Several Problems in Characterization of the Partitions Belonging to Graphs with the Given Properties

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Abstract

The paper deals with the problems of characterization of simple graphical partitions belonging to the perfect graphs and one class of stable graphs. The necessary and sufficient conditions for that the partition belonging to the perfect graph have been established.

Keywords: simple partition, perfect graph, stable graph.

1 Introduction

Studying of properties of the graphs connected with their partitions, is one of interesting and perspective directions of the graph theory. The present work is devoted to characterization of the one class of simple graphic partitions. Here we use concepts of simple graphic partitions [1], movings of edges of the graph [2], and besides, two definitions are introduced.

If in the graph there are no four of vertices such that it is possible some moving of edges incidental to them, that we name such graph perfect.
The graph with simple partition is named stable.

2 Characterization of perfect graphs.

Theorem 1 If $G = (X_1, U_1)$ and $H = (X_2, U_2)$ - two graphs with identical partitions then it is possible to receive H from G by means of finite number of movings of graph edges.

Proof. The proof is similar to the proof of the theorem of "semi-degrees" for the oriented graphs []. From theorem 1 follows that perfect graphs are stable.
Let's consider more in detail structure of perfect graphs. Lemma 1 follows directly from definition 1.

Lemma 1 Graph G is perfect if and only if the subgraph formed by any of its four vertices and edges connecting them, contains either a triangle, or three not adjacent in pairs vertices.

Lemma 2 The perfect graph contains no more than one unconnected components.
Indeed, if the graph contains two unconnected components, that, taking in each of them on two adjacent vertices, we will receive the four of vertices forbidden by lemma 1. This contradiction proves lemma 2.

Lemma 3 If $G=(X,U)$ is the perfect graph without isolated vertices then it is connected graph, and also the greatest of degrees of its vertices is equal $\Delta=|X|-1$.

Proof. Connectivity of the graph follows from lemma 2. The second statement will be proved by contradiction. We will assume that $\Delta<|X|-1$, and let be $\deg v_1=\Delta$, and $v_2,\ldots,v_{\Delta+1}$ are vertices, adjacent to v_1. Then, as the graph is connected, there is the vertex w such that $(w,v_1)\notin U$ and $(w,v_k)\in U$ for some vertex v_k such that $(v_1,v_k)\in U$. Applying lemma 1 to the four of vertices of graph $G: w,v_1,v_k,v_i$ where v_i is any of vertices, adjacent to v_1, but not coinciding with v_k, we obtain $(v_k,v_i)\in U$. From here it follows $\deg v_k\geq\Delta+1$ that is impossible, since Δ is the greatest degree. This contradiction proves lemma 3.

The lemma 4 follows from lemma1.

Lemma 4 After removing any vertex together with edges incidental to it from the perfect graph we obtain the graph that also will be perfect.

Let's use further two forms of graphic partitions:

1) the not increasing sequence of degrees of vertices $\Pi=d_1,d_2,\ldots,d_n$;

2) the form $\tilde{\Pi}=a_1,a_2,\ldots,a_{n-1}$, where a_i is the number of vertices of the graph having degree i.

Perfect graphs and their partitions are characterized by the following theorem

Theorem 2 The graphic partition $\Pi(\tilde{\Pi})$ is a partition of the connected perfect graph $G=(X,U)$ if and only if for all $d_j\geq j$ the following relations are fulfilled:

$$\left.\begin{aligned} d_1 &= |X|-1, \\ \underset{i\neq 1}{d_i} &= d_{i-1}-a_{i-1} \end{aligned}\right\}. \tag{1}$$

Proof. 1. *Necessity*. In the connected perfect graph we have $d_1=|X|-1$ (lemma 3). Let's remove from G the vertex v_1 of degree d_1 and all edges incidental to it too. Thus we will receive the perfect graph G' (lemma 4) with a_1 isolated vertices. If it contains also non-trivial component, then maximum degree of its vertices is $\Delta'=d_1-a_1-1$ (lemma 3). Returning to graph G, we have $d=\Delta'+1=d_1-a_1$. Deleting, thus, from graph G the vertices of degrees $d_1,d_2,\ldots,d_j$ until the graph consisting of isolated vertices will turn up, we receive at each stage equalities $d_i=d_{i-1}-a_{i-1}$ for all $d_j\geq j$.

2. *Sufficiency*. Let partition $\Pi(\tilde{\Pi})$ satisfying to conditions (1) be set. The algorithm for constructing the graph belonging to this partition consists of the following steps.

1. We build the star with the partition $\Pi=|X|-1,1,\ldots,1$.

2. From $(|X|-1)$ vertices of the degree 1 we chose any and it is connected with $(|X|-a_1-2)$ vertices of the same degree. Then we repeat this procedure with vertices of degree 2 etc., backward to how it was done at the proof of the necessity of condition

(1), yet we will receive the vertex of degree $d_j \geq j$ such that $d_{j+1} \leq j$. The constructed graph evidently belongs to the set partition.

Sample1. Graphic partition $\Pi = 9,7,5,4,4,3,2,2,1,1$ is set. Let's show that it is a partition of the perfect graph. We check performance of conditions (1): $d_1 = |X| - 1 = 9; d_2 = d_1 - a_1 = 7; d_3 = d_2 - a_2 = 5; d_4 = d_3 - a_3 = 4$.

2 One class of stable graphs

The problem of characterization of stable graphs (and simple graphic partitions) can be formulated in the matrix form.

Let A is a matrix of contiguities of some graph $G = (X, U)$. We form the sum $\sum_{i=1}^{|X|} C_i A^2 C_i$ where C_i is the square matrix of order $|X|$ in which the element c_{ij} is equal to 1, and other elements are equal to 0. Then it is clear that at corresponding enumerating of vertices of the graph we obtain the matrix (Π) in which the diagonal elements are degrees of vertices of the graph, and other elements are equal to zero.

$$\sum_{i=1}^{|X|} C_i A^2 C_i = (\Pi). \tag{2}$$

If A_1 and A_2 are matrixes of contiguities of two isomorphic graphs then they are connected among themselves by relations of type

$$A_2 = I_{i_1 j_1} \ldots I_{i_k j_k} A_1 I_{i_k j_k} \ldots I_{i_1 j_1} = \pi(A_1), \tag{3}$$

where I_{ij} is the matrix obtained from a single matrix by the permutation of i-th and j-th lines [3]. As $I_{ij}^2 = I$ then $A_2^2 = \pi(A_1^2)$.

If now we designate $A^2 = Y$ and will consider expression (2) as the matrix equation at the given Π then its solution can be given by matrixes of stable graphs in following two cases.

1. There is unique solution $Y = A^2$ of equation (2), where Y is the square of a symmetric matrix of order $|X|$ with a zero diagonal.
2. All solutions of equation (2) are connected among themselves by relations (3), but $A_2^2 = \pi(A_1^2) \neq A_1^2$.

It is obvious that any transformation of type (3) of the matrix of contiguities of graph G, keeping equality (2), will be equivalent to remarks of vertices of the graph G, consisting of cycles of the vertices having equal degrees.

Let's investigate case 1.

Theorem 3 If A is the matrix of contiguities of perfect graph G, and $\pi(A)$ is the remark of type (3) keeping relation (2), then $\pi(A^2) = A^2$.

Proof. Let's in the perfect graph $G = (X, U)$ the vertices v_i and v_j have equal degrees $d_i = d_j = d$. Further, let M_i be the set of vertices, adjacent to v_i, and let M_j be the set of vertices, adjacent to v_j. Then there are two different vertices $v_k ; v_l$ of the graph G such that $v_k \in M_i / M_j$ and $v_l \in M_j / M_i$.

If $(v_i, v_j) \notin U$ then v_k doesn't coincide with v_j, and v_l - with v_i.

However in this case the four-in-hand of vertices v_i, v_j, v_k, v_l does not satisfy the conditions of lemma 1 and consequently it cannot belong to the perfect graph. From here it follows that in the perfect graph for any two vertices having equal degrees, one of the following statements is correct:

a) $M_i \equiv M_j \ \& \ (v_i, v_j) \notin U$;

b) $v_j \in M_i / M_j \ \& \ v_i \in M_j / M_i$.

Extending our reasoning to some set $N_i = \{v_{i_1}, \ldots v_{i_l}\}$ of vertices of the graph G having equal degrees, we will receive that for one of the following systems of relations is also carried out:

$$\underset{r=1,\ldots,l}{M_{i_r}} \Big/ \bigcap_{s=1}^{l} M_{i_s} = \varnothing ; \tag{4}$$

$$\underset{r=1,\ldots,l}{M_{i_r}} \Big/ \bigcap_{s=1}^{l} M_{i_s} = N_i / \underset{r=1,\ldots,l}{v_{i_r}} . \tag{5}$$

Thus, in the perfect graph the subgraph formed by vertices with equal degrees, is either the complete graph, or completely unconnected.

As each element $a_{lm}^{(2)}$ of the matrix A^2 is equal to number of ways of length 2 from the vertex v_l to the vertex v_m, and remarks of type (3) consist of cycles of vertices with equal degrees, the theorem statement is easily deduced from conditions (4) and (5).
From the proved theorem it follows that if Π is the partition of the perfect graph, the equation $\sum_i C_i Y C_i = (\Pi)$ where $Y = A^2$, has the unique solution Y.

Let's assume now that the equation (2) has the unique solution Y, but Π is not the partition of the perfect graph. Let A_1 be the matrix of contiguities of the graph G_1 belonging to the partition Π, and A_2 -the matrix of contiguities of the graph G_2 obtained from G_1 by moving any pair edges and having the same partition Π.
Let's admit, for example, that such moving of edges is made: $(v_i, v_j) \to (v_i, v_l)$; $(v_k, v_l) \to (v_l, v_j)$. We will put for definiteness that $i > j > k > l$.
As a result of this moving the matrix of contiguities will change:

$$A_1 = \begin{pmatrix} \cdot & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 0 & 1 & \delta_1 & 0 \\ & & & 1 & 0 & 0 & \delta_2 \\ & & & \delta_1 & 0 & 0 & 1 \\ & & & 0 & \delta_2 & 1 & 0 \end{pmatrix} \to A_2 = \begin{pmatrix} \cdot & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 0 & 0 & \delta_1 & 1 \\ & & & 0 & 0 & 1 & \delta_2 \\ & & & \delta_1 & 1 & 0 & 0 \\ & & & 1 & \delta_2 & 0 & 0 \end{pmatrix}$$

Let's investigate, to what requirements the elements of matrixes A_1 and A_2 should satisfy that condition $A_1^2 = A_2^2$ was met.

1. As $a_{ij}^{(2)}\left(A_2^2\right) = A_{ij}^{(2)}\left(A_1^2\right) + \delta_1 + \delta_2$ then equalities $\delta_1 = 0; \delta_2 = 0$ should be fulfilled.
2. For any $m \neq j$ we get

$$a_{im}^{(2)}(A_1) = \Sigma + a_{jm}(A_1) \,\&\, a_{im}^{(2)}(A_2) = \Sigma + a_{lm}(A_1).$$

From this it follows $a_{jm}(A_1) = a_{lm}(A_1)$.

It is similarly proved that $a_{im}(A_1) = a_{km}(A_1) \,\&\, a_{jm}(A_1) = a_{km}(A_1)$. Here we consider moving $(v_i, v_j,) \to (v_i, v_k) \,\&\, (v_k, v_l,) \to (v_j, v_l)$ which is possible, since $\delta_1 = \delta_2 = 0$.
3. If M_i', M_j', M_k', M_l' are the sets of vertices, adjacent to vertices v_i, v_j, v_k, v_l accordingly, and these sets don't contain these vertices in themselves, then $M_i' \equiv M_j' \equiv M_k' \equiv M_l'$ follows from the previous consideration. Let's $\left|M_{ijkl}'\right| \geq 2$, and for some r, s we have $v_r \in M_{ijkl}' \,\&$ $v_s \in M_{ijkl}'$. Then we get $a_{js} = a_{rk} = a_{ks} = 1 \,\&\, a_{jk} = 0$.

As $a_{jk}(A_1) = 0$ then at $a_{rs}(A_1) = 0$ the moving $(v_k, v_r,) \to (v_k, v_j) \,\&\, (v_j, v_s,) \to (v_r, v_s)$ is possible in the graph G_1. But from here we will come to result $a_{ks} = 0$ by repeating point 1,. The received contradiction proves that $a_{rs}(A_1) = 1$, i.e. the subgraph formed by set of vertices M_{ijkl}', is complete**.**
4. We will consider now any edge (v_p, v_q) of the graph G_1. The following is obvious:
a) if the moving of edges (v_p, v_q) and $(v_{i;k}, v_{j;l})$ is impossible, then at least one of vertice- v_p or v_q - belongs to M_{ijkl}';
b) if the moving (v_p, v_q) and $(v_{i;k}, v_{j;l})$ is possible, then $M_p' \equiv M_q' \equiv M_{ijkl}'$.
5. From point 4 it follows that if (v_e, v_f) is such edge of the graph G_1 that $v_e \notin M_{ijkl}'$; $v_f \notin M_{ijkl}'$; v_e doesn't coincide with $v_i \vee v_j$, and v_f - with $v_i \vee v_j$ also, then the moving of edges (v_e, v_f) and (v_i, v_j) is possible and consequently $M_e' \equiv M_f' \equiv M_{ij}'$.
6. If u and w are two edges, each of which is incidental at least to one vertex from M_{ijkl}' then the moving of edges u and w is impossible. It follows from points 1 and 3.
7. From points 1-6 it follows that graphs G_1 can be realized in the form of superposition of three graphs. The first graph G_1' is formed by a subset of edges of the graph G_1 in which the each pair of edges supposes the moving. This graph consists of components of type K_2.

Removing from the graph G_1 all vertices and edges of the graph G_1', and also edges incidental to vertices of G_1' too, we obtain the second graph - G_1'' which is perfect.

The third graph G_1''' is formed by the edges connecting the each vertex of the graph G_1' with all vertices of some complete subgraph $\widetilde{G}$ of the graph G_1''; and other vertices of G_1'' form a trivial subgraph. (As appears from the proof of theorem 2, in the perfect graph all vertices of degree $d_i \geq i$ form the complete subgraph, and all vertices of degree $d_i < i$ - the trivial subgraph).

It is easy to show that the constructed graph G_1 is stable. From the reasoning spent in points 1-7, and theorems 2 and 3 we obtain the following theorem.

Theorem 4 If G is the stable graph such that at any remark π of its vertices keeping equality (2), the square of the matrix of contiguities of G doesn't change then partitions $\Pi(\tilde{\Pi})$ of this graph satisfy to following conditions:

1) $d_1 = n-1$;
2) for all $d_i > i$ it is true: $d_i = d_{i-1} - a_{i-1}$;
3) if there exists the term $d_i = i$ then the subset consisting of even number of terms of the partition such that $d_i = d_{i+1} = \ldots = d_{i+2s-1} = i$ exists also. Thus the changed partition $\Pi' = d_1 - 2s, d_2 - 2s, \ldots, d_{i-1} - 2s, d_{i+2s}, \ldots, d_n$, consisting of $(n-2s)$ terms, is the partition of the perfect graph.

The return to the theorem 4 statement also is correct.

Example 2 Let's graphic partition $\Pi = 12,11,10,9,5,5,5,5,4,4,3,2,1$ be given. We check performance of the first condition of theorem 3: $d_1 = n-1 = 12; d_2 = d_1 - a_1 = 11; d_3 = d_2 - a_2 = 10; d_4 = d_3 - a_3 = 9$.
In the partition Π there is the subset consisting of four terms: $d_5 = d_6 = d_7 = d_8 = 5$. Changed partition $\Pi' = 8,7,6,5,4,4,3,2,1$ is the partition of the perfect graph, as it is easy to check up. From this it follows that partition Π is simple, and for stable graph belonging to it by any π the equality $\pi(A^2) = A^2$ is carried out.

Interesting problem The following problem is very interesting. What is the criterion (or algorithm) for defining graphic partitions such that graphs belonging to them:

1) are planar without fail (strongly planar);
2) are non-planar without fail (strongly non-planar);
3) can be planar or non-planar.

For example, Kuratowski's graph K_5 (4,4,4,4,4) is strongly non-planar; however Kuratowski's graph $K_{3,3}$ (3,3,3,3,3,3) is neither strongly non-planar nor strongly planar.
We know the work of Ćhvatal [4], where the conditions for planarity of graphs belonging to the given partitions were found. But we don't know if the mentioned above problem is solved.

References

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