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RANDOM WALK AND BROWNIAN LOCAL TIMES IN WIENER SHEETS: A TRIBUTE TO MY ALMOST SURELY MOST VISITED 75 YEARS YOUNG BEST FRIENDS, ENDRE CSÁKI AND PÁL RÉVÉSZ

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Dedicated to Endre Csáki and Pál Révész
on the occasion of their 75th birthdays

Preamble

Many of us from all over the world and Hungary were happily gathered in Budapest on November 5–7, 2009 to celebrate the 75$^{\text{th}}$ birthdays of Endre Csáki and Pál Révész, attending an International Conference in their honour in the Alfréd Rényi Institute of Mathematics of the Hungarian Academy of Sciences. It pleases me very much to also have the opportunity to contribute to this Festschrift Volume of Periodica Mathematica Hungarica that is published in honour of these truly eminent scholars, who also happen to have been two of my most favourite cherished friends for a long time now. I am also glad to say that I already had the privilege to pay tribute to their achievements separately ([9], [10]), when they turned 65, and also on the occasion of celebrating them together on their 70$^{\text{th}}$ birthday ([11]).

I note in passing that reference numbers preceded by Cs in [ ] will refer to Endre Csáki's List of Publications, while those preceded by R in [ ] to Pál Révész's List of Publications. References in [ ] refer to those at the end of this exposition.

Back to 2004/2005, in [11] I concentrated mainly on highlighting some of the peaks of their formidable collaboration since 1979, when they published the first one

Mathematics subject classification numbers: 60-02, 60F15, 60F17, 60G15, 60G50, 60J55.
Key words and phrases: random walk, Brownian motion, strong approximations, random walk and Brownian local times, invariance principles, Wiener sheet, local times and additive functionals, iterated processes, limit theorems.

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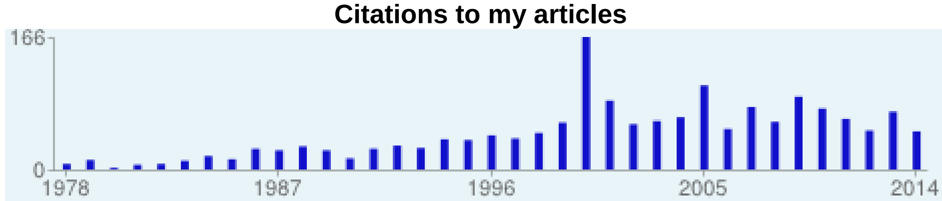

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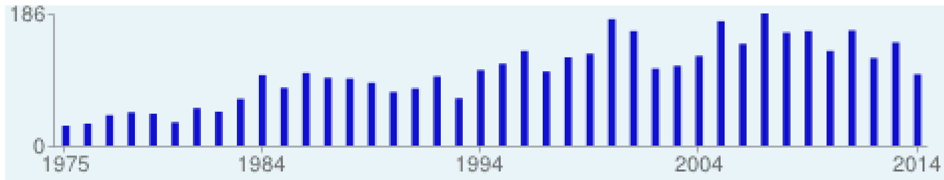

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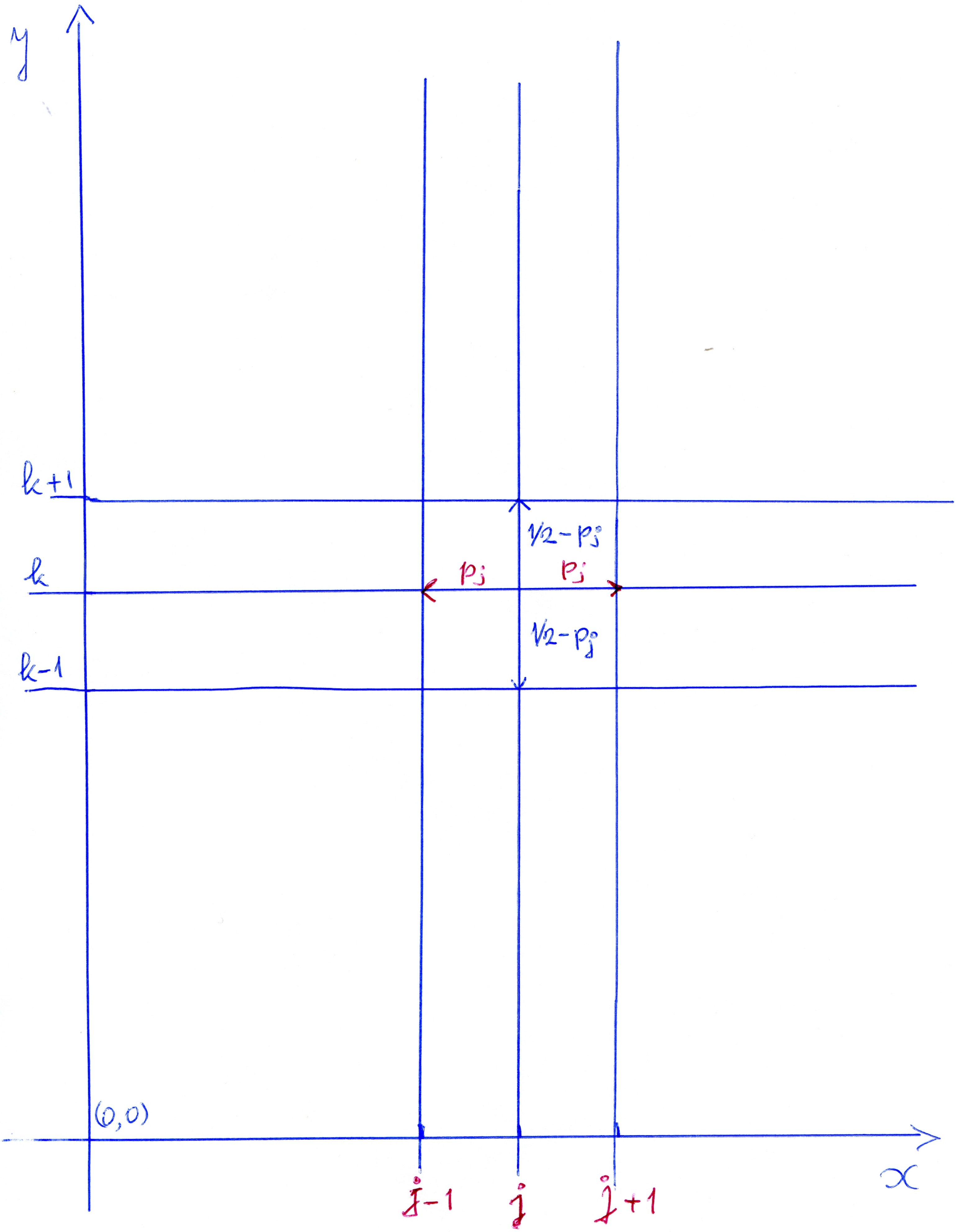

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Endre Csáki and Pál Révész are 80

Asymptotic results in probability and statistics

International conference in honor of the 80th birthday of Endre Csáki and Pál Révész

August 21-23, 2014

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Main Lecture Hall
Budapest, Hungary

August 21, 9:15–10:15: Miklós Csörgő

Life continues to be a random walk with a local time and difficulties are still welcome and almost surely overcome: A tribute to Pál Révész, already 80, and Endre Csáki, to be 80, both slowly varying as n goes to infinity.

Limit Theorems in Probability
and Statistics
Balatonielle, June 28-July 2, 1999

Limit Theorems in Probability and Statistics I.
(I. Berkes, E. Csáki, M. Csörgő, eds.)
Budapest, 2002, pp. 21-99.

LIFE IS A RANDOM WALK WITH A LOCAL TIME: A TRIBUTE TO PÁL RÉVÉSZ

Periodica Mathematica Hungarica Vol. 41 (1-2), (2000), pp.1-25

BE n FIXED OR GO TO INFINITY: DIFFICULTIES ARE WELCOME, AND ALMOST SURELY OVERCOME

A TRIBUTE TO ENDRE CSÁKI

Periodica Mathematica Hungarica Vol. 50 (1-2), 2005, pp.1-27

JOINT PATH PROPERTIES OF TWO OF MY MOST FAVOURITE FRIENDS IN MATHEMATICS:

A TRIBUTE TO ENDRE CSÁKI AND PÁL RÉVÉSZ

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Csáki, E., Csörgő, M. Földes, A and Révész, P.: Random walk on half-plane half-comb structure. *Ann. Mathematicae et Informaticae* **39** (2012) 29-44.

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Csáki, E., Földes, A and Révész, P.: On the number of cutpoints of the transient nearest neighbor random walk on the line. *J. Theoret. Probab.* **23** (2010) 624-638.

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MIKLÓS CSÖRGŐ (Carleton University, Ottawa)

Two-dimensional anisotropic random walks: fixed *versus* random column configurations for transport phenomena

Joint work with Endre Csáki, Antónia Földes, Pál Révész

Along the lines of Heyde (1982, 1993) and den Hollander (1994), and in view of [1], we consider random walks on the square lattice of the plane whose studies have in part been motivated by the so-called transport phenomena of statistical physics (cf., e.g., **1 Introduction** of [4] and **1.4 History** of [2], and their references). Two-dimensional anisotropic random walks with asymptotic density conditions *á la* [3] and [4] yield *fixed column configurations*, and nearest-neighbour random walks in a random environment on the square lattice of the plane as in [2] result in *random column configurations*. In both cases we will conclude simultaneous weak Donsker and strong Strassen type invariance principles in terms of appropriately constructed anisotropic Brownian motions on the plane. The style of presentation will be that of a semi-expository survey of related results in a historical context.

A subset of **References**

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1.1 Two-dimensional anisotropic random walks; Fixed column configurations with asymptotic density conditions

First, an **Overview** from Wikipedia, the free encyclopedia:

> "In physics, **transport phenomena** are all irreversible processes of statistical nature stemming from the random continuous motion of molecules, mostly observed in fluids. They involve a net macroscopic transfer of matter, energy or momentum in thermodynamic systems that are not in statistical equilibrium.
>
> Examples of transport processes include heat conduction (energy transfer), viscosity (momentum transfer), molecular diffusion (mass transfer), radiation and electric charge transfer in semiconductors."

Next, we quote from 1. Introduction of Heyde (1993):

> One area of application involves conductivity of various organic salts, such as tetrathiofuvalene (TTF)-tetracyanoquinodimethane (TCNQ), which show signs of superconductivity. They conduct strongly in one direction but not others. Indeed, the conductivity parallel to the structural axis is 100 times or more that perpendicular to it.
>
> The predominance of one dimension for the transport, say along rows, plus a possible lack of complete connectivity, can be modelled rather more generally by an anisotropic 2-dimensional random walk in which the transition mechanism depends only on the index of the column which is at present occupied. Thus, we consider a random walk which, if situated at a site on column j, moves with probability p_j to either horizontal neighbour and with probability $\frac{1}{2} - p_j$ to either vertical neighbour at the next step."

More formally, let X_n and Y_n denote the horizontal and vertical positions of the walk after n steps, starting from $X_0 = Y_0 = 0$, with transition probabilities as in (1) of Heyde (1993):

$$
\begin{aligned}
&P\{(X_{n+1}, Y_{n+1}) = (j-1, k) | (X_n, Y_n) = (j, k)\} = p_j \\
&P\{(X_{n+1}, Y_{n+1}) = (j+1, k) | (X_n, Y_n) = (j, k)\} = p_j \\
&P\{(X_{n+1}, Y_{n+1}) = (j, k-1) | (X_n, Y_n) = (j, k)\} = 1/2 - p_j \\
&P\{(X_{n+1}, Y_{n+1}) = (j, k+1) | (X_n, Y_n) = (j, k)\} = 1/2 - p_j
\end{aligned}
\tag{1.1}
$$

for $(j, k) \in \mathbb{Z}^2$, and $n = 0, 1, 2, \ldots$, a so-called anisotropic 2-dimensional random walk with possibly unequal **symmetric** horizontal and vertical step transition probabilities that depend only on the index of the column which is at present occupied.

Even though the transition probabilities depend on the position of the first coordinate of $\mathbf{Z}_n := (X_n, Y_n)$, $n \in \mathbf{N}$, both X_n and Y_n will be seen to behave like a simple symmetric random walk on $\mathbb{Z}$, except for a random time delay, and independently of each other.

We assume throughout that $0 < p_j \leq 1/2$ and $\min_{j \in \mathbb{Z}} p_j < 1/2$.

Thus, as in Heyde (1993), we consider a random walk which, if situated at a site on column j, moves with probability p_j to either horizontal neighbour, and with probability $1/2 - p_j$ to either vertical neighbour at the next step.

The case $p_j = 1/4$, $j = 0, \pm 1, \pm 2, \ldots$ corresponds to a simple symmetric random walk on the plane, (cf., e.g., Erdős-Taylor (1960), Dvoretzky-Erdős (1951), Révész (2005) ....).

Having $p_j = 1/2$ for some j, then the vertical line $x = j$ is missing, a so-called *non-connective column*.

If all $p_j = 1/2$, then the random walk takes place on the x axis, i.e., (X_n, Y_n) reduces to a simple symmetric random walk on the real line. This case is excluded from this investigation. Hence the assumption that $\min_{j \in \mathbb{Z}} p_j < 1/2$.

When $p_j = 1/2$, $j = \pm 1, \pm 2, \ldots$, but $p_0 = 1/4$, then all the vertical lines $x = j = \pm 1, \pm 2, \ldots$ are missing, except that of $x = 0$, i.e., the y axis. Thus we get what could be called a random walk on a "horizontal comb", whose vertical version was studied by Weiss and Havlin (1986), Bertacchi amd Zucca (2003), Bertacchi (2006), Csáki *et al.* (2009, 2011).

Heyde (1982) assumed the following asymptotic density condition for the transition probabilities of (1.1):

$$k^{-1}\sum_{j=1}^{k} p_j^{-1} = 2\gamma + o(k^{-\eta}), \quad k^{-1}\sum_{j=1}^{k} p_{-j}^{-1} = 2\gamma + o(k^{-\eta}) \tag{1.2}$$

as $k \to \infty$, for some constants $1 < \gamma < \infty$ and $1/2 < \eta < \infty$.

Under condition (1.2), for the first coordinate of $\mathbf{Z}_n$, Heyde (1982) concluded his Theorem 1 that reads as follows.

Theorem A *On an appropriate probability space for X_n there is a standard Wiener process $\{W(t), t \geq 0\}$ so that*

$$\gamma^{1/2} X_n = W(n(1+\varepsilon_n)) + O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/2}) \quad a.s. \tag{1.3}$$

as $n \to \infty$, with $\varepsilon_n \to 0$ a.s.

As to this theorem, Heyde (1982) writes:

> "If strong additional assumptions are made about the column types, for example if there is a finite number of different types which occur with fixed periodicity, then the random variable ε_n can safely be removed and relegated to the error in (2). We shall not pursue this question here because the presence of ϵ_n does not cause undue complication in the extraction of specific results from (2) as evidenced by Corollary 1 below."

We note in passing that (2) in this quote coincides with (1.3) above. The just mentioned Corollary 1 reads as follows.

Corollary A *Under condition* (1.2), *we have*

(i) $\gamma^{1/2}n^{-1/2}X_n \to_d N(0,1)$ *as* $n \to \infty$,

(ii) $\limsup_{n\to\infty}(2n\gamma^{-1}\log\log n)^{-1/2}X_n = 1$ *a.s.*

$\liminf_{n\to\infty}(2n\gamma^{-1}\log\log n)^{-1/2}X_n = -1$ *a.s.*

One of our aims is to show that no additional conditions are needed for removing and relegating the random variable ε_n to the error in (1.3).

Let $\sigma_0 = 0 < \sigma_1 < \sigma_2 < \cdots$ be the successive times at which the values of $X_i - X_{i-1}$, $i = 1, 2, \ldots$, are nonzero, and put $S_1(k) = X_{\sigma_k}$. By the assumed symmetry of the transition probabilities in (1.1), $S_1(k)$ is a simple symmetric random walk on $\mathbb{Z}$.

Also, $X_n = X_{\sigma_k}$ for $\sigma_k \le n < \sigma_{k+1}$. Now, for n fixed let

$$\sigma_{k(n)} := \max\left[j : 1 \le j \le n,\ X_j \ne X_{j-1}\right].$$

Then

$$X_n = X_{\sigma_{k(n)}} = S_1(k(n)) \tag{1.5}$$

is the horizontal position of the walk $\mathbf{Z}_n = (X_n, Y_n)$ after $k(n)$ horizontal steps in the first n steps of $\mathbf{Z}_n$. Under (1.2) Heyde (1982) concludes

$$n^{-1}k(n) \to \gamma^{-1} \text{ a.s.,} \quad \text{as } n \to \infty \tag{1.6}$$

The latter conclusion in combination with Strassen's invariance principle (cf. Strassen (1967)) as in Heyde (1982), results in his Theorem 1 (cf. Theorem A above).

Clearly, in view of (1.5), $\ell(n) := n - k(n)$ is the number of vertical steps in the first n steps of $\mathbf{Z}_n$ and, as a consequence of (1.6), under (1.2) we have also

$$n^{-1}\ell(n) \to 1 - \gamma^{-1} \text{ a.s., as } n \to \infty. \tag{1.7}$$

Under the weaker condition that $\eta = 0$ in (1.2), Heyde (1993) concluded (1.6) and (1.7) in probability, i.e., that as $n \to \infty$,

$$n^{-1}k(n) \to \gamma^{-1} \ \text{ and } \ n^{-1}\ell(n) \to 1-\gamma^{-1} \ \text{ in probability,} \tag{1.8}$$

and established the following asymptotic joint distribution for the random walk $\mathbf{Z}_n = (X_n, Y_n)$.

Theorem B *Assume* (1.2) *with* $\eta = 0$. *Then, as* $n \to \infty$,

(a) $$n^{-1/2}\mathbf{Z}_n = \left(\frac{X_n}{n^{1/2}}, \frac{Y_n}{n^{1/2}}\right) \to_d \left(\gamma^{-1/2}N_1, (1-\gamma^{-1})^{1/2}N_2\right),$$

where N_1 *and* N_2 *are independent standard normal random variables, and*

(b) $$EX_n^2 \sim \gamma^{-1}n \ \text{ and } \ EY_n^2 \sim (1-\gamma^{-1})n.$$

The symbol $\sim$ means that the ratio of the two sides tends to 1 as $n \to \infty$.

Thus, as noted by Heyde (1993), the asymptotic behaviour of $n^{-1/2}(X_n, Y_n)$ depends only on the macroscopic properties of the medium through which the walk takes place, i.e., of the $\{p_j,\ j \in \mathbb{Z}\}$ as in (1.2), with $\eta = 0$ in this particular case. In the same paper Heyde also notes that this is of considerable practical significance since, although many materials are heterogeneous on a microscopic scale, they are essentially homogeneous on a macroscopic or laboratory scale.

In conclusion Heyde (1993) writes (cf. **3. Final Remark**):

> "The so called **dimensional anisotropy** is $(1 - \gamma^{-1})/\gamma^{-1}$ and for a material like TTF-TCNQ we expect this to be, say 10^{-2}. It should be noted that this would be achieved, for example, if only every 100th column were connective. We would have $p_j = \frac{1}{2}$ for j a non-connective column and $p_j = \frac{1}{4}$ for j a connective column."

Remark 1 We note that condition (1.2) holds true if $\{p_j\}_{j\in\mathbb{Z}}$ is a periodic sequence. Namely, in this case, for a given positive integer $L \geq 1$, $p_{j+L} = p_j$ for all $j \in \mathbb{Z}$, and for $i = 0, 1, \ldots,$ we have

$$\frac{1}{L}\sum_{j=0}^{L-1}\frac{1}{p_{j+iL}} = \frac{1}{L}\sum_{j=0}^{L-1}\frac{1}{p_j} = 2\gamma, \tag{1.9}$$

with some constant $1 < \gamma < \infty$, as in (1.2). A particular periodic case, the so-called uniform case, when $p_j = 1/4$ if $|j| \equiv 0$ (mod L) and $p_j = 1/2$ otherwise yields (1.9) with $\gamma = (L+1)/L$. This uniform periodic case may serve as a model for describing dimensional anisotropies as in the just quoted example of Heyde (1993) via $(1-\gamma^{-1})/\gamma^{-1} = \gamma - 1 = 1/L$.

Csáki *et al.* [1] study the asymptotic behaviour of the anisotropic random walk $\mathbf{Z}_n = \{X_n, Y_n\}$ under both conditions (1.2) and (1.9).

In order to formulate our results here, consider $D([0,\infty),\mathbb{R}^2)$, the space of $\mathbb{R}^2$ valued *càdlàg* functions on $[0,\infty)$. For functions $(f(t),g(t)) = ((f_1(t),f_2(t)),(g_1(t),g_2(t)))$ in this function space, define for all fixed $T>0$

$$\Delta = \Delta_T(f,g) := \sup_{0\le t\le T} \|(f_1(t)-g_1(t)),(f_2(t)-g_2(t))\|, \tag{1.10}$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^2$, usually the $\|\cdot\|_p$ norm with $p=1$ or 2 in our case.

Based on Heyde's result as in (1.8) that is implied by condition (1.2) with $\eta = 0$, our first conclusion is a Donsker type weak invariance principle, an extension of Heyde's bivariate central limit theorem (CLT) as in (a) of Theorem B:

Theorem 1 *With $X_0 = Y_0 = 0$, let $\{X_n, Y_n, n\ge 0\}$ be the horizontal and vertical positions after n steps of the random walk on $\mathbb{Z}^2$ with transition probabilities as in* (1.1), *and assume that condition* (1.2) *holds true with $\eta = 0$. Then, on an appropriate probability space for this random walk $\mathbf{Z}_n := \{X_n, Y_n\}$ on $\mathbb{Z}^2$, one can construct two independent standard Wiener processes $\{W_1(t), t\ge 0\}$, $\{W_2(t), t\ge 0\}$ so that, as $n\to\infty$, with*

$$\{\mathbf{W}_n(t), t\ge 0\}_{n\ge 0} := \left\{\frac{W_1(nt\gamma^{-1})}{n^{1/2}}, \frac{W_2(nt(1-\gamma^{-1}))}{n^{1/2}}, t\ge 0\right\}_{n\ge 0} \tag{1.11}$$

we have

$$\begin{aligned}
&\sup_{0\le t\le T} \|n^{-1/2}\mathbf{Z}_{[nt]} - \mathbf{W}_n(t)\| \\
&= \sup_{0\le t\le T} \left\| \frac{X_{[nt]} - W_1(nt\gamma^{-1})}{n^{1/2}}, \frac{Y_{[nt]} - W_2(nt(1-\gamma^{-1}))}{n^{1/2}} \right\| \\
&= o_P(1) \quad \textit{for all fixed } T>0.
\end{aligned} \tag{1.12}$$

Moreover, based on (1.6) instead of (1.8) that is implied by assuming (1.2) as postulated with $1/2 < \eta < \infty$, we arrive at a strong Strassen type invariance principle for $\mathbf{Z}_n := \{X_n, Y_n\}$ on $\mathbb{Z}^2$ as follows.

Theorem 2 *With $X_0 = Y_0 = 0$, let $\{X_n, Y_n,\, n \geq 0\}$ be the horizontal and vertical positions after n steps of the random walk on the integer lattice $\mathbb{Z}^2$ with transition probabilities as in* (1.1) *and assume that condition* (1.2) *holds true. Then, on an appropriate probability space for this random walk $\mathbf{Z}_n := \{X_n, Y_n\}$ on $\mathbb{Z}^2$, one can construct two independent standard Wiener processes $\{W_1(t), t \geq 0\}$, $\{W_2(t),\, t \geq 0\}$ so that, as $n \to \infty$, with $\{\mathbf{W}_n(t),\, t \geq 0\}_{n\geq 0}$ as in* (1.11)*, we have*

$$\begin{aligned}
&\sup_{0\leq t\leq 1} \|(2n\log\log n)^{-1/2}\mathbf{Z}_{[nt]} - (2\log\log n)^{-1/2}\mathbf{W}_n(t)\| \\
&= \sup_{0\leq t\leq 1} \left\| \left(\frac{X_{[nt]} - W_1(nt\gamma^{-1})}{(2n\log\log n)^{1/2}}, \frac{Y_{[nt]} - W_2(nt(1-\gamma^{-1}))}{(2n\log\log n)^{1/2}} \right) \right\| \\
&= o(1) \ \ a.s. \qquad (1.13)
\end{aligned}$$

Remark 2 It will be seen when proving Theorems 1 and 2 that the respective conclusions of (1.12) and (1.13) hold simultaneously on the same probability space in terms of the there constructed two independent standard Wiener processes W_1 and W_2. Thus $\mathbf{W}_n$ in (1.13) coincides with that of (1.11) not only by notation, but also by construction.

Define now the measurable space $(D([0,\infty),\mathbb{R}^2),\mathcal{D})$, where $\mathcal{D}$ is the σ-field generated by the collection of all Δ-open balls for all $T > 0$ of the function space $D([0,\infty),\mathbb{R}^2)$.

As a consequence of (1.12) of Theorem 1, on taking $T = 1$, we may, for example, conclude weak convergence for the *càdlàg* process $n^{-1/2}\,\mathbf{Z}_{[n\cdot]}$ in $(D[0,1],\mathbb{R}^2)$ in terms of the following functional convergence in distribution statement.

Corollary 1 *Under the conditions of* Theorem 1, *as $n \to \infty$, we have*

$$h(n^{-1/2}\mathbf{Z}_{[nt]}) = h\left(\frac{X_{[nt]}}{n^{1/2}}, \frac{Y_{[nt]}}{n^{1/2}}\right) \to_d h(\mathbf{W}(t)\mathrm{diag}(\gamma^{-1/2},(1-\gamma^{-1})^{1/2}))$$
$$\stackrel{d}{=} h(W_1(t\gamma^{-1}), W_2(t(1-\gamma^{-1}))$$

for all $h : D(0,1],\mathbb{R}^2) \to \mathbb{R}^2$ that are $(D(]0,1],\mathbb{R}^2),\mathcal{D})$ measurable and Δ-continuous, or Δ-continuous except at points forming a set of measure zero on $(D[0,1],\mathbb{R}^2),\mathcal{D})$ with respect to the Wiener measure $\mathbb{W}$ of $(\mathbf{W}(t), 0 \le t \le 1) := ((W_1(t), W_2(t)), 0 \le t \le 1)$, where W_1 and W_2 are two independent standard Wiener processes, and $\stackrel{d}{=}$ stands for equality in distribution.

The Brownian motion $\mathbf{W}(t)\mathrm{diag}(\gamma^{-1/2},(1-\gamma^{-1})^{1/2})$ on $\mathbb{R}^2$ with the indicated diffusion matrix $\mathrm{diag}(\cdot,\cdot)$ is an example of an anisotropic Brownian motion.

On taking $t = 1$ in Corollary 1, it reduces to Theorem of Heyde (1993) (cf. (a) of Theorem B).

As another immediate consequence of Corollary 1, as $n \to \infty$, we conclude

$$
\begin{aligned}
&\left(n^{-1/2}\int_0^1 X_{[nt]}dt,\, n^{-1/2}\int_0^1 Y_{[nt]}dt\right)\\
&\quad\to_d \left(\gamma^{-1/2}\int_0^1 W_1(t)dt,\, (1-\gamma^{-1})^{1/2}\int_0^1 W_2(t)dt\right)\\
&\quad\stackrel{d}{=} (\gamma^{-1/2}N_1(0,1/3), (1-\gamma^{-1})^{1/2}N_2(0,1/3)),
\end{aligned}
$$

where $N_1(0,1/3)$ and $N_2(0,1/3)$ are independent normal random variables with mean 0 and variance 1/3.

Define now the continuous time process $\mathbf{Z}_{nt} := \{X_{nt}, Y_{nt}\}$ by linear interpolation for $t \in [0,1]$, i.e., $\mathbf{Z}_{nt}$ are random elements of $C([0,1],\mathbb{R}^2)$, the set of $\mathbb{R}^2$-valued continuous functions defined on $[0,1]$, and interpret the uniform Δ-norm as defined in (1.10) for functions in this function space.

Recall the definition of the two dimensional Strassen (1964) class of absolutely continuous functions:

$$
\begin{aligned}
\mathcal{S}^{(2)} = \Big\{&(f(x), g(x)),\, 0 \le x \le 1 : f(0) = g(0) = 0,\\
&\int_0^1 (\dot{f}^2(x) + \dot{g}^2(x))dx \le 1\Big\}.
\end{aligned}
$$

Fashioned after Corollary 1.1 of Csáki *et al.* [1], via Strassen (1964) and Theorem 2, we arrive at

Corollary 2 *For the random walk* $\mathbf{Z}_{n\cdot}$, *as a consequence of* Theorem 2, *we conclude that*

(i) *the sequence of random vector-valued functions*

$$\left(\gamma^{1/2}\frac{X_{nx}}{(2n\log\log n)^{1/2}},\left(\frac{\gamma}{\gamma-1}\right)^{1/2}\frac{Y_{nx}}{(2n\log\log n)^{1/2}},0\le x\le 1\right)_{n\ge 3}$$

is almost surely relatively compact in $C([0,1],\mathbb{R}^2)$ *in the uniform* Δ-*norm topology, and its limit points is the set of functions* $\mathcal{S}^{(2)}$ *(i.e., the collection of a.s. limits of convergent subsequences in uniform* Δ*-norm).*

(ii) *In particular, the vector sequence*

$$\left(\frac{X_n}{(2n\log\log n)^{1/2}}\frac{Y_n}{(2n\log\log n)^{1/2}}\right)_{n\ge 3}$$

is almost surely relatively compact in the rectangle

$$\left[-\frac{1}{\sqrt{\gamma}},\frac{1}{\sqrt{\gamma}}\right]\times\left[-\frac{\sqrt{\gamma-1}}{\sqrt{\gamma}},\frac{\sqrt{\gamma-1}}{\sqrt{\gamma}}\right]$$

and the set of its limit points is the ellipse

$$\left\{(x,y):\gamma x^2+\frac{\gamma}{\gamma-1}y^2\le 1\right\}.$$

(iii) *Moreover,*

$$\limsup_{n\to\infty}\frac{X_n}{\sqrt{2n\log\log n}}=\frac{1}{\sqrt{\gamma}}\quad \text{a.s.}$$

and

$$\limsup_{n\to\infty}\frac{Y_n}{\sqrt{2n\log\log n}}=\frac{\sqrt{\gamma-1}}{\sqrt{\gamma}}\quad \text{a.s.}$$

1.2 A nearest-neighbour random walk in a random environment on $\mathbb{Z}^2$; Random column configurations

Shuler (1979), *Physica* **94A**: 12-34, formulated three conjectures on the asymptotic properties of a nearest-neighbour random walk on $\mathbb{Z}^2$ that is allowed to make horizontal steps everywhere but vertical steps only on a random fraction of the columns.

Following den Hollander (1994), let

$$C = \{C(x)\}_{x \in \mathbb{Z}} \tag{1.16}$$

be a random $\{0,1\}$-valued sequence with probability law μ on $\{0,1\}^{\mathbb{Z}}$, satisfying the assumptions:

$$\begin{aligned} &\mu \text{ is stationary and ergodic (w.r.t. translations in } \mathbb{Z}), \\ &0 < q := \mu(C(0) = 1) \le 1, \end{aligned} \tag{1.17}$$

i.e., $q = E_\mu C(0)$, the expected value of $C(0)$ with respect to μ.

We note in passing that due to the assumed stationarity of the probability measure μ on $\{0,1\}^{\mathbb{Z}}$, we have (1.17) for $C(x)$ for all $x \in \mathbb{Z}$ and $E_\mu C(x) = q$.

Given the sequence $C = \{C(x)\}_{x \in \mathbb{Z}}$, a *random environment* is constructed the following way: (i) *the horizontal edges of the lattice $\mathbb{Z}^2$ are left unchanged*, i.e., all the horizontal edges in all the rows are kept, and (ii) *all the vertical edges in the column x are left or erased, depending on whether $C(x) = 1$ or $C(x) = 0$.* Thus, given $C = \{C(x)\}_{x \in \mathbb{Z}}$, all the rows are connected, but only a part of the columns are.

Now, given $C = \{C(x)\}_{x \in \mathbb{Z}}$, let

$$\{\mathbf{Z}(n)\}_{n \geq 0} = \{X(n), Y(n)\}_{n \geq 0} \tag{1.18}$$

be the random walk that starts from the origin at time 0 and chooses with equal probability one of the unerased edges of the lattice $\mathbb{Z}^2$ adjacent to the current position and jumps along it to a neighbouring site, i.e., given C, the random walk $\mathbf{Z}(n)$, $n \in \mathbf{N}$, on the path space $(\mathbb{Z}^2)^{\mathbf{N}}$ with probability law P_C as in (4) of den Hollander (1994) has the following transition probabilities:

$$P_C(\mathbf{Z}(n+1) = (x \pm 1, y) | \mathbf{Z}(n) = (x, y)) = 1/2 \quad \text{if } C(x) = 0$$

$$P_C(\mathbf{Z}(n+1) = (x \pm 1, y \pm 1) | \mathbf{Z}(n) = (x, y)) = 1/4 \quad \text{if } C(x) = 1. \tag{1.19}$$

On integrating over C with respect to μ, the thus obtained *random walk in random environment* process has probability law $P := \int P_C \mu(dC)$.

With $q = 1$, (1.19) becomes a simple symmetric random walk on $\mathbb{Z}^2$.

The three conjectures formulated by Shuler (1979), and referred to as “Ansätze” by him, relate to the asymptotic behaviour of $\mathbf{Z}(n)$ under the law P as $n \to \infty$. They concern the *total number of steps* and *the mean-square displacement in the* x (horizontal) *and* y (vertical) *directions, the probability of return to the origin*, and the *expected number of distinct sites visited.*

For n fixed, and keeping the notation used by den Hollander (1994), let

$$n_x(n) := |\{0 \le m < n : X(m+1) \ne X(m)\}|\,,$$

$$n_y(n) := |\{0 \le m < n : Y(m+1) \ne Y(m)\}|\,, \tag{1.20}$$

respectively denote the total number of horizontal and vertical steps in the first n steps of $\mathbf{Z}(n)$. Proved as *Ansatz 1* of Shuler (1979) in den Hollander (1994), with $C = \{C(x)\}_{x\in\mathbb{Z}}$ of (1.16) and μ as in (1.17), we have

$$\lim_{n\to\infty} \frac{n_x(n)}{n} = q_x \quad P\text{–a.s.}$$

$$\lim_{n\to\infty} \frac{n_y(n)}{n} = q_y \quad P\text{–a.s.} \tag{1.21}$$

Clearly, $n_x(n) + n_y(n) = n$ and, in view of (1.21), $q_x + q_y = 1$. Also, as noted by den Hollander (1994), q_x and q_y denote the density of horizontal and vertical bonds in the lattice $\mathbb{Z}^2$, i.e., $q_x + q_y = 1$ and $q_y/q_x = q$ by (1.17). Consequently, the respective limits in (1.21) are

$$q_x = 1/(1+q) \;\text{ and }\; q_y = q/(1+q). \tag{1.22}$$

We note in passing that the P–a.s. conclusions of (1.21) can be viewed as respective analogues of those of (1.6) and (1.7) for the anisotropic random walk $\mathbf{Z}_n$ under the asymptotic density condition (1.2) for its transition probabilities as in (1.1) in that, under (1.17), the asymptotic behaviour of the random walk $\mathbf{Z}(n)$ with its P_C transition probabilities as in (1.19) integrated over C behaves like an anisotropic random walk on $\mathbb{Z}^2$, with the anisotropy determined only by q via μ of (1.17). Moreover, q viewed as the density of the connected columns via $q = q_y/q_x$, it "coincides" with the so-called **dimensional anisotropy** (cf. the quoted **3. Final Remark** of Heyde (1993) right above Remark 1)

$$\lim_{n\to\infty} \frac{n^{-1}\ell(n)}{n^{-1}k(n)} = \frac{1-\gamma^{-1}}{\gamma^{-1}} = \gamma - 1 \tag{1.23}$$

of the anisotropic random walk $\mathbf{Z}_n$ under the condition (1.2) that, in turn, implies the respective a.s. conclusions of (1.6) and (1.7) of Heyde (1982).

Remark 4 In view of our conclusion in Remark 1 concerning the so-called uniform periodic case, and our lines right above on $q = q_y/q_x$ *versus* (1.23), it appears to be reasonable to say that (1.17) together with (1.19), in the long run amount to a stationary ergodic randomization of deleting columns in the uniform periodic case on an average $q = E_\mu C(0) = EC(x)$ times, instead of $(\gamma - 1)$ fraction of times as in (1.23) that, for a given positive integer $L \geq 1$, is equal to $1/L$ in the fixed column uniform periodic configuration case. On taking $q = 1/L$ in (1.17) in combination with the transition probabilities of (1.19) in the random column configuration model, in the long run amounts to realizing the dimensional anisotropy of a fixed column uniform periodic configuration $1/L$ times on an average, instead of having it equal to $(\gamma - 1) = 1/L$ eventually under the condition (1.2).

Now we are to spell out analogues of Theorems 1 and 2 under the (∗) den Hollander (1994) condition of (1.17) on μ. These analogues will constitute evidence to saying that, under the latter condition, asymptotically the random walk $\mathbf{Z}(n)$ in random environment with probability law $P = \int P_C \mu(dC)$ (cf. (1.16), (1.19) respectively for C and P_C) behaves like an anisotropic random walk on $\mathbb{Z}^2$ does, with the anisotropy determined only by q and not by any other parameters of μ, as if it were an anisotropic 2-dimensional random walk under the asymptotic density condition of Heyde (1982), with $\gamma = 1 + q$, $0 < q \le 1$, in (1.2) for the transition probabilities of (1.1).

As in den Hollander (1994), let $B_q = \{B_q(t)\}_{t\ge 0}$ be anisotropic Brownian motion on $\mathbb{R}^2$ with diffusion matrix

$$D = \begin{pmatrix} 1/(1+q) & 0 \\ 0 & q/(1+q) \end{pmatrix}, \tag{1.27}$$

i.e., $EB_q(t) = (0,0)$ and the finite dimensional distributions of $\{B_q(t)\}_{t\ge 0}$ are Gaussian with covariance matrix $\min(s,t)D$, $0 \le s,t < \infty$. Consequently, $\{B_q(t)\}_{t\ge 0}$ has the following representation

$$\begin{aligned} \{B_q(t)\}_{t\ge 0} &\stackrel{d}{=} \left\{\mathbf{W}(t)\ \mathrm{diag}\left((1/(1+q))^{1/2}, (q/(1+q))^{1/2}\right)\right\}_{t\ge 0} \\ &\stackrel{d}{=} \{W_1(t/(1+q)), W_2(tq/(1+q))\}_{t\ge 0} \end{aligned} \tag{1.28}$$

with $\{\mathbf{W}(t)\}_{t\ge 0} := \{(W_1(t),\, W_2(t))\}_{t\ge 0}$, where $\{W_1(t),\, t \ge 0\}$ and $\{W_2(t),\, t \ge 0\}$ are two independent standard Wiener processes.

We now reformulate the invariance principle of Theorem 1 of den Hollander (1994) for $\mathbf{Z}(n)$ *à la* Theorem 1 in our previous section.

Theorem C *Assume* (1.17) *for the probability law* μ *of the random* $\{0,1\}$*-valued sequence* $\{C(x)\}_{x\in\mathbb{Z}}$ *on* $\{0,1\}^{\mathbb{Z}}$*. Then* (1.21) *is true and, on an appropriate probability space for* $\mathbf{Z}(n)$ *on* $\mathbb{Z}^2$ *with transition probabilities as in* (1.19) *and probability law* $P := \int P_C\mu(dC)$*, one can construct two independent standard Wiener processes* $\{W_1(t),\, t\geq 0\}$*,* $\{W_2(t),\, t\geq 0\}$ *so that, as* $n\to\infty$*, with*

$$\{\mathbf{W}_n(t), t\geq 0\}_{n\geq 0} := \left\{\frac{W_1(nt/(1+q))}{n^{1/2}}, \frac{W_2(ntq/(1+q))}{n^{1/2}},\, t\geq 0\right\}_{n\geq 0} \tag{1.29}$$

we have

$$\begin{aligned}
&\sup_{0\leq t\leq T} \|n^{-1/2}\mathbf{Z}(nt) - \mathbf{W}_n(t)\| \\
&= \sup_{0\leq t\leq T}\left\|\frac{X([nt]) - W_1(nt/(1+q))}{n^{1/2}}, \frac{Y([nt]) - W_2(ntq/(1+q))}{n^{1/2}}\right\| \\
&= o_P(1) \quad \textit{for all fixed } T>0.
\end{aligned} \tag{1.30}$$

Moreover, *under the same conditions*, and on the same probability space with the same independent standard Wiener processes W_1 and W_2, we have also

Theorem 3

$$\begin{aligned}
&\sup_{0\leq t\leq 1}\left\|(2n\log\log n)^{-1/2}\mathbf{Z}(nt) - (2\log\log n)^{-1/2}\mathbf{W}_n(t)\right\| \\
&= \sup_{0\leq t\leq 1}\left\|\frac{X([nt]) - W_1(nt/(1+q))}{(2n\log\log n)^{1/2}}, \frac{Y([nt]) - W_2(ntq/(1+q))}{(2n\log\log n)^{1/2}}\right\| \\
&= o(1) \ \textit{a.s.}, \quad n\to\infty.
\end{aligned} \tag{1.31}$$

Remark 4 Under their respective conditions, Theorem 1 and Theorem C "coincide", and so do also Theorem 2 and Theorem 3. Thus, to the extent of simultaneously having a weak Donsker and a strong Strassen type asymptotic behaviour, the two random walks in hand behave similarly. Moreover, just like having Theorem 1 under the weaker conditions of (1.8) than that of (1.6), Theorem C will also be seen to be true under assuming only convergence in probability versions of the P–a.s. conclusions of (1.21). We do not however know how to go about weakening the assumption of (1.17) for the probability law μ so that it would only yield the desired weaker version of (1.21).

Remark 5 Mutatis mutandis, Corollaries 1 and 2 also hold true in the context of Theorems C and 3 respectively.

Let $B_\gamma = \{B_\gamma(t)\}_{t\geq 0}, 1 < \gamma < \infty$, be anisotropic Brownian motion on $\mathbb{R}^2$ with diffusion matrix

$$D_\gamma = \begin{pmatrix} 1/\gamma & 0 \\ 0 & 1-1/\gamma \end{pmatrix},$$

i.e., as in Theorems 1 & 2,

$$\begin{aligned}\{B_\lambda(t)\} &\stackrel{d}{=} \left\{\mathbf{W}(t)\ \mathrm{diag}(\gamma^{-1/2}, (1-\gamma^{-1})^{1/2})\right\}_{t\geq 0} \\ &\stackrel{d}{=} \left\{W_1(t\gamma^{-1}, W_2(t(1-\gamma^{-1}))\right\}_{t\geq 0}\end{aligned}$$

with $\{\mathbf{W}(t)\}_{t\geq 0} := \{(W_1(t),\ W_2(t))\}_{t\geq 0}$, where $\{W_1(t),\ t \geq 0\}$ and $\{W_2(t), t\geq 0\}$ are two independent standard Wiener processes.

In case of the *uniform periodic case*, i.e., when, in (1.1), $p_j = 1/4$ if $|j| \equiv 0$ (mod L) and $p_j = 1/2$ otherwise, we concluded that $\gamma = (L+1)/L$, $L \geq 1$ (cf. Remark 1)

$$D_{(L+1)/L} = \begin{pmatrix} L/(L+1) & 0 \\ 0 & 1/(L+1) \end{pmatrix}.$$

A similar conclusion holds true in this case for D of (1.27) with $q = 1/L$.

2 Proofs

2.1 Preliminaries and proofs of Theorems 1 and 2

Let again $\sigma_0 = 0 < \sigma_1 < \sigma_2 < \cdots$ be the successive times at which the values of the $X_i - X_{i-1}$, $i = 1, 2, \ldots$ are nonzero, and put again $S_1(k) = X_{\sigma_k}$. By the assumed symmetry of the transition probabilities in (1.1), $\{S_1(k),\, k \geq 0\}$ is a simple symmetric random walk on $\mathbb{Z}$. Also, $X_n = X_{\sigma_k}$ for $\sigma_k \leq n < \sigma_{k+1}$. For n fixed let

$$\sigma_{k(n)} := \max[j : j \leq n,\, X_j \neq X_{j-1}].$$

Then

$$X_n = X_{\sigma_{k(n)}} = S_1(k(n)) \tag{2.1}$$

is the horizontal position of the walk $\mathbf{Z}_n = (X_n, Y_n)$ after $k(n)$ horizontal steps in the first n steps of $\mathbf{Z}_n$.

Clearly, in view of (2.1), $\ell(n) := n - k(n)$ is the number of vertical steps in the first n steps of $\mathbf{Z}_n$. On its own, the vertical position Y_n of the walk $\mathbf{Z}_n = (X_n, Y_n)$ can be dealt with similarly to that of its horizontal position X_n. Let $\tau_0 = 0 < \tau_1 < \tau_2 < \cdots$ be the successive times at which the values of $Y_i - Y_{i-1}$, $i = 1, 2, \ldots$ are nonzero and put $S_2(k) = Y_{\tau_k}$. Then, again by the assumed symmetry of the transition probabilities (1.1), $\{S_2(k),\, k \geq 0\}$ is a simple symmetric random walk on $\mathbb{Z}$, and $Y_n = Y_{\tau_k}$ for $\tau_k \leq n < \tau_{k+1}$. For n fixed, let

$$\tau_{\ell(n)} := \max[j : j \leq n,\, Y_j \neq Y_{j-1}].$$

Then, for Y_n, the vertical position of the random walk $\mathbf{Z}_n = (X_n, Y_n)$ after n steps, we have

$$Y_n = Y_{\tau_{\ell(n)}} = S_2(\ell(n)) \tag{2.2}$$

after $\ell(n)$ vertical steps in the first n steps of $\mathbf{Z}_n$.

As above, with σ_k and τ_k standing for the times at which X and Y make their respective k^{th} steps ($\sigma_0 = \tau_0 = 0$), both

$$\{X_{\sigma_k}, k \geq 0\} = \{S_1(k), k \geq 0\} \tag{2.3}$$

$$\{Y_{\tau_k}, k \geq 0\} = \{S_2(k), k \geq 0\} \tag{2.4}$$

are simple symmetric random walks on $\mathbb{Z}$, and are *defined independently of each other*. Consequently, the random walk $\mathbf{Z}_n = (X_n, Y_n)_{n \geq 0}$ with transition probabilities as in (1.1) and viewed *à la* (2.1) and (2.2) can be studied in terms of

$$\mathbf{Z}_n = (X_n, Y_n) = (X_{\sigma_{k(n)}}, Y_{\tau_{\ell(n)}}) = (S_1(k(n)), S_2(\ell(n)) \tag{2.5}$$

where, as before, $k(n)$ and $\ell(n)$ are the respective numbesr of horizontal and vertical steps in the first n steps of $\mathbf{Z}_n$, $S_1(\cdot)$ and $S_2(\cdot)$ are independent simple symmetric random walks on $\mathbb{Z}$, and $k(n) + \ell(n) = n$.

Without changing their distribution, on an appropriate probability space for the two independent simple symmetric random walks $\{S_i(j), j \geq 0\}$, $i = 1, 2$, one can construct two independent standard Wiener processes $\{W_1(t), t \geq 0\}$ and $\{W_2(t), t \geq 0\}$ so that (cf. Komlós, Major and Tusnády [KMT] (1975))

$$|S_i(j) - W_i(j)| = O(\log j) \quad \text{a.s.}, \ i = 1, 2, \tag{2.6}$$

as $j \to \infty$, and

$$\sup_{0 \leq t \leq 1} |S_i([nt]) - W_i(nt)| = O(\log n) \quad \text{a.s.}, \ i = 1, 2, \tag{2.7}$$

as $n \to \infty$.

Consequently, as $n \to \infty$, the components of $\mathbf{Z}_n$ as in (2.5) can be studied independently via the approximations

$$|S_1(k(n)) - W_1(k(n))| = O(\log k(n)) \quad \text{a.s.}, \tag{2.8}$$

$$\sup_{0 \leq t \leq 1} |S_1(k(nt)) - W_1(k(nt))| = O(\log k(n)) \quad \text{a.s.}, \tag{2.9}$$

and

$$|S_2(\ell(n)) - W_2(\ell(n))| = O(\log \ell(n)) \quad \text{a.s.}, \tag{2.10}$$

$$\sup_{0 \leq t \leq 1} |S_2(\ell(nt)) - W_2(\ell(nt))| = O(\log \ell(n)) \quad \text{a.s.}, \tag{2.11}$$

where $k(nt)$ and $\ell(nt)$ are the respective numbers of horizontal and vertical steps in the first $[nt]$ steps of $\mathbf{Z}_{[nt]}$, $0 \leq t \leq 1$, and $k(nt) + \ell(nt) = [nt]$.

Under condition (1.2) we have (1.6) and, on writing $k(n) = n\gamma^{-1}(1+\varepsilon_n)$, with $\varepsilon_n \to 0$ a.s. as $n \to \infty$, we arrive at

$$\begin{aligned} |S_1(k(n)) - W_1(n\gamma^{-1}(1+\varepsilon_n))| &= O(\log n\gamma^{-1}(1+\varepsilon_n)) \ \text{ a.s.} \\ &= O(\log n) \ \text{ a.s.,} \end{aligned} \tag{2.12}$$

$$\sup_{0\le t\le 1} |S_1(k(nt)) - W_1(nt\gamma^{-1}(1+\varepsilon_n))| = O(\log n) \ \text{ a.s.} \tag{2.13}$$

as $n \to \infty$.

In view of (2.13), as $n \to \infty$, we have

$$\sup_{0\le t\le 1} \left| \frac{X_{[nt]}}{n^{1/2}} - \frac{W_1(nt\gamma^{-1}(1+\varepsilon_n))}{n^{1/2}} \right| = O\left(\frac{\log n}{n^{1/2}}\right) \ \text{ a.s.} \tag{2.14}$$

As to Y_n, the vertical position of the random walk $\mathbf{Z}_n = (X_n, Y_n)$, recall that we have $Y_n = Y_{\tau_{\ell(n)}} = S_2(\ell(n))$. On writing $\ell(n) = n(1-\gamma^{-1})(1+\tilde{\varepsilon}_n)$, with $\tilde{\varepsilon}_n \to 0$ a.s. as $n \to \infty$, *mutatis mutandis* in concluding (2.14), with a standard Wiener process $\{W_2(t), t \ge 0\}$ as in (2.6), we arrive at

$$\sup_{0\le t\le 1} \left| \frac{Y_{[nt]}}{n^{1/2}} - \frac{W_2(nt(1-\gamma^{-1})(1+\tilde{\varepsilon}_n))}{n^{1/2}} \right| = O\left(\frac{\log n}{n^{1/2}}\right) \ \text{ a.s.} \tag{2.15}$$

as $n \to \infty$.

We note again that the conclusions of (2.12)–(2.15) are based on the Heyde (1982) condition (1.2) yielding

$$k(n) = n\gamma^{-1}(1+\varepsilon_n), \ \ \ell(n) = n(1-\gamma^{-1})(1+\tilde{\varepsilon}_n) \tag{2.16}$$

with ε_n and $\tilde{\varepsilon}_n$ both converging almost surely to 0 as $n \to \infty$.

Under the weaker condition that $\eta = 0$ in (1.2), (cf. Heyde (1993)), in (2.16) ε_n and $\tilde{\varepsilon}_n$ both converge in probability to 0. This, in turn, results in having in probability versions of (2.14) and (2.15) as follows.

With the two independent standard Wiener processes W_1 and W_2 as in (2.6), and ε_n and $\tilde{\varepsilon}_n$ both converging in probability to 0 as $n \to \infty$, we arrive at

$$\sup_{0\le t\le 1}\left|\frac{X_{[nt]}}{n^{1/2}} - \frac{W_1(nt\gamma^{-1}(1+\varepsilon_n))}{n^{1/2}}\right| = o_P(1), \tag{2.17}$$

$$\sup_{0\le t\le 1}\left|\frac{Y_{[nt]}}{n^{1/2}} - \frac{W_2(nt(1-\gamma^{-1})(1+\tilde{\varepsilon}_n))}{n^{1/2}}\right| = o_P(1). \tag{2.18}$$

Lemma 1 *Under the condition* (1.2) *as is, that via* (2.16) *yields the a.s. convergence of both ε_n and $\tilde{\varepsilon}_n$ to 0 as $n \to \infty$, and also under the condition* (1.2) *with $\eta = 0$, that via* (2.16) *results in ε_n and $\tilde{\varepsilon}_n$ both converging in probability to 0 as $n \to \infty$, we have*

$$\sup_{0\le t\le 1}\left|\frac{W_1(nt\gamma^{-1}(1+\varepsilon_n)) - W_1(nt\gamma^{-1})}{n^{1/2}}\right| = o_P(1), \tag{2.19}$$

$$\sup_{0\le t\le 1}\left|\frac{W_2(nt(1-\gamma^{-1})(1+\tilde{\varepsilon}_n)) - W_2(nt(1-\gamma^{-1}))}{n^{1/2}}\right| = o_P(1). \tag{2.20}$$

Proof of Theorem 1 With W_1 and W_2 as in (2.6), and assuming (1.2) with $\eta = 0$, we combine (2.17) with (2.19) and (2.18) with (2.20), and thus conclude Theorem 1 with $T = 1$, without loss of generality, i.e., similarly for all fixed $T > 0$ as well. $\square$

Proof of Lemma 1 First assume (1.2). Then with $\varepsilon_n \to 0$ a.s. as $n \to \infty$, we have

$$
\begin{aligned}
& \sup_{0 \le t \le 1} \left| \frac{W_1(nt\gamma^{-1}(1+\varepsilon_n)) - W_1(nt\gamma^{-1})}{n^{1/2}} \right| \\
& \stackrel{d}{=} \sup_{0 \le t \le 1} \left| W_1(t\gamma^{-1}(1+\varepsilon_n)) - W_1(t\gamma^{-1}) \right|, \quad \text{for each} \quad n \ge 1, \\
& \stackrel{d}{=} \sup_{0 \le t \le 1} \left| \gamma^{-1/2}(W_1(t + t\varepsilon_n) - W_1(t)) \right|, \quad \text{for} \quad 1 < \gamma < \infty, \\
& \le \sup_{0 \le t \le 1} \sup_{0 < s \le |\varepsilon_n|} \gamma^{-1/2} \left| W_1(t+s) - W_1(t) \right| \\
& \le \sup_{0 \le t \le 1} \sup_{0 < s \le \varepsilon} \gamma^{-1/2} \left| W_1(t+s) - W_1(t) \right|, \text{ with any } \varepsilon > 0, \text{ however small,} \\
& \qquad \text{for all but a finite number of } n \text{ on account of } \varepsilon_n \to 0 \ \text{ a.s. as } n \to \infty, \\
& = O(1)(\varepsilon \log 1/\varepsilon)^{1/2} \ \text{ a.s.} \\
& = o(1) \ \text{ a.s.} \qquad (2.21)
\end{aligned}
$$

by the Lévy modulus of continuity ($n \to \infty$, $\varepsilon \downarrow 0$). Thus, via (2.21), we conclude (2.19) under the condition (1.2), and a similar argument yields (2.20) as well under the same condition.

Assuming (1.2) with $\eta = 0$, results in having $\varepsilon_n \to 0$ in probability, as $n \to \infty$. Consequently, and equivalently, every subsequence $\{\varepsilon_{n_k}\}$ contains a further subsequence, say $\{\varepsilon_{n_{k(m)}}\}$, so that $\varepsilon_{n_{k(m)}} \to 0$ a.s. as $n_{k(m)} \to \infty$. Hence the proof above in terms of these subsequences yields (2.19) in this case. A similar argument results in having (2.20) as well under the same condition. This also completes the proof of Lemma 1. $\square$

Lemma 2 *Under condition* (1.2), *as* $n \to \infty$, *we have*

$$\sup_{0 \le t \le 1} \left| \frac{W_1(nt\gamma^{-1}(1+\varepsilon_n)) - W_1(nt\gamma^{-1})}{(2n \log\log n)^{1/2}} \right| = o(1) \quad \text{a.s.} \tag{2.24}$$

as well as

$$\sup_{0 \le t \le 1} \left| \frac{W_2(nt(1-\gamma^{-1})(1+\tilde{\varepsilon}_n)) - W_2(nt(1-\gamma^{-1}))}{(2n \log\log n)^{1/2}} \right| = o(1) \quad \text{a.s.} \tag{2.25}$$

where ε_n *and* $\tilde{\varepsilon}_n$ *both converge almost surely to* 0, *as indicated in* (2.16).

Just like in the case of Lemma 1, here too, W_1 and W_2 can be any generic Wiener processes. The specific forms of (2.24) and (2.25) are only for the sake of convenient reference when proving Theorem 2 and, later on, when indicating the proof of Theorem 3.

In the proof of Lemma 2, we make use of the following large increment result of Csörgő and Révész (1979, 1981) for a standard Wiener process $\{W(t), t \ge 0\}$: *Let* a_T $(T \ge 0)$ *be a monotonically nondecreasing function of* T *so that* T/a_T *is nondecreasing and* $0 < a_T \le T$. *Define* $\beta_T = \{2a_T(\log \frac{T}{a_T} + \log\log T)\}^{-1/2}$. *Then*

$$\limsup_{T \to \infty} \sup_{0 \le t \le T - a_T} \sup_{0 \le s \le a_T} \beta_T |W(t+s) - W(t)| = 1 \quad \text{a.s.} \tag{2.26}$$

Proof of Lemma 2 We have

$$\begin{aligned}
&\sup_{0\le nt\gamma^{-1}\le n\gamma^{-1}} |W_1(nt\gamma^{-1}(1+\varepsilon_n)) - W_1(nt\gamma^{-1})| \\
&\le \sup_{0\le nt\le n} \sup_{0\le nt|\varepsilon_n|\le n|\varepsilon_n|} \gamma^{-1/2}|W_1(nt+nt\varepsilon_n) - W_1(nt)| \\
&\le \sup_{0\le nt\le n} \sup_{0\le s\le n\varepsilon} \gamma^{-1/2}|W_1(nt+s) - W_1(nt)| \qquad (2.27)
\end{aligned}$$

with any $\varepsilon > 0$, however small, for all but a finite number of n, on account of $\varepsilon_n \to 0$ a.s. as $n \to \infty$.

Moreover, with any $0 < \varepsilon < 1$,

$$\begin{aligned}
&\sup_{0\le nt\le n} \sup_{0\le s\le n\varepsilon} |W_1(nt+s) - W_1(nt)| \\
&\le \sup_{0\le nt\le n-n\varepsilon} \sup_{0\le s\le n\varepsilon} |W_1(nt+s) - W_1(nt)| \\
&\quad + \sup_{n-n\varepsilon\le nt\le n} \sup_{0\le s\le n\varepsilon} |W_1(nt+s) - W_1(nt)| \\
&= O\left(\left(2n\varepsilon\left(\log\frac{n}{n\varepsilon} + \log\log n\right)\right)^{1/2}\right) \\
&= O\left(\left(2n\varepsilon\log\frac{1}{\varepsilon} + \varepsilon 2n\log\log n\right)^{1/2}\right) \\
&= \left(\varepsilon\log\frac{1}{\varepsilon} + \varepsilon\right)^{1/2} O\left((2n\log\log n)^{1/2}\right) \text{ a.s.} \qquad (2.28)
\end{aligned}$$

as $n \to \infty$, on applying (2.26) twice.

On combining now (2.28) with (2.27) via letting $n \to \infty$ and $\varepsilon \downarrow 0$, we arrive at

$$\sup_{0\le t\le 1} \frac{|W_1(nt\gamma^{-1}(1+\varepsilon_n)) - W_1(nt\gamma^{-1})|}{(2n\log\log n)^{1/2}} = o(1) \quad \text{a.s.},$$

i.e., we have (2.24). Mutatis mutandis, the conclusion of (2.25) is seen to be true as well along similar lines. $\square$

Proof of Theorem 2 By (2.14), as $n \to \infty$,

$$\sup_{0 \le t \le 1} \left| \frac{X_{[nt]} - W_1(nt\gamma^{-1}(1+\varepsilon_n))}{(2n \log\log n)^{1/2}} \right| = O\left(\frac{\log n}{(2n \log\log n)^{1/2}} \right) \text{ a.s.}, \quad (2.29)$$

Putting (2.24) and (2.29) together, as $n \to \infty$, we conclude

$$\sup_{0 \le t \le 1} \left| \frac{X_{[nt]} - W_1(nt\gamma^{-1})}{(2n \log\log n)^{1/2}} \right| = o(1) \text{ a.s.} \quad (2.30)$$

In a similar fashion, as $n \to \infty$, one concludes

$$\sup_{0 \le t \le 1} \left| \frac{Y_{[nt]} - W_2(nt(1-\gamma^{-1}))}{(2n \log\log n)^{1/2}} \right| = o(1) \text{ a.s.} \quad (2.31)$$

as well. Consequently, via (2.30) and (2.31), we arrive at having Theorem 2. $\square$

2.2 Preliminaries and proofs of Theorem C and Theorem 3

$$\begin{aligned}\{\mathbf{Z}(n)\}_{n\geq 0} &= \{X(n), Y(n)\}_{n\geq 0}\\ &= \{S_x(n_x(n)), S_y(n_y(n))\}_{n\geq 0},\end{aligned}$$

$S_x(\cdot)$ and $S_y(\cdot)$ are simple symmetric random walks on $\mathbb{Z}$, independent of $C = \{C(x)\}_{x\in\mathbb{Z}}$ and of each other, and, as in (1.21),

$$\lim_{n\to\infty} \frac{n_x(n)}{n} = \frac{1}{1+q} \quad P\text{–a.s.}$$

$$\lim_{n\to\infty} \frac{n_y(n)}{n} = \frac{q}{1+q} \quad P\text{–a.s.}$$

where $n_x(n)$ and $n_y(n)$ respectively denote the total number of horizontal and vertical steps in the first n steps of $\mathbf{Z}(n)$.

Hence...