Elementary functions

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It was proved by D. Richardson in 1968 that the question, whether or not a given element of the class E of elementary functions has a real root is recursively undecidable. Following Richardson, several subclasses of E have been presented with the same property; the smallest known such class is the ring generated by the constant 1 function, x, $\sin x^n$ and $\sin(x \cdot \sin x^n)$ $(n = 1, 2, \ldots)$. Now the question is how far this class is from being optimal. Assuming Schanuel's conjecture of transcendental number theory we prove that in the ring generated by 1, x, $\sin x^n$ and $\cos x^n$ it is recursively decidable whether or not a given element has a root in a given interval, and that in the ring generated by 1, $\sin x^n$ and $\cos x^n$ it is recursively decidable whether or not a given element has a real root.

Richardson's theorem implies that it is recursively undecidable whether or not a given elementary function is identically zero; at least if $\arcsin x$ counts as an elementary function. This fact leads on to the topic of identifying the „largest" class that can be called elementary functions. We briefly review the known results and the open problems this topic involves, namely the growth of the solutions of algebraic differential equations defined on the real line.