A Note on Covering by Convex Bodies

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Abstract. A classical theorem of Rogers states that for any convex body K in n-dimensional Euclidean space there exists a covering of the space by translates of K with density not exceeding $n\log n + n\log\log n + 5n$. Rogers' theorem does not say anything about the structure of such a covering. We show that for sufficiently large values of n the same bound can be attained by a covering which is the union of $O(\log n)$ translates of a lattice arrangement of K.

A classical theorem of Rogers [3] states that for any convex body K in n-dimensional Euclidean space E^n there exists a covering of the space by translates of K with density not exceeding

$$n\log n + n\log\log n + 5n.$$

Erdős and Rogers [1] showed that such a covering exists with the additional property that no point is covered by more than $e(n\log n + n\log\log n + 5n)$ bodies. Recently, Füredi and Kang [2] used the Local Lemma of Lovász to prove a result only slightly weaker than that of Erdős and Rogers. They showed that for sufficiently large values of n there is a covering of E^n by translates of any convex body such that each point is covered at most $10n\log n$ times.

Neither of these results yields information about the structure of such a covering. Rogers [5] proved the existence of a lattice covering by translates of any convex body in E^n with density not exceeding

$$n^{\log_2 \log n + O(1)}$$

as $n \to \infty$. Here we show that with a slight modification of Rogers' proof of this latter result we can get yet another proof of the bound $O(n\log n)$ for the non-lattice case. Moreover, we show that this bound can be reached by a covering which is the union of $O(\log n)$ translates of a lattice arrangement.

Theorem. *For any convex body K in n-dimensional Euclidean space there exists a lattice arrangement of K such that $O(\log n)$ translates of this arrangement form a covering of the space with density not exceeding*

$$n\log n + n\log\log n + n + o(n)\,.$$

The proof is based on three lemmas. The first two are modifications of Lemma 3 and Lemma 4 from Rogers' paper [5], the third one is a direct consequence of a result of W. Schmidt.

Lemma 1. *Let K be a convex body and let Λ be a lattice in n-dimensional space E^n. Further let $\mathbf{b}_1, \ldots, \mathbf{b}_m \in E^n$ be such that the set*

$$S = \bigcup_{i=1}^{m} \bigcup_{\mathbf{g} \in \Lambda} K + \mathbf{g} + \mathbf{b}_i$$

has density $1-\sigma$. Then there is a point $\mathbf{a} \in E^n$ such that the set

$$S \cup (S + \mathbf{a})$$

has density at least $1 - \sigma^2$.

Let D be a fundamental domain of Λ and let $\chi(\mathbf{x})$ be the characteristic function of S, that is

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in S \\ 0 & \text{if } \mathbf{x} \notin S. \end{cases}$$

Then

$$\chi(\mathbf{x}) = \chi(\mathbf{x} - \mathbf{g}) \qquad \text{if } \mathbf{g} \in \Lambda$$

and

$$\frac{1}{V(D)} \int_{D+\mathbf{v}} (1 - \chi(\mathbf{x}))\, d\mathbf{x} = \sigma \quad \text{for every } \mathbf{v} \in E^n \,.$$

Further,

$$\begin{aligned} & \frac{1}{V(D)} \int_D \left\{ \frac{1}{V(D)} \int_D (1 - \chi(\mathbf{x}))(1 - \chi(\mathbf{x} + \mathbf{y}))\, d\mathbf{x} \right\} d\mathbf{y} = \\ & = \frac{1}{V^2(D)} \int_D (1 - \chi(\mathbf{x})) \left\{ \int_D (1 - \chi(\mathbf{x} + \mathbf{y}))\, d\mathbf{y} \right\} d\mathbf{x} = \\ & = \frac{1}{V^2(D)} \int_D (1 - \chi(\mathbf{x})) \left\{ \int_{D-\mathbf{x}} (1 - \chi(\mathbf{z}))\, d\mathbf{z} \right\} d\mathbf{x} = \\ & = \sigma^2 \,. \end{aligned}$$

Thus, the mean value of

$$\frac{1}{V(D)} \int_D (1 - \chi(\mathbf{x}))(1 - \chi(\mathbf{x} + \mathbf{y}))\, d\mathbf{x}$$

over D equals σ^2. Therefore we can choose a point $\mathbf{y} = -\mathbf{a}$ such that

$$\frac{1}{V(D)} \int_D (1 - \chi(\mathbf{x}))(1 - \chi(\mathbf{x} - \mathbf{a}))\, d\mathbf{x} \le \sigma^2 \,.$$

The proof of the lemma is now complete by observing that the integrand is nothing else but the characteristic function of the complement of the set $S \cup (S + \mathbf{a})$.

Lemma 2. *Let K be an n-dimensional convex body and let Λ be a lattice such that the set*

$$R = \bigcup_{\mathbf{g}\in\Lambda} K + \mathbf{g}$$

has density $1-\sigma$. If for some natural number h and for some vectors $\mathbf{b}_1,\dots,\mathbf{b}_m$ the density of

$$S = \bigcup_{i=1}^{m} \bigcup_{\mathbf{g}\in\Lambda} K + \mathbf{g} + \mathbf{b}_i$$

is at least $1-h^{-n}(1-\sigma)$, then the sets

$$\left\{(1+h^{-1})K + \mathbf{g} + \mathbf{b}_i\right\}_{\mathbf{g}\in\Lambda, 1\le i\le m}$$

form a covering of the whole space.

Let $\mathbf{a}_1,\dots,\mathbf{a}_n$ be a basis of Λ. Write

$$T = \bigcup_{\mathbf{g}\in\Lambda} h^{-1}K + \mathbf{g}$$

and observe that

$$\bigcup_{i=1}^{h} \bigcup_{k=1}^{n} T + \frac{i}{h}\mathbf{a}_k = \frac{1}{h}R.$$

The density of the set on the right-hand side is the same as that of R, that is $1-\sigma$. On the left-hand side we have the union of h^n translates of T. It follows that the density of T is at least $h^{-n}(1-\sigma)$. The same lower bound holds for the density of $-T+\mathbf{x}$ for each $\mathbf{x}$. On the other hand, the density of S is at least $1-h^{-n}(1-\sigma)$. Since the union of $-T+\mathbf{x}$ and S is closed and the total density of the two sets is at least 1, they have a point in common. Hence there are points $\mathbf{k}_1$ and $\mathbf{k}_2$ in K, points $\mathbf{g}_1$ and $\mathbf{g}_2$ in Λ and an index i such that

$$-h^{-1}\mathbf{k}_1 - \mathbf{g}_1 + \mathbf{x} = \mathbf{k}_2 + \mathbf{g}_2 + \mathbf{b}_i\,.$$

Consequently

$$\mathbf{x} = h^{-1}\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{g}_1 + \mathbf{g}_2 + \mathbf{b}_i\,.$$

As K is convex and Λ is a lattice, we have $h^{-1}\mathbf{k}_1 + \mathbf{k}_2 \in (1+h^{-1})K$ and $\mathbf{g}_1 + \mathbf{g}_2 \in \Lambda$. Since $\mathbf{x}$ is arbitrary, this proves the lemma.

The next lemma asserts that there is a reasonably thin lattice arrangement of copies of an arbitrary Borel set leaving only a very small portion of the space uncovered.

Lemma 3. *Let $c_0 = 0.278\dots$ be the root of the equation $1+x+\log x = 0$. If $c < c_0$ then to any positive number ε there is a natural number $N(\varepsilon)$ such that for $n \ge N(\varepsilon)$ to any Borel set $S \subset E^n$ there is a lattice-arrangement of S with density cn covering the whole space with the exception of a set whose density is at most $(1+\varepsilon)e^{-cn}$.*

Let S be a Borel set and let Λ be a lattice with determinant 1. Let $\kappa(S, \Lambda)$ be the density of the part of the space left uncovered by the sets $\{S + \mathbf{g}\}_{\mathbf{g} \in \Lambda}$. Let $\mu(\Lambda)$ denote Siegel's measure in the space of lattices of determinant 1, normalised so that the whole space has measure 1 [7]. The mean value

$$M(S) = \int \kappa(S, \Lambda)\, d\mu(\Lambda)$$

of $\kappa(S, \Lambda)$ over all lattices of determinant 1 was first investigated by Rogers [4] who observed that an upper bound on $M(S)$ can be used to derive existence theorems for thin coverings of the space by translates of a convex body K. Rogers' bound on $M(S)$ was sharpened in [6] by Schmidt. Theorem 10* on page 212 in [6] states the following. To any positive number δ there is a natural number n_0 such that if $n > n_0$ and S is a Borel set in E^n, with volume $V(S) = V \leq n - 1$, then

$$M(S) = e^{-V}(1 - R^*),$$

where

$$|R^*| < V^{n-1} n^{-n+1} e^{V+n}(1 + \delta) + \delta\,.$$

Observe that if $c < c_0$ and $V = cn$, then $V^{n-1} n^{-n+1} e^{V+n}$ approaches zero as $n \to \infty$. Thus to any positive number ε there is a natural number $N(\varepsilon)$ such that if $n \geq N(\varepsilon)$ we have

$$M(S) \leq e^{-V(S)}(1 + \varepsilon)$$

for any Borel set $S \subset E^n$ with $V(S) = cn < c_0 n$.

Let now S be an arbitrary Borel set in E^n, $n \geq N(\varepsilon)$, and consider the set $S' = \lambda S$, where $\lambda = (\frac{cn}{V(S)})^{1/n}$. Then $V(S') = cn$ and there is a lattice Λ' with determinant 1 such that $\kappa(S, \Lambda) \leq e^{-V(S)}(1 + \varepsilon)$. Letting $\Lambda = \lambda^{-1}\Lambda'$, the arrangement $\{S + \mathbf{g}\}_{\mathbf{g} \in \Lambda}$ has the property stated in Lemma 3.

Proof of the Theorem. We set $\varepsilon = e - 1$ and choose n so large that it satisfies the inequalities

$$n \geq N(e-1), \quad e^{-c_0 n/2+1} < \frac{1}{2}, \quad \text{and} \quad \left(n \log n e^{\frac{-c_0 n + 1}{\log n}}\right)^{1/c_0} < \frac{1}{2}\,, \qquad (*)$$

where the function $N(\varepsilon)$ and the constant c_0 are those appearing in Lemma 3. Further we choose c so that $c_0/2 \leq c < c_0$ and

$$k = \log_2\left((\log n + \log\log n + 1/\log n)/c\right)$$

is an integer. By Lemma 3, for any convex body K in E^n we can find a lattice Λ such that the sets

$$\left\{(1 + \lfloor n \log n \rfloor^{-1})^{-1} K + \mathbf{g}\right\}_{\mathbf{g} \in \Lambda}$$

have density

$$\delta_0 = cn$$

and they cover the whole space with the exception of a set whose density is at most

$$\sigma_0 = e^{-cn+1}\,.$$

Next we apply Lemma 1 successively k times to this arrangement. At each step the density of the arrangement doubles, while the density of the part of the space left uncovered by the new arrangement is at most the square of the corresponding quantity for the previous arrangement. In the k-th step we obtain an arrangement

$$\left\{\left(1+\lfloor n\log n\rfloor^{-1}\right)^{-1} K+\mathbf{g}+\mathbf{b_i}\right\}_{\mathbf{g}\in\Lambda, 1\le i\le m}$$

consisting of

$$m = 2^k = c^{-1}(\log n + \log\log n + 1/\log n)$$

translates of the lattice arrangement

$$\left\{\left(1+\lfloor n\log n\rfloor^{-1}\right)^{-1} K+\mathbf{g}\right\}_{\mathbf{g}\in\Lambda}$$

whose density is

$$\delta_k = m\delta_0 = n(\log n + \log\log n + 1/\log n)$$

and which leaves uncovered a set of density at most

$$\sigma_k = \sigma_0^m = (e^{-cn+1})^{(\log n+\log\log n+1/\log n)/c} =$$
$$= \left(n\log n e^{\frac{-cn+1}{\log n}}\right)^{1/c}(n\log n)^{-n}\,.$$

In view of the assumptions imposed on n in $(*)$ we have

$$\sigma_0 = e^{-cn+1} \le e^{-c_0 n/2+1} < \frac{1}{2}$$

and

$$\sigma_k = \left(n\log n e^{\frac{-cn+1}{\log n}}\right)^{1/c}(n\log n)^{-n} <$$
$$\left(n\log n e^{\frac{-c_0 n+1}{\log n}}\right)^{1/c_0}(n\log n)^{-n} < \frac{1}{2}(n\log n)^{-n}\,,$$

hence

$$\sigma_k < (n\log n)^{-n}(1-\sigma_0) < \lfloor n\log n\rfloor^{-n}(1-\sigma_0)\,.$$

Therefore we can apply Lemma 2 with $h = \lfloor n\log n\rfloor$ showing that the sets

$$\{K+\mathbf{g}+\mathbf{b_i}\}_{\mathbf{g}\in\Lambda, 1\le i\le m}$$

form a covering of the whole space. Their density is

$$\begin{aligned}\delta &= \left(1 + \lfloor n \log n \rfloor^{-1}\right)^n \delta_k = \\ &= \left(1 + 1/\log n + O\left(1/\log^2 n\right)\right) n(\log n + \log\log n + 1/\log n) = \\ &= n(\log n + \log\log n + 1 + o(1)) .\end{aligned}$$

This completes the proof of our theorem. In the proof a crucial role was played by Lemma 3 which was deduced from the deep result of W. Schmidt. We note that Theorem 1 in [4] by Rogers gives only a slightly weaker bound on $M(S)$ than Schmidt's bound, however its proof is considerably simpler. From Rogers' bound immediately follows Lemma 2 in [5] which, used in place of our Lemma 3, still yields the existence of a covering of density $\Theta(n \log n)$.

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