *For any finite set $\Omega \subset$ bd$(\mathcal{B})$ and any δ with $0 < \delta < 1$, if $v \in$ Conv$(\Omega(\delta)) \cap$ bd$(\mathcal{B})$, then $v \in \Omega$.*

Proof. Let $\Omega = \{x_1, x_2, \cdots, x_s\}$. Then $v = \sum_{i=1}^s a_i x_i + \sum_{i=1}^s b_i \delta x_i$ for some non-negative a_i's and b_i's with $\sum_{i=1}^s (a_i + b_i) = 1$. Then

$$1 = \|v\| \leq \sum_{i=1}^s (a_i + b_i \delta)\|x_i\| = 1 - (1 - \delta) \sum_{i=1}^s b_i \leq 1. \tag{1}$$

Since $\delta < 1$, $\sum_{i=1}^s b_i = 0$ and thus $b_i = 0$ for $i = 1, \cdots, s$. That the equality of (1) holds implies $v = a x_i$ for some $a > 0$ and some i; further, $1 = \|v\|$ and thus $v = x_i \in \Omega$. ∎

A consequence of Lemma 1 is for any finite sets $\Omega^1, \Omega^2 \subset \text{bd}(\mathcal{B})$ with $\Omega^1 \not\subseteq \Omega^2$, $\Omega^1(\delta) \not\subseteq \text{Conv}(\Omega^2(\delta))$; otherwise, $\Omega^1 \subset \text{Conv}(\Omega^2(\delta))$ and then $\Omega^1 \subseteq \Omega^2$, a contradiction.

Lemma 2. *Suppose $\Omega^1, \Omega^2 \subset bd(\mathcal{B}) \subset \mathbb{R}^d$ are finite and distinct, and Cone(Ω^i) is pointed for $i = 1, 2$. For $Q \in \{CvS, NC\}$, $\{\Omega^1(\delta), \Omega^2(\delta)\}$ satisfies Q for all δ with $0 < \delta < 1$ if and only if $\{\Omega^1, \Omega^2\}$ satisfies CnS. For $Q = NP$, the statement holds except for the necessary condition under $d \geq 3$.*

Proof. Let $\Omega^1 = \{x_1, x_2, \cdots, x_s\}$ and $\Omega^2 = \{y_1, y_2, \cdots, y_r\}$. For the sufficient condition, suppose $\{\Omega^1, \Omega^2\}$ satisfies CnS. Then there exists a nonzero d-vector C such that $Cx > 0 > Cy$ for all $x \in \Omega^1$ and $y \in \Omega^2$. Then

$$C\sum_{i=1}^{s}(a_i x_i + a_i'\delta x_i) = \sum_{i=1}^{s}(a_i + a_i'\delta)Cx_i > 0$$

$$> \sum_{i=1}^{r}(b_i + b_i'\delta)Cy_i = C\sum_{i=1}^{s}(b_i y_i + b_i'\delta y_i)$$

for any $a_i, a_i' \geq 0$ but not all zero, $b_i, b_i' \geq 0$ but not all zero, and any $0 < \delta < 1$. Hence, $\{\Omega^1(\delta), \Omega^2(\delta)\}$ satisfies CvS and thus NC and NP by implications in Figure 1.

For the necessary condition, it is easy to derive from Lemma 1 that neither $\Omega^1(\delta) \subset \text{Conv}(\Omega^2(\delta))$ nor $\Omega^2(\delta) \subset \text{Conv}(\Omega^1(\delta))$; hence, it suffices to consider $Q = CvS$. Suppose to the contrary that $\text{Cone}(\Omega^1) \cap \text{Cone}(\Omega^2)$ contains a non-zero point v. Then $v = \sum_{i=1}^{s} a_i x_i = \sum_{i=1}^{r} b_i y_i$ for some non-negative a_i's and b_i's with $\sum_{i=1}^{s}(a_i + b_i) = 1$. Let $a = \sum_{i=1}^{s} a_i$ and $b = \sum_{i=1}^{r} b_i$. If $a = b$, then $\sum_{i=1}^{s} \frac{a_i}{a} x_i = \sum_{i=1}^{r} \frac{b_i}{b} y_i$ whose coefficients sum to 1, respectively. Thus $v \in \text{Conv}(\Omega^1(\delta)) \cap \text{Conv}(\Omega^2(\delta))$ for any $0 < \delta < 1$, a contradiction. Suppose, w.l.o.g, $a > b$. Let $\delta = \frac{b}{a}$ (< 1). Then $\sum_{i=1}^{s} \frac{a_i}{a} x_i = \sum_{i=1}^{r} \frac{b_i}{a\delta} \delta y_i$ where $\sum_{i=1}^{r} \frac{b_i}{a\delta} = 1$. Thus $\text{Conv}(\Omega^1(\delta)) \cap \text{Conv}(\Omega^2(\delta)) \neq \emptyset$ when $\delta = \frac{b}{a}$, a contradiction.

For $Q = NP$ and $d = 2$, along bd$(\mathcal{B})$ order points in $\Omega^1 \cup \Omega^2$ clockwise. Suppose there exist $x, z \in \Omega^i$ and $y \in \Omega^j$ such that x, y, z are ordered for $i \neq j$. Let $\delta' y$ be the intersection point of the $0y$ segment and the xz segment. Then $\delta' y \in \text{Conv}(\Omega^1(\delta)) \cap \text{Conv}(\Omega^2(\delta))$ whenever $0 < \delta < \delta'$, a contradiction. Hence, points in Ω^i are consecutive along the order for $i = 1, 2$; further, their convex hulls are pointed and thus they are cone-disjoint. For $d \geq 3$, let $\Omega^1 = \{(\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}, \cdots, \frac{1}{\sqrt{d}}), (\frac{-1}{\sqrt{d}}, \frac{1}{\sqrt{d}}, \cdots, \frac{1}{\sqrt{d}})\}$ and $\Omega^2 = \{x_i : (x_i)_j = \delta_{\{j=i+1\}}$ for $i = 1, \cdots, d-1\}$. It is easy to verify that $\{\Omega^1(\delta), \Omega^2(\delta)\}$ satisfies NP for any $0 < \delta < 1$, but Ω^1 and Ω^2 are not cone-separable.

∎

For a partition $\pi = \{\pi_1, \cdots, \pi_p\}$, let $\pi(\delta)$ denote $\{\pi_i(\delta) : i = 1, \cdots, p\}$; for a family Π of partitions, let $\Pi(\delta)$ denote $\{\pi(\delta) : \pi \in \Pi\}$. Then we have

Lemma 3. *For $Q \in \{CvS, NC\}$, $l =$ weak or sort-specific, and $\Theta \subset bd(\mathcal{B}) \subset \mathbb{R}^d$, if Π is an (l, k, t)-invariant family of partitions of Θ not satisfying CnS, then $\Pi(\delta)$ is an (l, k, t)-invariant family not satisfying Q for some $0 < \delta < 1$. For $Q = NP$, the statement holds for $d = 2$.*

Proof. Let π_i and π_j be any two parts of a partition $\pi \in \Pi$. It is easy to verify that if $\{\pi_i(\delta'), \pi_j(\delta')\}$ does not satisfy Q for some $0 < \delta' < 1$, then $\{\pi_i(\delta), \pi_j(\delta)\}$ does not satisfy Q for any $0 < \delta < \delta'$. Thus by Lemma 2, there exists a δ with $0 < \delta < 1$ such that for any $\pi \in \Pi$, $\{\pi_i, \pi_j\}$ satisfies CnS if and only $\{\pi_i(\delta), \pi_j(\delta)\}$ satisfies Q for any two parts π_i and π_j of π. Let $\pi(\delta) \in \Pi(\delta)$ be a partition not satisfying Q. Then for any k-subpartition $K(\delta)$ of $\pi(\delta)$, $K(\delta)$ does not satisfy Q if and only if K does not satisfy CnS. Besides, a CnS-k-t-sorting of K provides a Q-sorting of $K(\delta)$ while v and δv are always sorted into the same part, guaranteeing $\Pi(\delta)$ being a (l, k, t)-invariant family not satisfying Q. ∎

Theorem 4. *NP is not (strong, 2, t)-sortable.*

Proof. By Lemma 3 and the non-(strong, $2, t$)-sortability of CnS [5, 6], the theorem follows. ∎

For example, the (weak, 2, shape)-invariant family not satisfying NP constructed from the invariant family not satisfying CnS in [5] is shown in Figure

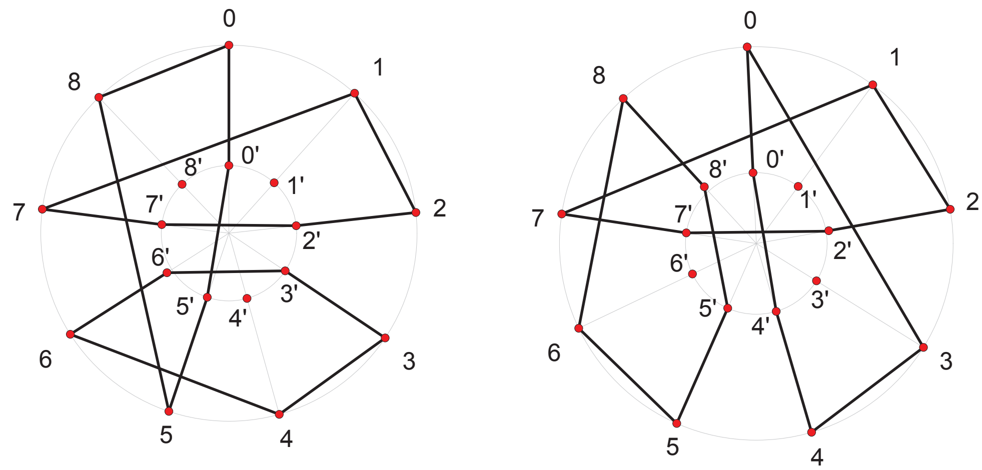


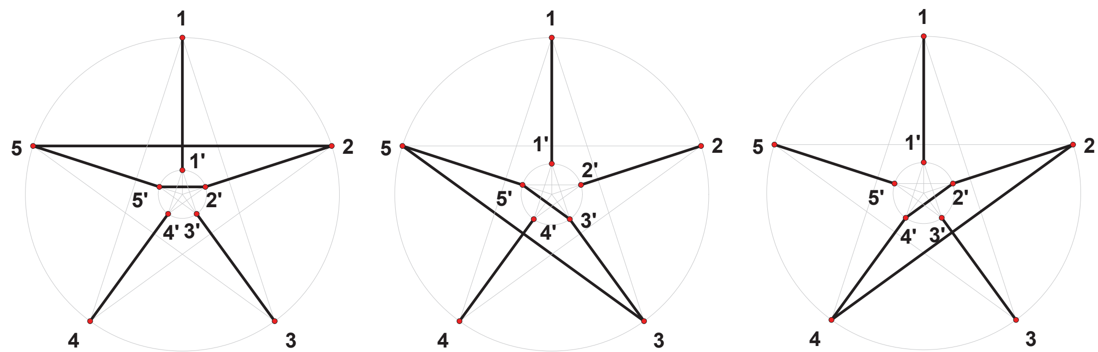
Figure 2: We start on the partition $\pi^1 = (\{1, 1', 2, 2', 7, 7'\}, \{3, 3', 4, 4', 6, 6'\}, \{0, 0', 5, 5', 8, 8'\})$ which does not satisfy NP. Sort π^1_2 and π^1_3, two parts not satisfying NP, to obtain a partition $\pi^2 = (\ \{1, 1', 2, 2', 7, 7'\}, \{0, 0', 3, 3', 4, 4'\}, \{5, 5', 6, 6', 8, 8'\})$. π^2 can be viewed as a rotation of π^1 by an angle $\frac{2\pi}{9}$; thus, π^1 will be encountered again if the corresponding sortings are implemented nine times.

2. Notice that the pattern of this invariant family is much more concise than the example in [5].

Theorem 5. *CvS, NC and NP are not (part-specific, k, t)-sortable for any t and any $k \geq 3$.*

Proof. Obtained from Lemma 3 and the non-(part-specific, k, t)-sortability of CnS for $k \geq 3$ [6]. ∎

For example, the (sort-specific, 3, shape)-invariant family not satisfying $Q \in \{CvS, NC, NP\}$ is shown in Figure 3 which is constructed from the invariant family not satisfying CnS in [6].



(a) (b) (c)

Figure 3: Each figure shows a partition of $\{1, 1', 2, 2', 3, 3', 4, 4', 5, 5'\}$ and the convex hull of each part. Let us discuss unordered partitions. We can easily extend the arguments to ordered partitions. In figure (a), all 3-subpartitions not satisfying Q are $K_1 = \{\{1, 1'\}, \{2, 2', 5, 5'\}, \{3, 3'\}\}$ and $K_2 = \{\{1, 1'\}, \{2, 2', 5, 5'\}, \{4, 4'\}\}$. (b) is obtained from a Q-3-shape-sorting of K_1 and (c) is obtained from a Q-3-shape-sorting of K_2. Either case keeps the same pattern as in Figure (a). Thus continuing the same sorting rule would produce partitions of the same pattern and thus generate a (sort-specific, 3, shape)-invariant family not satisfying Q.

Acknowledgements
The author would like to thank Frank K. Hwang for introducing me to this field.

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