A nonamenable "factor" of a euclidean space

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Abstract

Answering a question of Benjamini, we show an isometry-invariant partition of the euclidean space $\mathbb{R}^3$ into infinite connected indinstinguishable pieces, such that the adjacency graph defined on the pieces is the 3-regular infinite tree. The existence of such an invariant decomposition (from amenable to nonamenable) is rather unexpected, e.g. there is no way to define a random copy of the 3-regular infinite tree on the vertices of $\mathbb{Z}^d$ in such a way that it is invariant under isometries.

Along the way, it is proved that any finitely generated amenable Cayley graph can be represented in $\mathbb{R}^3$ as an *invariant* collection of polyhedral domains, a 3-dimensional analogue of planar maps. A new technique is developed to prove indistinguishability for certain constructions, connecting this notion to factor of iid's.

1 Introduction

Definition 1. Let X and G be two graphs. Say that a partition $\mathcal{P}$ of $V(X)$ is a G**-partition of** X, if the graph $H = (V(H), E(H))$, $V(H) = \mathcal{P}$, $E(H) = \{(C, C') : C, C' \in \mathcal{P}, \exists x \in C, x' \in C', (x, x') \in E(X)\}$ is isomorphic to G. Say that it is a *connected* G-partition if every $C \in \mathcal{P}$ induces a connected graph in G.

Let G be a graph as before. Say that $\mathcal{P}$ is a G**-partition of** $\mathbb{R}^3$ if the following hold.

- Every element of $\mathcal{P}$ is an open subset of $\mathbb{R}^3$.

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- The elements of $\mathcal{P}$ are pairwise disjoint, the closure of their union is $\mathbb{R}^3$.
- Say that two elements of $\mathcal{P}$ are adjacent, if the intersection of their closures contains a two dimensional smooth subset. (We say that a subset of $\mathbb{R}^3$ is **2-dimensional** if it is diffeomorphic to $[0,1]^2$.) Then the graph defined on $\mathcal{P}$ this way is isomorphic to G.

If further every partition class (*piece*) in $\mathcal{P}$ is connected then we say that $\mathcal{P}$ is a **connected G-partition** of $\mathbb{R}^3$. Denote by T_3 the 3-regular (infinite) tree.

Question 1.1. (Itai Benjamini) Is there an invariant measurable random connected T_3-partition of $\mathbb{R}^3$ such that the pieces of the partition are indistinguishable?

It follows from [3] that there exists no such partition for $\mathbb{R}^2$. Furthermore, it is easy to show that there does not exist a random invariant map of the vertices of T_3 to $\mathbb{Z}^d$. So one may expect that a partition that satisfies the question does not exist. However, we prove that the answer to the question is positive. Our construction works for any $\mathbb{R}^d$, $d \geq 3$, we state it for $d = 3$ to make the proof a bit simpler.

Theorem 1.2. There exists an invariant connected T_3-partition $\mathcal{P}$ of $\mathbb{R}^3$ with indistinguishable pieces.

By the indistinguishability of the pieces we mean the following. Let X be the set of all open subsets of the euclidean space $\mathbb{R}^3$, and consider the Hausdorff metric on it. Suppose that some $A \subset X$ is Borel measurable, and is closed under isometries of $\mathbb{R}^3$ (i.e., if a set is in A then all its isometric translates are in it too). We say that the **pieces of a random partition of $\mathbb{R}^3$ are indistinguishable** if for any such A either every piece of the partition is in A almost surely, or none of them.

The requirement that the pieces are indistinguishable is natural, otherwise there is a simple counterexample using the sizes of the (bounded) pieces as the property to distinguish them. See Remark 3.5.

Below is a sketch of the proof. The rigorous reader may skip this sketch and proceed to the next section right away. By a **random copy** of graph H on graph G we mean a random bijective map of $V(H)$ to $V(G)$. We say that a random copy is invariant, if its distribution is invariant with respect to the automorphisms of G. By an invariant random copy of graph G in $\mathbb{R}^3$ we mean a random map ϕ from $V(G)$ to $\mathbb{R}^3$ whose distribution is invariant under the isometries of $\mathbb{R}^3$, and such that $\phi(V(G))$ has no accumulation points.

Sketch of the proof:

1. Let G be some arbitrary amenable transitive graph. One can take an isometry-invariant random copy of $\mathbb{Z}^3$ in $\mathbb{R}^3$ by taking random rotations and shifts. Then one can define a random copy of G on $\mathbb{Z}^3$, invariant under the isomorphisms of $\mathbb{Z}^3$. Combining these two, we get a random isometry-invariant copy of G in $\mathbb{R}^3$.

2. We show that any graph G that has an isometry-invariant copy ϕ in $\mathbb{R}^3$ has a representation by polyhedra. Moreover, this representation can be defined as a *factor of iid from the vertices of G modulo ϕ*, by which we mean the following. Let $\lambda(v)$ be iid Lebesgue$[0,1]$ variables as $v \in V(G)$. Then a representation by polyhedra is a factor of iid modulo ϕ, if for every $v \in V(G)$, the polyhedron representing v can be determined as a function of the configuration in $B(\phi(v), r) \cap \phi(V(G)) =: B_r$ and the labels $(\lambda(w))_{w \in B_r}$, up to a small error. Here small error means that for any $\epsilon > 0$ with probability at least $1 - \epsilon$, the polyhedron that we assign to v based on the r-neighborhood has Hausdorff distance at most ϵ from the polyhedron that the representation by polyhedra assigns to it.

3. Choose G to be the Baumslag-Solitar group BS(1,2), which is known to be amenable. This group has an invariant partition into "fibers" that are connected, and whose adjacency graph is T_3. In the invariant polyhedral representation of G (as in the previous item), take the union of polyhedra that correspond to cells in the same fibre. Do this separately for each fiber, to obtain pieces whose adjacency graph is T_3. The resulting pieces may not be indistinguishable, if we were not careful enough. However, if the images of the fibers in the random copy of G in $\mathbb{R}^3$ are indistinguishable (as sets in $\mathbb{R}^3$, in the natural sense, and including the scenery), then the collections of polyhedra that correspond to these fibres will also be indistinguishable. This will be a consequence of that the polyhedra are determined as factors of iid on G, and using Lemma 3.3. In that lemma we show that indistinguishable components remain indistinguishable if a factor of iid decoration is applied to them. Fibers are indistinguishable components of a (trivial) invariant percolation on G (which is supported on a single configuration).

4. So, our goal is to find an invariant copy of G in $\mathbb{R}^3$ with the images of fibers being indistinguishable. This problem will turn out to be the same as finding an invariant copy of G on $\mathbb{Z}^3$ where the images of the fibers are indistinguishable (as subsets of $\mathbb{Z}^3$, including scenery). We will view this invariant copy as a "decoration" of $\mathbb{Z}^3$ by G. It is not a priory clear how to find such an invariant copy of G on $\mathbb{Z}3$. However, if we interchange the roles of the underlying graph and the decoration, we are able to do something similar: there is a decoration of G by $\mathbb{Z}^3$ such that the decorated fibers are indistinguishable. This decoration will be a factor of iid from G (hence indistinguishability of the decorated fibers will follow again from Lemma 3.3).

5. This kind of duality (decorating G by a copy of $\mathbb{Z}^3$ instead of decorating $\mathbb{Z}^3$ by a copy of G) will turn out to be useful. Invariance of the decoration will be preserved when switching to the dual (see Lemma 3.4 for the more precise formulation, with proper conditions). We define indistinguishability of decorated subgraphs in a new way, which is stronger than the usual definition, and does not depend on the underlying graph G (Definition 2). Thanks to the duality, the indistinguishable decorated fibers

of G can be taken as indistinguishable decorated subsets of $\mathbb{Z}^3$.

6. $\mathbb{Z}^3$ has an isometric embedding into $\mathbb{R}^3$, which can be made invariant by a random rotation and translate. This random embedding preserves the indistinguishability of the fibers, this time in $\mathbb{R}^3$. Now we can proceed as in item 3, and get the T_3-partition with indistinguishable pieces.

The existence of a representation as in Step 2 above will turn out to be true for any Cayley graph that can be invariantly embedded in $\mathbb{R}^3$, as shown in the next section. The class of such graphs is that of amenable graphs, hence the following theorem follows:

Theorem 1.3. For every amenable Cayley graph G there is an invariant G-partition of $\mathbb{R}^3$ where every piece of the partition is a bounded polyhedron. Moreover, the set of domain-accumulation points can be chosen to be 1-dimensional.

FOLYTATNI A TETELT: Moreover, there exists a MOD... A TETEL ALA BEIRNI REFERENCIAT MOD DEFINICIOJARA, AMIBE BEVENNI, HOGY AZON PONTOK HALMAZA, AHOVA VEGTELEN SOK DOMAIN TORLODIK, EGY DIMENZIOS. DEF: domain-accumulation point, 1-dimensional set.

The discrete version of our question under investigation, also by Benjamini, remains open.

Question 1.4. (Itai Benjamini) Is there an invariant measurable random connected T_3-partition of $\mathbb{Z}^3$ with indistinguishable pieces?

As we have mentioned, there exist countable amenable groups where the answer is positive, see Theorem 3.1.

2 From graph embeddings to representations by domains

Denote the closure of a subset A of $\mathbb{R}^3$ by $\bar{A}$, its boundary by ∂A, and its complement by A^c. For sets $A, B \subset \mathbb{R}^3$, let $\text{dist}(A, B)$ be the euclidean distance of $\bar{A}$ and $\bar{B}$. For $A \subset \mathbb{R}^3$, $c > 0$, let $N(A, c) = \{x \in \mathbb{R}^3 : \text{dist}(x, A) < c\}$ be the open c-neighborhood of A. If M is a collection of sets, let $\cup M$ be the short notation for $\cup_{A \in M} A$. Say that a collection M of subsets of $\mathbb{R}^3$ is a **map of domains (MOD)** if the elements of M are pairwise disjoint, connected and bounded, open polyhedrons in $\mathbb{R}^3$. A map of domains defines a graph: say that $A \in M$ and $B \in M$ are *adjacent* if A and B has some faces that intersect each other in some 2-dimensional set (in other words, if $\bar{A} \cap \bar{B}$ contains a 2-dimensional subset). If G is the graph defined by the map of domains M, we also say that M is a **representation** of G.

Given a map of domains M representing the graph G and vertex $v \in V(G)$, denote by $M(v)$ the domain representing v; for $x \in \mathbb{R}^3$ let $M^{-1}(x)$ be the vertex $v \in V(G)$ such that $x \in M(v)$. Given a map of domains

M, denote by $O(M)$ the interior of $\mathbb{R}^3 \setminus \cup_{D\in M} D$, and call it the *ocean.* If $D \in M$ is a domain and the intersection of its boundary with the boundary of the ocean is 2-dimensional, then we say that D is adjacent with the ocean. We think about map of domains as the 3-dimensional analogue of planar maps, and the ocean corresponds to the part of the space that is not covered by any region.

Note that the notions of a connected G-partition of $\mathbb{R}^3$ and of a MOD that represents G are very close. There are two differences: the $\mathbb{R}^3$-partition may have more general (e.g., not necessarily polyhedral) pieces than a MOD. On the other hand, a MOD may not fill in the space $\mathbb{R}^3$ (the closure of the union of domains may not be $\mathbb{R}^3$), but it is a requirement for $\mathbb{R}^3$-partitions.

Given a transitive graph G, we say that a function from G to some space is a **factor of iid** or **fiid**, if the function is determined by the random iid Lebesgue($[0,1]$) labelling of $V(G)$, so we can write it as f_ω with $\omega \in [0,1]^{V(G)}$, it is a measurable function of ω, and for any isomorphism γ of G, $f_{\gamma\omega}(\gamma x) = f_\omega(x)$.

Consider an invariant sequence of coarser and coarser partitions $(\mathbf{P}_n)_{n=0}\infty$ of $\mathbb{R}^3$, where any two points of $\mathbb{R}^3$ are in the same cell of $\mathbf{P}_n$ if n is large enough. We will refer to the classes of these partitions as **cells**. For other partitions of $\mathbb{R}^3$ (e.g. G-partitions) we will refer to the classes as **pieces**. Suppose now that G is some infinite graph whose vertices are mapped in $\mathbb{R}^3$ by a map ϕ. Let G_0 be the empty graph on vertex set $V(G)$. Let G_n be the subgraph of G consisting of edges $\{x, y\}$ such that there is some cell C in $\mathbf{P}_n$ such that both $\phi(x)$ and $\phi(y)$ are in the interior of C. Then every connected component of G_n is finite; furthermore, $G_n \to G$. Call the graph G_n the *subgraph of G defined by $\mathbf{P}_n$*. More generally, if $\mathcal{Q}_n$ is any partition of $V(G)$ then $G_n = (V(G), E(G_n))$ is the *subgraph of G defined by $\mathcal{Q}_n$*, if $E(G_n) = \{\{x, y\} \in E(G) : x$ and y are in the same class of $\mathcal{Q}_n\}$.

Lemma 2.1. Suppose that G is amenable, transitive, and that it has an isometry-invariant random map ϕ of G to $\mathbb{R}^3$. Then there is a sequence of coarser and coarser partitions $(\mathbf{P}_n)_n$ of $\mathbb{R}^3$ that is a factor of iid from G modulo ϕ.

Proof. We can regard the invariant random copy of G in $\mathbb{R}^3$ together with the iid labels on the vertices as an invariant point process with an index function as in [7]. Then the claim follows from Theorem 4.1 in [7].

We will denote Lebesgue measure by λ, and by λ_2 the area for 2-dimensional submanifolds. The symmetric difference of sets A and B is written as $A\Delta B$. EZ KELL VEGUL?

Theorem 2.2. Suppose that the graph G has a random invariant copy ϕ in $\mathbb{R}^3$ such that $\phi(V(G))$ has no accumulation point. Then there is a random invariant map of domains in $\mathbb{R}^3$ that represents G a.s. Moreover, we can assign this MOD to the random invariant copy of G as a factor of iid from G (modulo ϕ).

See the second item in the Sketch of the proof in Section 1 for the definition of a factor of iid modulo ϕ. Observe that the definition commutes with the automorphisms G, because it only uses the image set

$\phi(V(G))$. Also note that the random invariant copy ϕ of G in $\mathbb{R}^3$ may not be a fiid, but once we have that random invariant copy, the rest, the MOD can be added to it as a fiid.

Let $\epsilon > 0$ be arbitrary, $A, B \subset \mathbb{R}^3$. Define $v(A, B, \epsilon) := \{x \in \mathbb{R}^3 : \text{dist}(x, A) < \min(\text{dist}(x, B), \epsilon)\}$, and let $\text{Vor}(A, B, \epsilon)$ be the connected component of $v(A, B, \epsilon)$ containing A.

Proof. For simplicity, we will say factor of iid instead of factor of iid modulo ϕ in this proof.

Let $\lambda : V(G) \to [0, 1]$ be the iid random labelling of the vertices that we will use. Consider the set of points $\omega = \phi(V(G))$ in $\mathbb{R}^3$. For each $x \in \omega$, let $r_x = \frac{1}{4}\min(\{dist(x, y) : y \in \omega\} \cup \{\text{dist}(x, \partial C) : C \in P_0\})$. Define $D_0 := \{C(x, r_x, \lambda(x)) : x \in \omega\}$, where $C(x, r_x, \lambda(x))$ is a cube with center x, contained in the ball $B(x, r_x)$ and maximal such, and finally, positioned (rotated around x) in some way which is deterministically given by $\lambda(x)$ and ω. Denote by $D_0(x)$ the element of D_0 that contains $x \in \omega$. Let $\mathbf{P}_n$ be the fiid sequence of partitions of $\mathbb{R}^3$, as given by Lemma 2.1. Let $\mathbf{E}_n$ be a subset of $E(G)$ where $\{x, y\} \in \mathbf{E}_n$ if $\phi(x)$ and $\phi(y)$ are contained in the same cell of $\mathbf{P}_n$. Since $\mathbf{P}_n$ contains bounded classes, every component of $\mathbf{E}_n$ is finite.

If the degree of x is the same in $\mathbf{E}_n$ as in G, say that x *has full degree* in $\mathbf{E}_n$. If x has full degree in $\mathbf{E}_n$ and furthermore, the component K of x in $\mathbf{E}_n$ is such $K \setminus x$ is connected, then say that x is *internal* in $\mathbf{E}_n$. Note that both the property of having full degree and of being internal are monotone under n. For a domain representation of $\mathbf{E}_n$ we call a domain internal if the corresponding vertex is internal.

Consider the following properties that a map of domains D_n may satisfy:

1. arises as a factor of iid from G;

2. the adjacency graph given by D_n is isomorphic to $\mathbf{E}_n$;

3. if $\phi(x) \in C$ for some $C \in \mathbf{P}_n$ then $D_n(x) \subset C$;

4. a cell $D_n(v)$ is adjacent to the ocean if and only if v does not have full degree;

5. the closure of a cell $D_n(v)$ has a nonempty intersection with the closure of the ocean if and only if v is not internal;

6. if v is internal then $D_n(v) = D_m(v)$ for every $m > n$;

7. the intersection of the ocean with any of the $C \in \mathbf{P}_n$ is connected, that is, $C \setminus \cup_x D_n(x)$ is connected.

A FELSOROLASBAN KAOSZ VAN x ES v ILLETVE ABBAN, HOGY $V(G)$ vagy ω eleme a pontunk!

First note that D_0 satisfies all the requirements above (with D_{-1} defined as D_0). Suppose, inductively, that for every $k \leq n$ we have a map of domains D_k that satisfies the properties. We will define D_{n+1}. Consider an edge $\{x, y\}$ of $\mathcal{E}_{n+1} \setminus \mathbf{E}_n$, and pick one of its endpoints according to some rule determined by the iid labels. We may assume by symmetry that we chose x. The domains $D_n(x)$ and $D_n(y)$ are contained in the same $\mathbf{P}_n$-cell by item 3 above, and hence also in the same $\mathbf{P}_{n+1}$-cell C. Both of these domains are neighbors of the ocean $O(D_n)$, by item 4. By item 7, $C \cap O(D_n)$ is connected. Hence there is a broken line path between $D_n(x)$ and $D_n(y)$ within $C \cap O(D_n)$, and we can choose such a path $P(x,y)$ in a way that the collection of the paths over all such pair x, y is pairwise disjoint. (This is because every cell $C \in \mathbf{P}_n$ contains only finitely many pairs). Then we can replace the $P(x,y)$ by a polyhedron $\pi(x,y) \subset O(D_n)$ that contains $P(x,y)$ and such that the closures of the $\pi(x,y)$ are pairwise disjoint over all such pairs x, y that correspond to C. Define $D'_{n+1}(x) := D_n(x) \bigcup \cup_y \pi(x,y)$. If we did all the choices (of $x, P(x,y), \pi(x,y)\ldots$) as some deterministic function of the labels, over all the $C \in \mathbf{P}_{n+1}$, the D'_{n+1} is a factor of iid. It is straightforward to check that it also satisfies property 2, 3 and with a suitable choice of the $\pi(x,y)$, one can easily satisfy item 7. We want to construct a $D"_{n+1}$ that also satisfies item 4, and finally modify that to obtain D_{n+1} which further satisfies 5 and 6, while all other properties remain valid.

First, we may assume that the $\pi(x,y)$ were chosen so that item 4 holds for vertices that does not have full degree. (One just needs the $\pi(x,y)$ to be contained in a small enough neighborhood of the $P(x,y)$ for this to hold, for all x, y as above.) To satisfy item 4 for the set of vertices that have full degree, let S be their set and apply Lemma 2.4.A C-n BELUL MARADAS IGENYEL EGY KIS GONDOT, ILLETVE INTERNAL CSUCSOK DOMAINJEN AT NEM FURHATUNK ALAGUTAT. The resulting new map of domains will be $D"_{n+1}$.

Finally, we want to get rid of the 1-dimensional intersections that the closures of some internal domains may have with the closure of the ocean.

. If v does not have full degree in $\mathcal{E}_{n+1}$, then we could have chosen the ..... (This is doable by making the $\pi(x,y)$ "thinner", if necessary.) Hence D'_{n+1} also satisfies property 5 for vertices that do not have full degrees. We are going to modify D'_{n+1} now in such a way that the resulting D_{n+1} also has property 3 and both parts of property 5, while the other properties are also remain valid.

□

Lemma 2.3. Let M be an invariant map of domains representing a graph H in $\mathbb{R}^3$, and suppose that all the components of H are finite. Let $\hat{H} \supset H$ be a graph containing H, $V(\hat{H}) = V(H)$, and still having only finite components. Also suppose that the pair $(H, \hat{H})$ is unimodular. Then for any $\epsilon > 0$ there is an invariant map of domains $\hat{M}$ such that

1. $\hat{M}$ represents the graph $\hat{H}$, and if $M(v)$ is adjacent to the ocean of $\hat{M}$ then $\hat{M}(v)$ is also adjacent to the ocean of $\hat{M}$;

2. Suppose that every $\{x, y\} \in E(\hat{H} \setminus H$ is such that $M(x)$ and $M(y)$ are both adjacent to the ocean in

M. Then $\hat{M}$ can be chosen in such a way that $M(v) = \hat{M}(v)$ for every vertex that is not an endpoint of any edge in $E(\hat{H} \setminus H$.

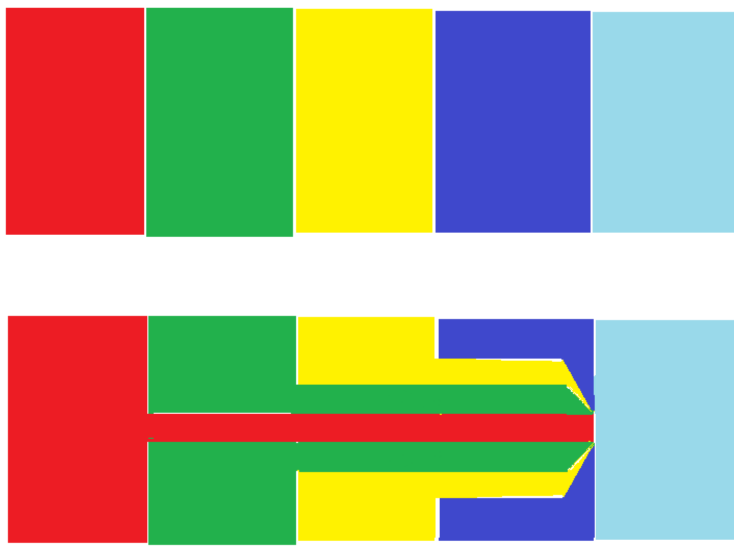

Figure 2.1: Making two domains (the leftmost and the rightmost) adjacent, with all other adjacencies unchanged. Before (top) and after (bottom) the operation of "growing a tunnel", with the original domains D_j on top and the modified ones D_i' on bottom. Note that in dimension 3, the slim tunnel between the two domains does not disconnect the other ones.

Proof. Fix $\epsilon > 0$ arbitrarily. Define $M_0 := M$, $H_0 := H$. The set $V(H) = V(\hat{H})$ is partitioned by $\mathbf{P}_n$: say that $x, y \in V(H)$ are in the same class if $M(x)$ and $M(y)$ are both fully contained in some cell C of $\mathbf{P}_n$. For $n \geq 1$, let H_n be the subgraph of $\hat{H}$ defined by this partition (that is, H_n consists of the edges of $\hat{H}$ induced by the pieces of the partition). Then H_n only has finite components, and $H_n \to \hat{H}$. For each $e = \{x, y\} \in E(H_n) \setminus E(H_{n-1})$ we will modify $M_{n-1}(x)$ and $M_{n-1}(y)$, and maybe some other domains, in such a way that every modification takes place in the cell C of $\mathbf{P}_n$ that contains $M_{n-1}(x)$ and $M_{n-1}(y)$. Doing this for every e in C iteratively, and over every $C \in \mathbf{P}_n$ in parallel, we will arrive to a new map of domains M_n such that

1. M_n represents the graph H_n,
2. for every $v \in V(H)$, $\lambda(M_n(v)\Delta M_{n-1}(v))/\lambda(M_n(v)) < \epsilon/2^n$.
3. for every $v, w \in V(H)$, $\lambda_2(\partial\hat{M}_n(v) \cap \partial\hat{M}_n(w)\Delta\partial M_{n-1}(v) \cap \partial M_{n-1}(w))/\lambda_2(\partial M_{n-1}(v) \cap \partial M_{n-1}(w)) < \epsilon/2^n$

So, consider some $C \in \mathbf{P}_n$, and let $e_1, \ldots, e_k$ be a listing of the edges in $E(H_n) \setminus E(H_{n-1})$ with the property that for the endpoints x_i, y_i of e_i, C contains both $M_{n-1}(x_i)$ and $M_{n-1}(y_i)$. As $i = 1, \ldots, k$, we will modify the domains to create the new adjacency defined by e_i, without changing any other adjacencies. See Figure 2.1 for a picture that grabs the idea of doing so. For the formal explanation, first fix some broken line path γ between $\partial M_{n-1}(x_i)$ and $\partial M_{n-1}(y_i)$, such that $\gamma \subset C$, and such that γ intersects the boundary of any cell in finitely many points. Pick one of the two points x_i, y_i uniformly at random, by symmetry we assume it is x_i. Let $D_1, \ldots, D_m$ be the domains that γ crosses, in this consecutive order as we go from x_i to y_i. We may assume that every domain occurs at most once in this listing (because the domains are connected, thus we may replace γ so that this holds). Use notation $D_0 = M(x_i)$ and $D_{m+1} = M(y_i)$. Define $\gamma_j := \gamma \cap \bar{D}_j$. Let g_j be the endpoint of γ_j that is an endpoint of γ_{j+1}, or, in case of $j = m$, let g_m be the point of γ in $\bar{M}(y_i)$. See the upper part of Figure 2.1 for an example (where γ happens to be a straight line segment).

Now choose $0 < \epsilon' < \epsilon/2^n$ so that

1. $\gamma_{\epsilon'} := \cup_j N(\gamma_j, \epsilon')$ is simply connected, its complement in any domain of M is connected,
2. for every domain D_j, $\lambda(D_j \cap C \cap \gamma_{\epsilon'})/\lambda(D_j \cap C) < \epsilon/2^n k$,
3. for every domain D_j and $D \in M_{n-1}$, $\lambda_2(\partial D \cap \partial D_j \setminus \gamma_{\epsilon'})/\lambda_2(\partial D \cap \partial D_j) > 1 - \epsilon/2^n k$.

Define $\Gamma_0 = N(\gamma, \epsilon'/2) \setminus M(y_i)$ and $D'_0 = D_0 \cup \Gamma_0$. Suppose Γ_{j-1} and D'_{j-1} have been defined. Let $\Gamma_j = (\cup_{\ell=j}^m D_\ell) \cap \mathrm{Vor}(\Gamma_{j-1}, M(y_i), (1 - 2^{-j})\epsilon')$ and $D'_j = D_j \cup \Gamma_j \setminus \Gamma_{j-1}$. See Figure 2.1 for an illustration. Redefine each D_j to be D'_j, and increase i by 1 as long as it is less then k, otherwise stop. When the procedure stops, we redefined the domains within C in such a way that the edges in $(E(H_n) \setminus E(H_{n-1}))|_{\cup_{i=1}^k \{x_i, y_i\}}$ were added to their adjacency graph, and the domains were only changed by controlled small amounts. Doing all these changes in parallel for every C, we obtain the new map of domains M_n. Then the M_n have a limit, because the domains change little when going from M_{n-1} to M_n (by item 2 above). Let us call the limit $\hat{M}$. What remains to show is that $\hat{M}$ represents $\hat{H}$, or in other words, that $\hat{M}$ represents the limit graph H' of the H_n. Two domains will be neighbors in $\hat{M}$ if they are neighbors in some (and hence all but finitely many) M_n, by item 3. Therefore $H' \subset \hat{H}$. On the other hand, any two neighbors in $\hat{M}$ have to be neighbors in all but finitely many of the M_n. To see this, let n be the smallest number such that the (finite) component of x in H_n is the same as in $\hat{H}$. By construction, $M_n(x) = M_m(x)$ for every $m > n$.

□

The next lemma shows that one can turn countries by the ocean into continental countries (i.e. countries nonadjacent to the ocean), with a relatively small modification of the map of domains, and without changing the adjacency structure in any other way. The construction is a bit technical, but Figure 2.2 shows the idea behind it (in a simplified way). For a graph H, a vertex $v \in V(H)$ and a set $S \subset V(H)$, denote by $N_H(v)$

the set of neighbors of v and by $N_H(S)$ the set of vertices that are neighbors of at least one element of S in H.

Lemma 2.4. Let M be a map of domains representing a graph H in $\mathbb{R}^3$, and suppose that all the components of H are finite. Suppose further that $\mathbb{R}^3 \setminus \cup_{v \in V(H)} M(v)$ is connected. Let $\epsilon > 0$. Let $S \subset V(H)$ be such that the corresponding set of domains, $\{M(v) : v \in S\}$, is invariant in $\mathbb{R}^3$, and such that for any finite component K of H_n we have $V(K) \setminus S \neq \emptyset$. Then there is a map of domains M', such that $\mathbb{R}^3 \setminus \cup_{v \in V(H)} M'(v)$ is connected, and further:

1. M' represents H;
2. no domain $M'(v)$, $v \in S$, is adjacent to the ocean;
3. $M'(u) = M(u)$ whenever $u \notin N_G(S) \cup S$.

Figure 2.2: One way to make the boundary shared with the ocean disappear, first for the blue then for the red domain. The actual construction is a bit more complicated, to cover the case when the boundaries are "less suitably" arranged.

Proof. Let K be an arbitrary component of H, let n be such that the set $M(K) := \cup_{v \in K} M(v)$ is contained in some cell C of $\mathbf{P}_n$. By the condition that $\mathbb{R}^3 \setminus \cup_{v \in V(H)} M(v)$ is connected, $\partial M(K)$ is connected. Let ϵ_K be such that $N(M(K), 2\epsilon_K) \cap M(x) = \emptyset$ for every $x \in V(H) \setminus K$. We will change the domains inside $N(M(K), \epsilon_K)$ without changing anything outside of this set, and also in such a way that if we redefine M by applying this change, $\partial M(K)$ remains connected (and hence $\mathbb{R}^3 \setminus \cup_{v \in V(H)} M(v)$ remains connected). We will perform the changes in some deterministic measurable way, so that if we do them for every connected component K of H in parallel, the redefined M stays invariant.

So let K be a component. First we transform the elements of $\{M(v) : v \in V(K)\}$ in $N(M(K), \epsilon_K)$ to get a set $M_{\text{new}}(v)$ from each $M(v)$ in such a way that $\partial M_{\text{new}}(v) \setminus \cup\{\partial M_{\text{new}}(w) : w \in V(K) \setminus \{v\}\}$ is connected for every $v \in V(K)$, and M_{new} represents K. To do so, we simply grow tunnels between the connected components of $\partial M(v) \cap \partial O(M)$ inside $O(M)$, and grow them close to $\partial M(v)$ so that they stay in $N(M(K), \epsilon_K)$. After having grown the tunnels, for all of the finitely many $v \in V(K)$ simultaneously and so that they do not intersect each other, we can redefine M (for each such K). So we may assume that all the $\partial M(v) \cap \partial O(M)$ are connected. Then we do the following. Let $v_1, \ldots, v_k$ be a listing of all the vertices in $K \cap S$ such that for every i, $N_H(v_i)$ (the set of neighbors of v_i in the graph H) is not in $\{v_1, \ldots, v_{i-1}\}$. Such a listing exists: choose first v_k ($k = |V(K)|$), to be a vertex with a neighbor outside of $K \cap S$. Such a vertex exists by the assumption of S. Then let $v_{k-1}, v_{k-2}, \ldots, v_1$ be the set of vertices on an exploration path in the component of v_k (which is K), in the order of their occurence. Now, as $i = 1, \ldots, k$, let u_i be a neighbor of v_i not in $\{v_1, \ldots, v_{i-1}\}$. Consider $\nu(M(v_i), c) := \text{Vor}(\partial M(v_i) \cap \partial O(M), \partial M(v_i) \setminus \partial O(M), c) \setminus M(v_i)$, and define $M_{\text{new}}(u_i) = M(u_i) \cup \nu(M(v_i), \epsilon_K) \cup t_i$, where t_i is a "tunnel" (a small neighborhood of a broken line path) inside $M(v_i)$ from $\partial M(v_i) \cap \partial M(u_i)$ to $\partial M(v_i) \cap \partial O(M)$. Redefine $M(u_i)$ as $M_{\text{new}}(u_i)$ and redefine all the other $M(v)$ as $M(v) \setminus (\nu(\partial M(v_i), \epsilon_K) \cup t_i)$. Apply the procedure for every $C \in \mathbf{P}_n$ in parallel, and iterate it as $i = 1, \ldots, k$. By the choice of ϵ_K, the total change of $M(v)$ (over all steps n and cubes C) takes place within a set whose measure is a factor of ϵ, as required by the lemma. Similarly for the total change of the λ_2 measures of shared boundaries, as in item 3 of the lemma.

□

The next claim is almost trivial.

Lemma 2.5. Let M be an invariant map of domains representing the graph H, and $\epsilon > 0$ be arbitrary. Then there is an invariant map of domains M' that also represents H, and such that for every $v \in V(H)$ we have $M(v) \subset M'(v)$, and finally, such that for a fixed $x \in \mathbb{R}^3$ we have $\mathbb{P}(x \in O(M)) < \epsilon$.

Proof. Consider any $n > 0$ and the partition $\mathbf{P}_n$. For each $C \in \mathbf{P}_n$, if $\lambda(C \setminus (\cup M))/\lambda(C) < \epsilon$, then do nothing. Otherwise attach pieces of $C \setminus (\cup M)$ to the domains of M intersecting C (in some arbitrary, but deterministically predefined measurable way), in such a way that every augmented domain (that is, after attachment) is connected, and the complement of the domains in C is also connected. The pieces attached this way should be large enough so that the part of C not in any augmented domain has density at most ϵ. Doing this over every $C \in \mathbf{P}_n$ we obtain M' as required.

□

Now we are ready to prove Theorem 2.2. Denote by $\deg(v, H)$ the degree of vertex v in graph H.

Proof of Theorem 2.2. Let $\phi(\omega) = \phi : V(G) \to \omega$ be the map defining G as a (factor) graph on ω, invariantly. Let G_n be the subgraph of G defined by $\mathbf{P}_n$. From the graph G_0 (the empty graph on ω), define a MOD M_0. Namely, for every $v \in V(G)$ fix an $\epsilon_v := \frac{1}{4}\min\{|\phi(v) - \phi(w)| : w \in V(G)\}$. Then the collection of balls $\{B(\phi(v), 2\epsilon_v) : v \in V(G)\}$ is pairwise disjoint. Let $M_0(v)$ be $B(\phi(v), \epsilon_v)$. The resulting map of domains is in fact a representation of G_0.

Suppose inductively that for G_{n-1} we have a MOD. To obtain a MOD for G_n, apply Lemma 2.3 with $H = G_{n-1}$ and $\hat{H} = G_n$, $\epsilon = 2^{-n-1}$. Next, apply Lemma 2.4 with $\epsilon = 2^{-n-1}$ and S being the set of vertices v that have $\deg(v, G_n) = \deg(v, G)$ but $\deg(v, G_{n-1}) < \deg(v, G)$. Finally, to finish step n, apply Lemma 2.5 with $\epsilon = 1/n$. As a result of this last step, the domains will occupy the entire space as $n \to \infty$. Note that after the application of the two lemmas, $\mathbb{P}(x \in M_{n-1}(v), x \in M_n(u), u \neq v) \leq 2^{-n}$. In other words, if a point becomes part of a domain for a certain n then it will stay in that domain with probability tending to 1 with n. Since points stabilize in the same domain with probability 1, M_n weakly converges to a map of domains M. What remains is to show that M is a representation of G. First, if $\{x, y\} \in E(G_n)$ and hence $M_n(x)$ and $M_n(y)$ share some 2-dimensional boundary, then in every later application of Lemmas 2.3 and 2.4 they will share the same boundary up to some small error, by items 3 of the quoted lemmas. Hence this boundary cannot vanish in the limit, $M(x)$ and $M(y)$ share a 2-dimensional boundary. Second, if for some $x, y \in V(G)$ the domains $M(x)$ and $M(y)$ share a 2-dimensional boundary then $M_n(x)$ and $M_n(y)$ share a 2-dimensional boundary for n large enough. To see this, make the following observation. We only apply Lemma 2.5 to domains $M(v)$ where the degree of v in G_n is not the same yet as its degree in G, because only such domains can be neighbors of the ocean, by the way we repeatedly apply Lemma 2.4. Therefore, for every $v \in V(G)$ there are only finitely many n's such that $M_n(v)$ is changed when applying Lemma 2.5. The same is true for Lemma 2.4 and for Lemma 2.3. Therefore the only way $M(x)$ and $M(y)$ could share a 2-dimensional boundary in the limit is if $M_n(x)$ and $M_n(y)$ shared it already for every large enough n. We have shown that M needs to represent G, which finishes the proof of Theorem 2.2.

□

Proof of Theorem 1.3. First note that one can define a copy of the amenable graph G on the Poisson point process as a *factor* of the point process. This follows from the extension of the method of [7], or (as observed by Hutchcroft) from the fact that all amenable groups are orbit equivalent. In other words, there is an equivariant copy of G on the Poisson point process ω as vertex set. Now we can apply Theorem2.2 to G, to get a MOD representing G.

3 A T_3-partition of $\mathbb{R}^3$

Our original proof was using the lamplighter group, but the BS(1,2) group makes the construction a lot simpler, by an observation of Gaboriau.

Theorem 3.1. (Gaboriau) The Baumslag-Solitar group BS(1,2) has a connected T_3-partition.

For any amenable G we have a MOD of $\mathbb{R}^3$ that represents G, by the previous section. Suppose that there is a connected T_3-partition of G. Then, composing these two, we could arrive to a T_3-partition of $\mathbb{R}^3$. To prove this, let $\mathcal{M}$ be an invariant G-partition of $\mathbb{R}^3$, and consider the following T_3-partition. Let α be a piece (fiber) in the T_3-partition of G, and take the union of all domains in $\mathcal{M}$ that represent vertices in α. Doing this over all fibers, we get the T_3-partition of $\mathbb{R}^3$. Choosing BS(1,2) to be G, one could "almost" conclude Theorem 1.2. However, one thing is not clear: why the resulting pieces would be indistinguishable. One can check that in fact there exist map of domains as in Theorem 2.2 that would give distinguishable pieces after taking the union over fibers. Even if everything is defined as a a factor of iid (or factor of the point process), that need not necessarily imply any kind of indistinguishability, as shown by Remark 3.2. So, to pursue the above line of proof, one would have to apply Theorem 2.2 in a more refined way. Note that showing indistinguishability of the components is highly nontrivial even in the case of the infinite clusters of a Bernoulli percolation, and so far the method developed by Lyons and Schramm in [6] seems to be an important element of all proofs of indistinguishability. (See the definition of indistinguishability for percolations after the remark.)

The major goal in this section is to construct a MOD representing BS(1,2) as provided by the previous section, but with the further property that the unions of domains over the fibers are indistinguishable.

Remark 3.2. There exist factor of iid processes on transitive graphs whose infinite clusters do not satisfy indistinguishability. Consider some transitive graph G whose critical probability for site percolation is smaller than 1/3. Let $\lambda(v)$ be a Lebesgue($[0,1]$) label on $v \in V(G)$, independently over all the v. Define a bond percolation: an edge $\{x,y\}$ is open in this percolation if $\lambda(x) < 1/2$ and $\lambda(y) < 1/2$, or if both $\lambda(x)$ and $\lambda(y)$ are between 1/2 and 5/6. It is easy to check that some infinite clusters look like those of Bernoulli(1/2) site percolation, and some look like Bernoulli(1/3) site percolation. The two are distinguishable (as can be seen, say, by the fact that their critical probabilities differ).

First define indistinguishability for percolation. This definition is seemingly stronger than the usual one (see [6]), but it is easy to check that they are in fact equivalent.

Definition 2. Let $\mathcal{G}$ be the space of all locally finite connected rooted multigraphs up to rooted isomorphisms, and consider the distance $d((G,o),(G',o')) = \min\{1/r : B_G(o,r)$ and $B_{G'}(o',r)$ are rooted isomorphic$\}$. Suppose now that some $\mathcal{A} \subset \mathcal{G}$ is Borel measurable, and it is closed under the change of root (that is, if $(G,o) \in \mathcal{A}$ and $x \in V(G)$, then $(G,x) \in \mathcal{A}$). Such an $\mathcal{A}$ will be called an *invariant property.* Say that a percolation on a transitive graph has indistinguisable infinite components, if for any such $\mathcal{A}$, either every infinite component belongs to $\mathcal{A}$ or none.

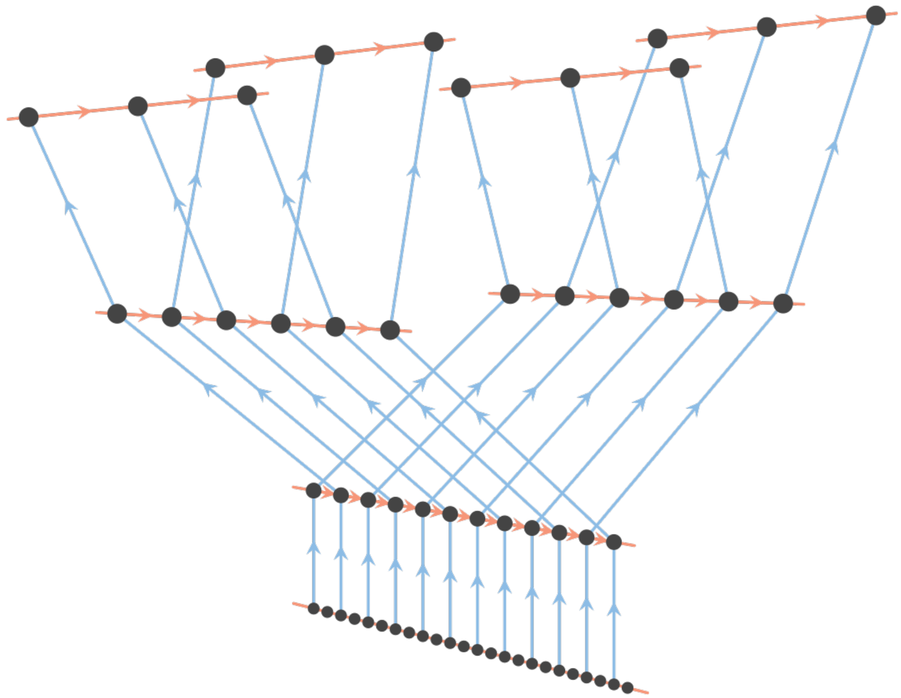

Figure 3.1: Part of the Cayley graph of BS(1,2). Red lines ("fibers") chosen as partition classes provide a T_3-partition of the Cayley graph. (Image by Jens Bossaert.)

Now, fix some metric space X. Let $\mathcal{G}^{\mathrm{deco}}$ be the space of all elements (G, o) of $\mathcal{G}$ together with some decoration of (G, o), where a decoration means some coloring of the vertices of G with elements of X. A distance on $\mathcal{G}^{\mathrm{deco}}$ can be defined similarly to $\mathcal{G}$. If (G, o, δ) and (G', o', δ') are rooted graphs with decorations δ and δ' respectively, then let their distance be $\min\{c : d((G, o), (G', o')) \leq c, \mathrm{dist}_X(\delta(v), \delta'(v)) < c$ for every $v \in B_G(o, 1/c)\}$. Now, with $\mathcal{G}^{\mathrm{deco}}$ and the metric on it, we can define indistinguishability of the infinite components of some percolation with decoration just the same way we did for the non-decorated case.

What we will be interested in, is actually a seemingly broader type of decoration: when extra edges can also be added to G. However, this can be represented as a vertex coloring (by coloring every vertex with the set of its new neighbors, and defining any two distinct sets of vertices to have distance 1 from each other). We will refer to graphs with extra edges as decorated graphs because of this correspondance, without explicitly transforming the extra edges into vertex decorations.

Suppose that there is some invariant percolation ω on a transitive graph G. Denote by $C_v(\omega) = C_v$ the component of v in ω. One can define a decoration δ_ω for G with color set $\mathcal{G}$: let $\delta_\omega(v) := (C_v, v)$. Finally, if besides the percolation ω there is also some decoration $\delta : V(G) \to X$ mapping to some space X, then we define a new decoration, called the *full decoration*, as follows. Define $X' := X \times \mathcal{G}^{\mathrm{deco}}$ as the space of colors, and let the decoration δ' map every vertex v to the new color $\delta'(v) = (\delta(v), \delta_\omega(v))$. The point of defining the full decoration is that this defines the most refined collection of properties that could be distinguishing between two clusters.

Lemma 3.3. *(FIID decoration preserves indistinguishability)* Let G be a transitive graph and ω be a percolation. Let δ be some factor of iid decoration on G, where the independent labels $\lambda : V(G) \to [0,1]$ used for the factor are independent from the percolation. Suppose that the infinite components of ω are indistinguishable. Then they are also indistinguishable in the full decoration.

Given some group and a finite set of generators, a *Cayley diagram* is defined on the vertex set being the set of group elements, and oriented edges labelled by a generator if the tail of the edge multiplied from the right by this generator gives the group element in the head of the edge. We say that two Cayley diagrams are isomorphic if there is a graph isomorphism between them that also preserves the orientations and labels of the edges. A *diagram* in general will be a graph with oriented and labelled edges. The lemma will be applied to a very particular (and boring) invariant percolation, the set of edges labelled by b in the Cayley diagram of BS(1,2)=$< a, b|a^{-1}ba = b^2 >$. We stated the lemma in a more general form because we believe that it might be of interest beyond the current application, as a new potential technique to prove indistinguishability.

Proof. Proving by contradiction, suppose that there is some $\mathcal{A}$ invariant property in the fully decorated space such that with positive probability some infinite (fully decorated) component is in it and some is not. For every $\epsilon > 0$ there exists an R_ϵ such that the restriction of ω and λ to $B(o, R_\epsilon)$ determines up to an error at most ϵ whether C_o with its full decoration is in $\mathcal{A}$ or not. (By "determining up to an error ϵ we mean that looking at the configuration in the ball, one can make a guess whether C_o is in $\mathcal{A}$ or not, and the probability of the guess being wrong is less than ϵ.) Let $A(R_\epsilon)$ be the set of those such configurations in $B(o, R_\epsilon)$ where the guess is that $C_o \in \mathcal{A}$. In particular, if we run two-sided delayed simple random walk w_n conditioned on that w_0 is in an infinite component C of ω, with $C \in \mathcal{A}$, the probability that the configuration of ω and λ restricted to $B(w_n, R_\epsilon)$ is in $A(R_\epsilon)$ is at least $1 - \epsilon$. This is true by the stationarity of this random walk (see [6]). Similarly, the probability that the (ω, λ)-configuration on $B(w_n, R_\epsilon)$ is in $A(R_\epsilon)$ is at most ϵ when $w_0 \in C \notin \mathcal{A}$. Since λ is iid on the vertices, the restriction of λ to $B(w_n, R_\epsilon)$ has to have the same distribution no matter if we are walking on a C in $\mathcal{A}$ or not. Therefore the different behavior has to be caused by the restriction of ω to the $B(w_n, R_\epsilon)$. But then the infinite ω components could be distinguished by some invariant property, contradicting the assumption on the percolation.

□

Suppose that there are two, possibly random graphs, G and H, defined on the same vertex set. We will use three, equivalent ways to jointly represent G and H. First, we will think about the pair (H, G) as one multigraph, with edges of two colors: let the vertex set of (H, G) be $V(H) = V(G)$, the edge set be

$E(H) \cup^* E(G)$ (where $\cup^*$ denotes multiset union, in particular, we may have parallel edges), and color edges of $E(H)$ by one color, and edges of $E(G)$ by another color. Secondly, we can think about (H, G) as the graph H decorated by G, or, thirdly, as the graph G decorated by H. Denote these by $\mathrm{Dec}(H, G)$ and $\mathrm{Dec}(G, H)$ respectively. We will say that $\mathrm{Dec}(G, H)$ is the *dual* of $\mathrm{Dec}(H, G)$. The next lemma says that this kind of duality preserves invariance, whenever H is a Cayley diagram.

By the $\mathrm{Aut}(G)$-invariance of H, the rooted graph $((G, \phi^{-1}(H)), \mathrm{id}_G)$ is unimodular. This random decorated rooted graph is naturally equivalent to $(\mathrm{Dec}(G, H), \mathrm{id}_G)$. Now, $((G, \phi^{-1}(H)), \mathrm{id}_G)$ is also equivalent to $((H, \phi(G)), \mathrm{id}_H)$, by the map ϕ (using the symmetry of colors); and finally, the latter is equivalent to $(\mathrm{Dec}(H, G), \mathrm{id}_H)$.

Lemma 3.4. *(Duality lemma)* Let G be a Cayley graph or Cayley diagram, and let H be a Cayley diagram. Let ϕ be a random map $\phi : V(G) \to V(H)$ that is bijective and satisfies $\phi(\mathrm{id}_G) = \mathrm{id}_H$. Suppose that the diagram $\phi^{-1}(H)$ is $\mathrm{Aut}(G)$-invariant. Then $\phi(G)$ is $\mathrm{Aut}(H)$-invariant.

In other words, for the decorations defined by ϕ, the $\mathrm{Aut}(G)$-invariance of $\mathrm{Dec}(G, H)$ implies the $\mathrm{Aut}(H)$-invariance of $\mathrm{Dec}(H, G)$.

Proof. Let's assume that H is a right-Cayley diagram, so its automorphisms are all of the form hH, where $h \in H$. For every vertex a in H there is a unique automorphism of H that takes id_H to a, namely, multiplication from the left by a. Consequently, the rooted graph (H, id_H) does not have any nontrivial rooted isomorphisms preserving the root, and therefore the same is true about the pair $((H, \phi(G)), o) = (\mathrm{Dec}(H, G), o)$. Therefore we may ignore the rooted equivalences when talking about $((H, \phi(G)), o)$ as a random element of $\mathcal{G}^{\mathrm{deco}}$ and just use the representative rooted graph. The pair $(\mathrm{Dec}(H, G), o)$, thought of as a rooted multigraph with colored edges, is the same as the pair $(\mathrm{Dec}(G, H), o) = ((G, \phi^{-1}(H)), o)$. This latter is unimodular, by the $\mathrm{Aut}(G)$-invariance of the distribution of $\phi^{-1}(H)$ (check that the MTP needs to hold). Again, this is the same as $((H, \phi(G)), o)$ being unimodular.

The fact that a unimodular decoration of a transitive network is automorphism invariant is well known, but we prove it for completeness. Pick an arbitrary event $\mathcal{E}$, with respect to the Borel σ-algebra on the equivalence classes of rooted graphs with the usual metric. Define the following mass transport. Let every vertex x send mass 1 to the vertex xa (the vertex that an a-labeled edge is pointing to from x) if the event $x\mathcal{E}$ (i.e., the translate of $\mathcal{E}$ to root x) holds. Then the expected mass sent out is just $\mathbb{P}(\mathcal{E})$. By unimodularity and the MTP, this is the same as the expected mass received, which is $\mathbb{P}(a^{-1}\mathcal{E})$. That is, the probability of event $\mathcal{E}$ is the same for o and for the vertex where it is mapped by the automorphism of left-multiplying by a^{-1}. Since this is true for any $\mathcal{E}$, we obtain that $(H, \phi(G))$ is invariant under the automorphism by a^{-1}. As a ranges over all generators and their inverses, we obtain invariance as claimed.

□

The above lemma can be generalized to G unimodular graph and H unimodular Schreier graphs, while it is not true if we only assume H to be a Cayley graph. The conditions under which the lemma can be generalized will be detailed in a followup note, [2].

Proof of Theorem 1.2. Let G be the Cayley graph of BS(1,2) in the representation $< a, b | a^{-1}ba = b^2 >$. Let $\mathcal{P}$ be the (deterministic) percolation consisting of edges of G that would be labelled by b in the Cayley diagram. The infinite clusters (that we will call fibers) are all biinfinite paths, and they are obviously indistinguishable in the scenery, because any of them can be taken to any other by an automorphism that preserves the clusters of $\mathcal{P}$. Now, one can define a copy of $\mathbb{Z}^3 =: H$ on G as a factor of iid (in fact, this would work for any 1-ended amenable unimodular transitive graph G), as follows. First define a fiid 1-ended tree subgraph of G, as in Theorem 5.3 of [4]. Then use it to define a fiid *clumping* on $V(G)$ (i.e., a sequence of coarser and coarser partitions such that any two points are in the same partition set after a while) with 2^n points in each piece of the n'th partition, by a slight natural modification of Theorem 4.1 in [7]. Then use this to define the fiid $\mathbb{Z}^3$-factor, as in the proof of Theorem 1.2 in [7]. By Lemma 3.3, the fibers remain indistinguishable with this decoration $\mathrm{Dec}(G, \mathbb{Z}^3)$ of $\mathbb{Z}^3$ on G.

By Lemma 3.4, if we consider $\mathbb{Z}^3 = H$ with the decoration G on it given as the dual of $\mathrm{Dec}(G, \mathbb{Z}^3)$, then it will be invariant with respect to $\mathrm{Aut}(\mathbb{Z}^3)$. Now embed $\mathbb{Z}^3$ into $\mathbb{R}^3$ periodically, and invariantly, by using a random rotation and shift of the natural embedding $\{(x, y, z) \in \mathbb{Z}^3\} \subset \{(x, y, z) \in \mathbb{R}^3\}$. Let the (random) embedding be ϕ. Construct the map of domains M corresponding to this embedding as a factor of iid on the vertices of $\mathbb{Z}^3$, as provided by Theorem 1.3. Denote by X the space of open subsets of $\mathbb{R}^3$ with Hausdorff metric. Suppose now that the sets $\cup\{M(\phi(v)) : v \in F\} =: M(F)$ over the fibers F of $\mathcal{P}$ do not satisfy indistinguishability. Then use $M(F)$ as a (fiid) decoration of every point of F in G, meaning that we color every vertex of F by $M(F) \in X$. Then the distinguishability of the $M(F)$ would imply the distinguishability of the F's in this fiid decoration, contradicting Lemma 3.3.

We have concluded that the pieces $M(F)$ are indistinguishable. Their adjacency graph is T_3 by definition, which fact finishes the proof. □

Remark 3.5. There is a simple example of a T_3-partition of $\mathbb{R}^3$ if we do not require the pieces to be indistinguishable. It has bounded partition classes of different scales (which provides distinguishability right away). For simplicity, we do the construction of a T_5-partition. Consider a sequence of random vectors $(v_i)_{i=-\infty}^{\infty}$, such that $v_i \in [0, 2^i]^3$ and $v_{i+1} - v_i \in 2^i\mathbb{Z}^3$. Define sets of the form $v_i + w + 2^i(1/5, 2/5)$ with $w \in 2^i\mathbb{Z}^3$ and $v_i + w + 2^i(3/5, 4/5)$ with $w \in 2^i\mathbb{Z}^3$. Let the collection of all such sets be $\mathcal{K}_i$. Finally, define $\mathcal{S}_i$ to be the collection of sets $K \setminus \cup\{\bar{L} : L \in \mathcal{K}_{i-1}\}$, as K ranges over $\mathcal{K}_i$. Then $\cup_i \mathcal{S}_i$ is a translation-invariant

T_5-partition. By applying a random rotation, we can make it isometry-invariant.

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