On the Locating Chromatic Number of Cartesian Product of Graphs

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Abstract

Let c be a proper k-coloring of a connected graph G and $\Pi = (C_1, C_2, \ldots, C_k)$ be an ordered partition of $V(G)$ into the resulting color classes. For a vertex v of G, the color code of v with respect to Π is defined to be the ordered k-tuple

$$c_{\Pi}(v) := (d(v, C_1), d(v, C_2), \ldots, d(v, C_k)),$$

where $d(v, C_i) = \min\{d(v, x) \mid x \in C_i\}, 1 \leq i \leq k$. If distinct vertices have distinct color codes, then c is called a locating coloring. The minimum number of colors needed in a locating coloring of G is the locating chromatic number of G, denoted by $\chi_L(G)$. In this paper, we study the locating chromatic number of the cartesian product of paths and complete graphs.

Keywords: Cartesian product, Locating coloring, Locating chromatic number.

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1 Introduction

Let G be a graph without loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$. A proper k-coloring of G is a function c defined from $V(G)$

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onto a set of colors $C = \{1, 2, \ldots, k\}$ such that every two adjacent vertices have different colors. In fact, for every i, $1 \leq i \leq k$, the set $c^{-1}(i)$ is a nonempty independent set of vertices which is called the color class i. The minimum cardinality k for which G has a proper k-coloring is the chromatic number of G, denoted by $\chi(G)$. For a connected graph G, the distance $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them, and for a subset S of $V(G)$, the distance between u and S is given by $d(u, S) := \min\{d(u, x) \mid x \in S\}$. A set $W \subseteq V(G)$ is called a *resolving set*, if for each two distinct vertices $u, v \in V(G)$ there exists $w \in W$ such that $d(u, w) \neq d(v, w)$, see [6,8]. The minimum cardinality of a resolving set in G is called the *metric dimension* of G, and denoted by $\dim_M(G)$. The vertices of a connected graph G could be represented by other means, namely, through partitions of $V(G)$ and the distances between each vertex of G and the subsets in the partition. Dividing the vertex set of a graph into classes according to some prescribed rule is a fundamental process in graph theory. Perhaps the best known example of this process is graph coloring, where the vertex set of a graph is partitioned into classes each of which is an independent set.

Definition 1.1 [1] Let c be a proper k-coloring of a connected graph G and $\Pi = (C_1, C_2, \ldots, C_k)$ be an ordered partition of $V(G)$ into the resulting color classes. For a vertex v of G, the color code of v with respect to Π is defined to be the ordered k-tuple

$$c_{\Pi}(v) := (d(v, C_1), d(v, C_2), \ldots, d(v, C_k)).$$

If distinct vertices of G have distinct color codes, then c is called a resolving or locating coloring of G. The locating chromatic number, $\chi_L(G)$, is the minimum number of colors in a locating coloring of G.

The concept of locating coloring was first introduced by Chartrand et al. in [1] and studied further in [2] and [3]. Note that since every locating coloring is a proper coloring, $\chi(G) \leq \chi_L(G)$. For more results in the subject and related subjects, one can see [1, 2, 3, 4, 7].

2 Main results

First we give an upper bound for the locating chromatic number of cartesian product of two graphs. Recall that $G \square H$ is a graph with vertex set $V(G) \times V(H)$ in which two vertices (a, b) and (a', b') are adjacent in it just when $a = a'$ and $bb' \in E(H)$, or $aa' \in E(G)$ and $b = b'$.

Theorem 2.1 *If G and H are two connected graphs, then*

$$\chi_L(G\square H) \leq \chi_L(G)\chi_L(H).$$

Proof. let $m := \chi_L(G)$ and let $A_1, A_2, ..., A_m$ be the color classes of a locating coloring of G. Also let $n := \chi_L(H)$ and let $B_1, B_2, ..., B_n$ be the color classes of a locating coloring of H. for each $i \in [n]$ and $j \in [m]$, $A_i \times B_j$ is an independent set in $G\square H$ and so the partition $\{A_i \times B_j \mid i \in [n],\ j \in [m]\ \}$ can be considered as the color classes of a proper coloring of $G\square H$. we show that this is a locating coloring of $G\square H$. Let (a, b) and (a', b') be two distinct vertices in the color class $A_i \times B_j$ and, without loss of generality, assume that $a \neq a'$. Then there exists $k \in [n]$ such that $d_G(a, A_k) \neq d_G(a', A_k)$ and so

$$\begin{aligned} d((a, b), A_k \times B_j) &= d(a, A_k) + d(b, B_j) \\ &= d(a, A_k) + 0 \\ &\neq d(a', A_k) \\ &= d(a'\ A_k) + 0 \\ &= d(a', A_k) + d(b', B_j) \\ &= d((a', b'), A_k \times B_j) \end{aligned}$$

Thus this coloring is a locating coloring. □

For $G = H = K_2$ we have

$$\chi_L(K_2\square K_2) = \chi_L(C_4) = 4 = 2 \times 2 = \chi_L(K_2)\chi_L(K_2)$$

and so the inequality is sharp. But in general this upper bound is not so good. First we will compute the exact value of the locating chromatic number of an m by n grid.

Theorem 2.2 *If $n \geq m \geq 2$, then $\chi_L(P_m\square P_n) = 4$.*

For the locating chromatic number of $P_n\square K_t$ we have the following obvious cases:

(a) $\chi_L(P_n\square K_2) = \chi_L(P_n\square P_2) = 4$

(b) $\chi_L(P_n\square K_1) = \chi_L(P_n) = 3$

(c) $\chi_L(P_1\square K_t) = \chi_L(K_t) = t$

and for the general case we have the following theorem.

Theorem 2.3 *Let $n \geq 2, t \geq 3$ be two positive integers. Then*

$$\chi_L(P_n\square K_t) = \begin{cases} t+1 & \text{if } n \leq t+1, \\ t+2 & \text{if } n \geq t+2. \end{cases}$$

For the locating chromatic number of cartesian product of complete graphs K_m and K_n, Theorem 2.1 will also give a bad upper bound, mn. If $m=n$, then every proper coloring of $K_m\square K_n$ is equivalent to an n by n Latin square. Similarly, a locating coloring of $K_m\square K_n$ is equivalent to an m by n Latin rectangle in which every two blocks with the same symbol in it, have different neighbors in their rows and columns.

Theorem 2.4 *For two positive integers $2 \leq m \leq n$, let*

$$m_0 := \max\{m_1 \mid m_1(m_1-1)-1 \leq n,\ m_1 \in \mathbb{N}\}.$$

(a) If $m \leq m_0 - 1$, then $\chi_L(K_n\square K_m) = n+1$,

(b) If $m_0+1 \leq m \leq \frac{n}{2}$, then $\chi_L(K_n\square K_m) = n+2$.

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