Universal Algebraic Logic

(Dedicated to the Unity of Science)

Incomplete preliminary version!!

Comments are welcome

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Preface

Preliminary text — it will be strongly reviesed!

The idea of solving problems in logic by first translating them to algebra, then using the powerful methodology of algebra for solving them, and then translating the solution back to logic, goes back to Leibnitz and Pascal. Papers on the history of Logic (e.g. Anellis–Houser [19], Maddux [51], Pratt [65]) point out that this method was fruitfully applied in the 19th century not only to propositional logics but also to quantifier logics (De Morgan, Peirce, etc. applied this method to quantifier logics, too). The number of applications grew ever since. (Though some of these remained unnoticed, e.g. the celebrated Kripke–Lemmon completeness theorem for modal logic w.r.t. Kripke models was first proved by Jónsson and Tarski in 1948 using algebraic logic.)

For brevity, we will refer to the above method or procedure as "applying Algebraic Logic (AL) to Logic". This expression might be somewhat misleading since AL itself happens to be a part of logic, and we do not intend to deny this. We will use the expression all the same, and hope, the reader will not misunderstand our intention.

In items (i) and (ii) below we describe two of the main motivations for applying AL to Logic.

(i) This is the more obvious one: When working with a relatively new kind of problem, it has often proved to be useful to "transform" the problem into a well understood and streamlined area of mathematics, solve the problem there and translate the result back. Examples include the method of Laplace Transform in solving differential equations (a central tool in Electrical Engineering). This general method is discussed in considerable detail in Madarász [49], under the keyword "duality theories". Cf. e.g. Appendix A therein which describes algebraic logic as a special duality theory.

Ide vagy valahova a Prefacebe kéne egy szelíd szoveg arról, hogy SZEMANTIKAVAL RENDELKEZŐ logikákat algebraizálunk, tehát a szemantikának is megvan az algebrai megfelelője, ami nemcsak a szintaxistól függ.

In the present book we define the algebraic counterpart $\mathsf{Alg}(\mathcal{L})$ of a logic $\mathcal{L}$ together with the algebraic counterpart $\mathsf{Alg}_m(\mathcal{L})$ of the semantical-model theoretical ingredients of $\mathcal{L}$. Then we prove equivalence theorems, which to essential logical properties of $\mathcal{L}$ associate natural and well investigated properties of $\mathsf{Alg}(\mathcal{L})$ such that if we want to decide whether $\mathcal{L}$ has a certain property, we will know what to ask from our algebraician colleague about $\mathsf{Alg}(\mathcal{L})$. The same devices are suitable for finding out what one has to change in $\mathcal{L}$ if we want to have a variant of $\mathcal{L}$ having a desirable property (which $\mathcal{L}$ lacks). To illustrate these applications we include several exercises (which deal with various concrete Logics). For all this, first we have to define what we understand by a logic $\mathcal{L}$ in general (because otherwise it is impossible to define e.g. the function Alg associating a class $\mathsf{Alg}(\mathcal{L})$ of algebras to each logic $\mathcal{L}$).

(ii) With the rapidly growing variety of applications of logic (in diverse areas like computer science, linguistics, artificial intelligence, law, the logic of spacetime, relativity theory etc.) there is a growing number of new logics to be investigated. In this situation AL offers us a tool for economy and a tool for unification in various ways. One of these is that $\mathsf{Alg}(\mathcal{L})$ is always a class of algebras, therefore we can apply the same machinery namely Universal Algebra to study all the new logics. In other words we bring all the various logics to a kind of "normal form" where they can be studied by uniform methods. Moreover, for most choices of $\mathcal{L}$, $\mathsf{Alg}(\mathcal{L})$ tends to appear in the same "area" of Universal Algebra, hence specialized powerful methods lend themselves to studying $\mathcal{L}$. There is a fairly well understood "map" available for the landscape of Universal Algebra. By using our algebraization process and equivalence theorems we can project this "map" back to the (far less understood) landscape of possible logics.

* * *

The approach reported here is strongly related to works of Blok and Pigozzi cf. e.g. [22], [21], [23], [64], Czelakowski [28], Font–Jansana [30]. Strongly related material by the present authors appeared in [15] and [62]. To keep the present work self-contained, we had to repeat here some of the things we wrote there. Another strongly related paper is Andréka–Németi–Sain–Kurucz [17], using a somewhat different setting. That setting gives a broader perspective, however, the investigations of Hilbert-style inference systems done herein are not yet pushed through in that setting. A common root of all the work mentioned in this paragraph is Henkin–Monk–Tarski [37] §5.6. Applications and other extensions of the present work include van Benthem [80], [18], [5], [6], [61], [62], [71], [72], [73], [1], [49].

After having read the present work, the reader might get interested in studying those classes $\mathbf{Alg}(\mathbf{L})$ of algebras which show up most frequently as algebraic counterparts of distinguished logics. [15] was designed to provide an in-depth study of these algebras as well as a guide to their literature. In fact, the most basic book on algebraic logic is [37]. The authors of that book suggested to the present authors to write [15] as an easier to read and more up-to-date introduction to the subject which, among others, could serve as a "prelude" to [37]. In the meantime, the somewhat more specialized book [40] by R. Hirsch and I. Hodkinson also appeared, nicely complementing [15], and [37], [38] in various ways.

This book grew out of course materials, like [16], [14] (cf. also [13]) used first at the Logic Graduate School, Budapest, at the beginning of the 1990's. Therefore our style often remained that of a lecturer writing to her/his students.

Contents

Chapter 1

Notation, Elementary Concepts

In Chapter 1 we introduce our basic notation we will use throughout this book — and, at the same time, we present a brief reminder/summary of the set theoretic foundation we use, in an informal way. Besides that some readers may find the latter useful, the former becomes less boring and better organized this way.

1.1 Sets, classes, tuples, simple operations on sets

As it is wide-spread when working in a classical branch of mathematics, we use the *Zermelo-Fraenkel set theory* with the *Axiom of Choice* (ZFC for short) as our foundation for mathematics. We do not assume familiarity with this formal system, our approach to set theory is informal. What we do assume is that the reader is familiar with naive set theory. If the reader would find out that s/he is unfamiliar with the material reviewed below, s/he is advised to consult e.g. Halmos [35] which is an outstanding elementary introduction to set theory. Another, more recent reference here is Devlin [29].

In set theory, we accept two things to be such basic concepts that we do not define. These concepts are: that of a *set* and the *membership relation*. Intuitively, sets are certain collections. For example, intuitively, the collection of all apples (in a given moment of time) is a set, the elements or members of it are the apples. In this case we say that an apple and the set of all apples are in the membership relation. If A denotes an apple and S denotes the set of all apples then

$$A \in S$$

abbreviates the statement read as "A and S are in the membership relation" or "A is an element of S" or "A belongs to S". The statement that an element B does not belong to C is abbreviated as

$$B \notin C\,,$$

for any sets B and C. Other intuitive examples are: all the dogs living in Budapest form a set, its elements are the dogs living in Budapest (and nothing else). Sets may be elements of other sets. For example, the collection of all classes in a school can be regarded as a set, and the classes themselves are sets again, their elements being the students.

Examples for sets from classical mathematics are the set ω of *natural numbers*, the set $\mathbf{Q}$ of *rational numbers*, the set $\mathbf{R}$ of *real numbers*.

Next we list our basic assumptions on sets and the membership relation.

• *Two sets are said to be equal iff they have exactly the same elements.*

• *We assume that there is a set having no elements.* Then there is exactly one such set, by our definition of the equality of sets above. We call this set the *empty set* and denote it by $\emptyset$.

• Once we have some sets, we can "construct" new sets from them. For example, if we have "finitely many" sets, say $A_1, \ldots, A_n$, then *we assume the existence of a set having exactly $A_1, \ldots, A_n$ as its elements.* We denote this set by $\{A_1, \ldots, A_n\}$. In particular, if A is a set then $\{A\}$ denotes the set having A as its only element. Such sets are called *singleton*s. For example, $\{\emptyset\}$ is the singleton having $\emptyset$ as its only element. Notice that the sets A and $\{A\}$ are different in general. E.g., while $\{\emptyset\}$ has exactly one element, $\emptyset$ has none. Thus $\{\emptyset\} \neq \emptyset$, according to our definition of the equality of sets.

• Another way of forming new sets from old ones is *taking subsets of a set.* If A and B are sets then A is a subset of B iff every element of A is an element of B as well. $A \subseteq B$ is our notation for A is a subset of B. For example, the set of red apples is a subset of the set of all apples; the set of puppies living in Budapest is a subset of the dogs living in Budapest. Notice that every set is a subset of itself; further, the empty set $\emptyset$ is a subset of every set. $\subseteq$ is called the *inclusion* relation. We denote *proper inclusion* by $\subsetneqq$; that is, $A \subsetneqq B$ iff $A \subseteq B$ but $A \neq B$. By our definition of the equality of sets, it is easy to see that

$$A \subseteq B \text{ and } B \subseteq A \text{ together imply } A = B$$

for arbitrary sets A and B.

• We postulate that *the collection of all subsets of a given set A forms a set.* We call it the *powerset* of A and denote it by $\mathcal{P}(A)$.

• Next we are looking at one of the most basic "set forming" constructions. Let $\varphi(x)$ denote a property of sets such that for every set x, $\varphi(x)$ is either true or false. *If A is a set then the collection of those elements of A for which $\varphi(x)$ is true, forms a set.* This is denoted by

$$\{x \in A \ : \ \varphi(x)\}\,. \tag{1.1}$$

Thus we can define the set of even numbers as

$$\{x \in \omega \ : \ x \text{ is divisible by } 2\}\,.$$

Notice that we did not define what we mean by a *property* $\varphi(x)$. This can be done using mathematical logic. We do not go into this now. We just think of ordinary statements used in those branches of mathematics where we do not use strict axiomatizations anyway.

To make our text shorter, we will often write

$$\{x \ : \ \varphi(x)\} \quad \text{instead of} \quad \{x \in A \ : \ \varphi(x)\}$$

when A is known, or easy to find, from the context. However, if such a set A does not exist, the collection of the x's for which $\varphi(x)$ holds, does not form a set in general. For a counterexample, see Russell's paradox below.

Theorem 1.1.1. (Russell's paradox) *The collection of all sets does not form a set.*

Proof. Assume, in the contrary, that there exists a set H the elements of which are the sets. Let $\varphi(x)$ denote the property that (x is a set and $x \notin x$). Then

$$H' \stackrel{\text{def}}{=} \{x \in H \ : \ \varphi(x)\} \tag{1.2}$$

is a set, by our convention above. Then either

$$H' \in H' \tag{1.3}$$

or

$$H' \notin H'\,. \tag{1.4}$$

If (1.3) holds then $H' \in H'$. Then $\varphi(H')$ by (1.2), thus $H' \notin H'$. Since both $H' \in H'$ and $H' \notin H'$ cannot hold, (1.3) must be false, and (1.4) must be the case. But then $H' \in H'$ by (1.2) — we got $H' \in H'$ and $H' \notin H'$ again. Since there are no more cases, by the above we derived a logical contradiction from our assumption (that there exists a set the elements of which are the sets). Thus this assumption must be false, which proves the theorem. $\square$

• We postulate that *if A is a set then the collection of the elements of its elements form a set as well.* This set is called the *(unary) union* of A and is denoted by $\bigcup A$. See Figure 1.1. In particular, if A and B are sets then $\bigcup\{A, B\}$ is a set again. For $\bigcup\{A, B\}$ we use the alternative notation $A \cup B$, and call it the *(binary) union* of A and B.

From $\bigcup$, using the schema (1.1), we define the *(unary) intersection* of a set A, as follows.

$$\bigcap A \stackrel{\text{def}}{=} \{a \in \bigcup A \ : \ a \in B \text{ for every } B \in A\}\,.$$

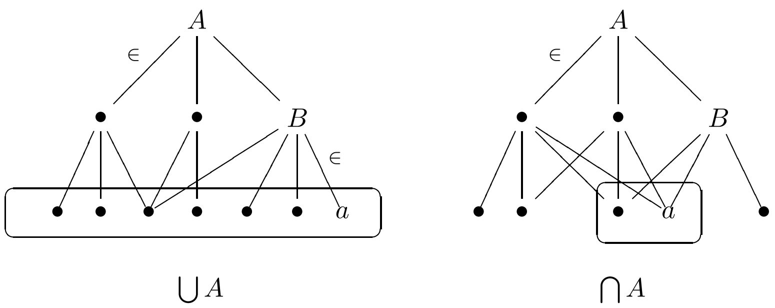


Figure 1.1: Unary union and intersection

Next, for any two sets A and B,

$$A \cap B \stackrel{\text{def}}{=} \{a \in A \;:\; a \in B\} (= \{a \in B \;:\; a \in A\}) .$$

$A \cap B$ is called the *(binary) intersection* of A and B. If $A \cap B = \emptyset$ then we say that A and B are *disjoint*. The symbol $\dot{\cup}$ stands for *disjoint union*, that is, $H = A \,\dot{\cup}\, B$ iff $H = A \cup B$ and $A \cap B = \emptyset$, for any set H.

The (set theoretic) *difference* of two sets A and B is defined as

$$A \smallsetminus B \stackrel{\text{def}}{=} \{a \in A \;:\; a \notin B\} .$$

We will need to use *(ordered) pair*s and *(ordered) n-tuple*s (n a natural number). The idea of a pair is that it is an object consisting of a *first member* and a *second member*. We denote a pair with first member a and second member b as $\langle a, b\rangle$. We want to regard two pairs equal iff their first members as well as their second members coincide. That is,

$$\langle a, b\rangle = \langle c, d\rangle \quad \text{iff} \quad (a = c \text{ and } b = d) . \tag{1.5}$$

The concept of an n*–tuple* ($n \in \omega$) is a natural generalization of that of a pair. An n–tuple ($n \in \omega$) has a first member, a second member, ..., an n–th member. $\langle a_1, \ldots, a_n\rangle$ denotes the n–tuple the first member of which is a_1, ..., the n–th member of which is a_n. $\langle a_1, \ldots, a_n\rangle = \langle b_1, \ldots, b_n\rangle$ iff ($a_1 = b_1$, ..., $a_n = b_n$). If $n = 3$ or 4 then we call an n–tuple, respectively, a *triple* or a *quadruple*. The set theoretic "coding" (that is, formal definition) of a pair $\langle a, b\rangle$ is the following.

$$\langle a, b\rangle \stackrel{\text{def}}{=} \{\{a\}, \{a, b\}\} . \tag{1.6}$$

See Figure 1.2. Checking that $\langle a, b\rangle$ defined as in (1.6) satisfies condition (1.5) is left to the reader.

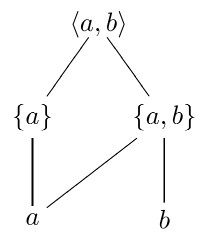


Figure 1.2: Ordered pair

Collections $\{x : \varphi(x)\}$ are called *class*es. Every set can be thought of as a class, since for any set H, $H = \{x : x \in H\}$. But, as Russell's paradox shows, some classes are not sets. Classes that are not sets are called *proper class*es. There may be collections of sets which do not form neither sets nor classes. Formulating and proving the class version of Russell's paradox is left to the reader.

• Though many of the "set forming" constructions described above generalize to classes in a straightforward way (everything we described above do generalize), some such constructions do *not* work for classes in general. Sets are more "coherent" than other kinds of classes. By "coherent" we mean that when we are forming new collections from "coherent" ones, we always get classes again. For example, *the collection of all functions going from a "coherent" class (set) into another "coherent" class (set) is again a class.* This is not true for classes in general.

An example for a proper class is the class V of all sets. Other typical examples for proper classes in mathematics are: the class of all Boolean algebras, the class of all semigroups.

Summing up what we said so far, ZFC is a set theory in which the existence of the empty set $\emptyset$ is postulated and then the rest of the sets are built up from $\emptyset$ via relatively simple "set forming constructions" (e.g. the construction of the powerset of a set). Some more set forming constructions will be introduced in the following sections (the less trivial of which is the Axiom of Choice in section 1.5). We also illustrated that, in forming new collections from sets, we can "fall into certain traps", see Russell's paradox (Thm.1.1.1). To avoid these traps, the concept of a class has been introduced. Thus in ZFC, two kinds of entities exist: *sets* and *classes* (and nothing else). While sets can be elements of other sets or of (possibly proper) classes, proper classes cannot be elements of classes.

Some of our concepts naturally extend to proper classes from sets without causing anomalies. Such are e.g. inclusion $\subseteq$, union $\cup$, intersection $\cap$, difference $\setminus$. See also our later comment in the paragraph at the end of section 1.6.

We close this subsection by introducing the set theoretic "coding" of natural numbers the way von Neumann did.

By the *successor* of a set x we mean $S(x) = x \cup \{x\}$. The natural numbers are generated from the empty set $\emptyset$ using the operation of successor S, as follows.

$$0 = \emptyset,\ 1 = S(0),\ 2 = S(1),\ \ldots,\ n = S(n-1),\ \ldots. \tag{1.7}$$

It is left to the reader to check that this definition implies that

$$n = \{0, 1, 2, \ldots, n-1\} \tag{1.8}$$

for every natural number n. Thus $k \in n$ for every $k \leqslant n$ (where $\leqslant$ is the usual ordering of natural numbers).

1.2 Binary relations, equivalence relation, functions

For any two sets A and B, $A \times B$ denotes the *Cartesian product* (or *direct product*) of A and B, and it is defined as follows:

$$A \times B \stackrel{\text{def}}{=} \{\langle a, b\rangle \ :\ a \in A \text{ and } b \in B\}\,.$$

If $R \subseteq A \times B$ for some sets A and B then R is called a *binary relation*. If R is a binary relation then $Dom(R)$ and $Rng(R)$ denote its *domain* and *range* respectively, that is,

$$Dom(R) = \{a \ :\ \langle a, b\rangle \in R \text{ for some } b\} \quad \text{and} \tag{1.9}$$
$$Rng(R) = \{b \ :\ \langle a, b\rangle \in R \text{ for some } a\}\,. \tag{1.10}$$

On Figure 1.3 we show two ways of "drawing" (or illustrating) binary relations. R on Figure 1.3 was drawn in "coordinate system style": a horizontal line and a vertical line (two "coordinate axes") represent, respectively, (sets containing) $Dom(R)$ and $Rng(R)$. A point of the plain determined by the two lines represents an ordered pair the first member of which comes from $Dom(R)$, the second one from $Rng(R)$ (see, e.g., the point $\langle a, b\rangle$ on Figure 1.3). Thus R itself is represented by a collection of points of the plain (see shaded area).

The second way we draw binary relations differs from the first one in that we represent ordered pairs differently: an ordered pair is represented by two points connected by a line segment. We have already used this method on Figure 1.2, to represent (a small part of) the membership relation. On Figure 1.3, we assume that the left–hand–side point is the first member of the pair. Thus the relation S on Figure 1.3 is represented by a collection of pairs drawn as line segments connecting points.

If R is a binary relation then instead of writing $\langle a, b\rangle \in R$, we will often write $R(a, b)$ or $a\ R\ b$ (the latter form is called *infix notation*). $a \not R\ b$ stands for $\langle a, b\rangle \notin R$.

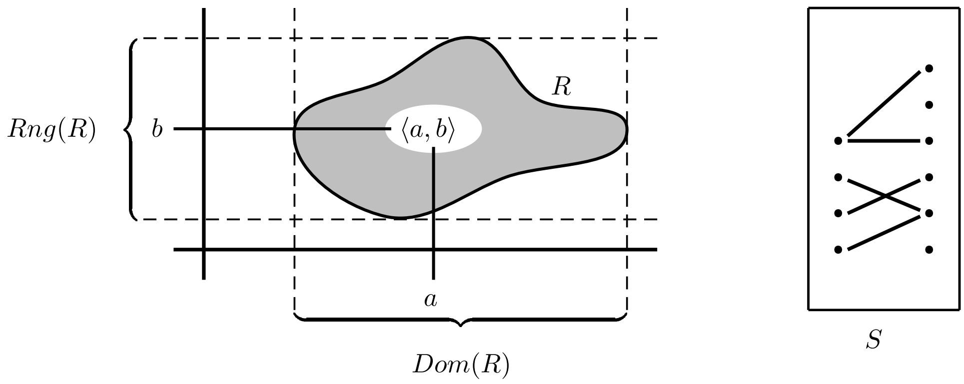


Figure 1.3: Binary relation

For two binary relations R and S, their *composition* $R \circ S$ is defined by

$$R \circ S \stackrel{\text{def}}{=} \{\langle a, b\rangle \ : \ \langle a, c\rangle \in R \text{ and } \langle c, b\rangle \in S \text{ for some set } c\}\,. \tag{1.11}$$

See the illustration of $R \circ S$ on Figure 1.4. The *converse* (or *inverse*) $R^{\smile}$ of R is:

$$R^{\smile} \stackrel{\text{def}}{=} \{\langle a, b\rangle \ : \ \langle b, a\rangle \in R\}\,. \tag{1.12}$$

If A is a set then the *identity relation* Id_A on A is defined as follows:

$$Id_A \stackrel{\text{def}}{=} \{\langle a, a\rangle \ : \ a \in A\}\,. \tag{1.13}$$

When there is no danger of confusion, we omit the subscript A from Id_A and write simply Id.

On Figure 1.4 we illustrate composition, converse and identity. On that figure, $A = 4 = \{0, 1, 2, 3\}$ and $R, S \subseteq A \times A$.

Let $R \subseteq U \times U$ for some set U. R is called an *equivalence relation* iff (i)–(iii) below hold.

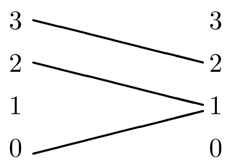
(i) R is *reflexive*, that is, $Id_{Dom(R) \cup Rng(R)} \subseteq R$;

(ii) R is *symmetric*, that is, $R^{\smile} \subseteq R$;

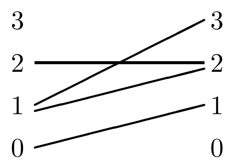
(iii) R is *transitive*, that is, $R \circ R \subseteq R$.

(You may want to consult Exercise 1.3.1 1,2,3 at this point.)

An equivalence relation R is called an equivalence relation *on* U iff $U = Dom(R) \cup Rng(R)$.

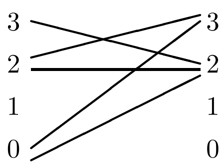


$R = \{\langle 0,1\rangle, \langle 2,1\rangle, \langle 3,2\rangle\}$

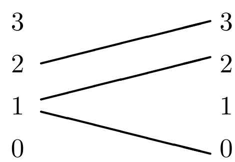


$S = \{\langle 0,1\rangle, \langle 1,2\rangle, \langle 1,3,\rangle, \langle 2,2\rangle\}$

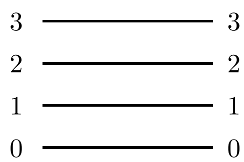
Now:



$R \circ S =$
$\{\langle 0,2\rangle, \langle 0,3\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$



$R^{-1} =$
$\{\langle 1,0\rangle, \langle 1,2\rangle, \langle 2,3\rangle\}$



$Id_4 =$
$\{\langle n,n\rangle : n < 4\}$

Figure 1.4: Operations on relations

For every $u \in U$ we let

$$u/R \stackrel{\text{def}}{=} \{v \in U \ : \ \langle u,v\rangle \in R\}\,. \tag{1.14}$$

The set u/R is called the R–*equivalence class* of u. Any element of an equivalence class is called a *representative* (or representing element) of that class.

Exercise 1.2.1. (uniqueness of representative) Prove that if $v \in u/R$ for some equivalence relation R and $u, v \in Dom(R)$, then $v/R = u/R$. That is, an equivalence class is determined by its arbitrary representative.

Exercise 1.2.2. (partition) Prove that if R is an equivalence relation on U then R determines a *partition* of U, that is, there are sets U_i ($i \in I$ for some set I) satisfying (i)–(iii) below.

(i) $\bigcup\{U_i \ : \ i \in I\} = U$

(ii) $(\forall i,j \in I)(i \neq j \Longrightarrow U_i \cap U_j = \emptyset)$

(iii) $(\forall u_1, u_2 \in U)\big(\langle u_1,u_2\rangle \in R \Longleftrightarrow (\exists i \in I)(u_1 \in U_i \text{ and } u_2 \in U_i)\big)$.

Hint: Use the equivalence classes defined above.

Let R be an equivalence relation on U. Then the *partition* U/R *determined* *by* R is defined as

$$U/R \stackrel{\text{def}}{=} \{u/R \ : \ u \in U\}\,. \tag{1.15}$$

A binary relation f is called a *function* iff $(\langle a,b\rangle, \langle a,c\rangle \in f \implies b = c)$. E.g., Id_A defined above is a function for any set A; on Figure 1.4, R and Id_4 are functions, but S, $R \circ S$, $R^{\smile}$ are not functions. For any $x \in Dom(f)$ of a function f, $f(x)$ denotes the unique element y for which $\langle x, y\rangle \in f$. Instead of $f(x)$, sometimes we write fx or f_x. When defining a function, say f, we often use the following notation.

$$f = \langle f(x) \ : \ x \in Dom(f)\rangle = \langle f(x)\rangle_{x \in Dom f} \tag{1.16}$$

and some slight variations of these. An example for using this notation: Consider the traditional definition "f is a function, $Dom(f) = \omega$, and $(\forall n \in \omega) f(n) \stackrel{\text{def}}{=} n+3$". Instead of this formulation we will write concisely "$f = \langle n+3 \ : \ n \in \omega\rangle$". The similar notation $g \stackrel{\text{def}}{=} \langle n+3 \ : \ n \in \omega \text{ and } n \text{ is even}\rangle$ says that g is the restriction of the function $\langle n+3 \ : \ n \in \omega\rangle$ to the even numbers. That is,

$$\langle n+3 \ : \ n \in \omega, n \text{ is even}\rangle = \{\langle 2k, 2k+3\rangle \ : \ k \in \omega\}\,. \tag{1.17}$$

For a function f and sets A, B, "$f : A \longrightarrow B$" means that $Dom(f) = A$ and $Rng(f) \subseteq B$. Sometimes we write $A \xrightarrow{f} B$ instead of $f : A \longrightarrow B$. If $f : A \longrightarrow B$ and $C \subseteq A$ then $f \lceil C$ denotes the *restriction* of f to C, that is,

$$f\lceil C \stackrel{\text{def}}{=} \langle f(x) \ : \ x \in C\rangle = \{\langle x, y\rangle \in f \ : \ x \in C\}\,. \tag{1.18}$$

A function $f : A \longrightarrow B$ is called *surjective* (or *onto*) iff $Rng(f) = B$; *injective* (or *one–one*) iff $(\forall a, b \in A)(f(a) = f(b) \implies a = b)$; *bijective* iff it is both surjective and injective. We introduce the following notation for surjective, injective and bijective functions. Each of

$$f : A \twoheadrightarrow B \qquad \text{and} \qquad A \stackrel{f}{\twoheadrightarrow} B \tag{1.19}$$

denotes that f is a surjective function (*surjection*) from A onto B; each of

$$f : A \rightarrowtail B \qquad \text{and} \qquad A \stackrel{f}{\rightarrowtail} B \tag{1.20}$$

denotes that f is an injective function (*injection*) from A into B; finally, each of

$$f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B \qquad \text{and} \qquad A \stackrel{f}{\rightarrowtail\!\!\!\!\twoheadrightarrow} B \tag{1.21}$$

denotes that f is a bijective function (*bijection*) from A otno B.

Let A, B be sets. Then

$$^{A}B \stackrel{\text{def}}{=} \{f \ : \ f \text{ is a function with } Dom(f) = A \text{ and } Rng(f) \subseteq B\}\,. \tag{1.22}$$

Thus $^{\emptyset}B = \{\emptyset\} = 1$ and $^{A}\emptyset = \emptyset$ if $A \neq \emptyset$.

If $f \in {}^{A}B$ and $g \in {}^{B}C$ for some sets A, B, C then, since functions are binary relations, it is meaningful to compose them using relation composition $\circ$ as defined by (1.11). It is easy to see that

- $f \circ g$ is again a function,
- $(f \circ g)(x) = g(f(x))$ for every $x \in Dom(f)$,
- $f \circ g : A \longrightarrow C$.

Besides relation composition $f \circ g$, we will speak about their *function–composition* as well. The function–composition fg of f and g is defined as

$$fg \stackrel{\text{def}}{=} g \circ f\,. \tag{1.23}$$

Hence $(g \circ f)(x) = (fg)(x) = f(g(x))$.

Exercise 1.2.3. Prove that if both f and g are injective then so is $f \circ g$. Assume $f \circ g$ is injective. Does this imply that so is f? Does this imply that so is g?

Let f be a function and $X \subseteq Dom(f)$. Then the *f–image* $f[X]$ *of* X taken pointwise is defined as

$$f[X] \stackrel{\text{def}}{=} \{f(y) \ : \ y \in X\}\,. \tag{1.24}$$

Exercise 1.2.4. (1) In our notation, we deliberately distinguish between $f[X]$ and $f(X)$ because it may happen that both $X \subseteq Dom(f)$ and $X \in Dom(f)$, but $f[X] \neq f(X)$. Define a function f and a set X such that both $f[X]$ and $f(X)$ exist and they differ.
Hint: If $Dom(f) = \omega$ then $3 \in \omega$ and $3 = \{0, 1, 2\} \subseteq \omega$ is a good start.
(2) Prove that $f[X] = Rng(f \lceil X)$.

Synonyms for "function" are *operation, mapping, map, system* and *family*. The word "system" and "family" are used when we, intuitively, think of the values of the function as collections of elements. E.g., we speak sometimes about a system or family $\langle A_i \ : \ i \in I\rangle$ of sets with index set I — this is just a function with domain I.

Sometimes we also call functions *sequences*, we will return to this in section 1.4.

1.3 Orderings, ordinals, cardinals

Let $R \subseteq A \times A$ for some set A. This binary relation R is called a *partial ordering* of A iff R is transitive, reflexive and anti–symmetric (i.e., $R \cap R^{\smile} = Id$). E.g., every set of sets is partially ordered by the inclusion relation $\subseteq$. Partial orderings are often denoted by $\leqslant$. Instead of $a \leqslant b$, we sometimes write $b \geq a$. Both $a < b$ and $b > a$ abbreviate ($a \leqslant b$ and $a \neq b$). If $\leqslant$ is a partial ordering of some nonempty set A then the pair $\langle A, \leqslant \rangle$ is called a *partially ordered set* (*poset* for short).

Let $\langle A, \leqslant \rangle$ be a poset, and let $a, b \in A$. We say that b *covers* a, in symbols $a \prec b$, iff $a < b$ and $\{c \in A \ : \ a < c < b\} = \emptyset$. By drawing the covering relation $\prec$ in the style the relation S was drawn on Figure 1.3 or all relations on Figure 1.4, it is possible to draw diagrams of finite posets and certain infinite posets (this technique has been developed and successfully used in lattice theory). In more detail, a diagram of a poset $\langle A, \leqslant \rangle$ is obtained by arranging the elements of A as points on the plane in such a way that if $a < b$ then the point representing b is above the point representing a. Then a line segment is drawn between any two points a and b whenever $a \prec b$ or $b \prec a$. On Figure 1.5 we give examples for poset diagrams.

By a *linear* (or *simple* or *total*) *ordering* of a set A we mean a partial ordering R of A which is *connected* (i.e., $A \times A \subseteq R \cup R^{\smile}$). If $\leqslant$ is a linear ordering of A then $\langle A, \leqslant \rangle$ is called a *linearly ordered set*. By a *well–ordered set* we mean a linearly ordered set $\langle A, \leqslant \rangle$ such that every nonempty subset $B \subseteq A$ has a least element ($\ell \in B$ is a *least element* of B iff $\ell \leqslant x$ for all $x \in B$). A subset $C \subseteq A$ of a poset $\langle A, \leqslant \rangle$ is called a *chain* in the poset iff $\langle C, \leqslant \cap (C \times C) \rangle$ is a linearly ordered set. By an *upper bound* of a chain C we mean an element $u \in A$ for which $c \leqslant u$ for every $c \in C$.

Zorn's Lemma is the statement that if $\langle A, \leqslant \rangle$ is a poset in which every chain has an upper bound, then $\langle A, \leqslant \rangle$ has a maximal element (m is a *maximal element* of $\langle A, \leqslant \rangle$ iff ($m \in A$ and $m \leqslant x \in A$) implies $m = x$).

We take this statement as an axiom.

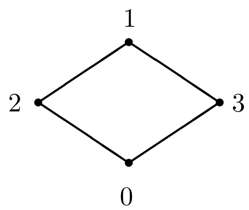
Recall from section 1.1 that by the successor $S(x)$ of a set x we mean $x \cup \{x\}$. *Ordinals* are generated from the empty set $\emptyset$ using the operations of successor S and union $\bigcup$ (the union of any *set* of ordinals is an ordinal). The *finite ordinals* are the *natural numbers*: $0 = \emptyset$, $1 = S(\emptyset)$, $2 = S(S(\emptyset))$, $\ldots$. Every set of ordinals is well–ordered by the following ordering $\leqslant$. For any ordinals α and β we set $\alpha \leqslant \beta$ iff ($\alpha = \beta$ or $\alpha \in \beta$). Then $0 \leqslant 1 \leqslant 2 \ldots$, and we have $n = \{0, 1, \ldots, n-1\}$ for each finite ordinal $n \geq 1$. The least *infinite ordinal* is

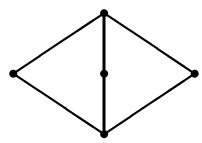
$$\omega = \bigcup \{\alpha \ : \ \alpha \text{ is a finite ordinal}\} = \{0, 1, 2, \ldots\} .$$

Now, clearly, $n \in \omega$ iff $n \leqslant \omega$ for any set n. (Cf. end of section 1.1)

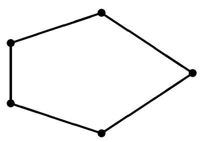
$\mathfrak{A} = \big\langle \{0,1\}, Id_A \cup \{\langle 0,1\rangle\}\big\rangle$

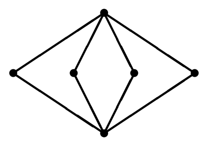
$\mathfrak{B} = \big\langle \{0,1,2\}, Id_B \cup \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle\}\big\rangle$





$\mathfrak{C} = \big\langle \{0,1,2,3\},$
$\quad Id_C \cup \{\langle 0,a\rangle : a \in C\} \cup \{\langle 2,1\rangle, \langle 3,1\rangle\}\big\rangle$





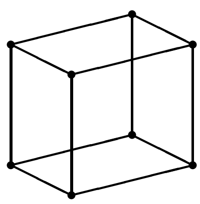


Figure 1.5: Orderings

Two sets A and B are said to *have the same cardinality* iff there is a bijection from A onto B. The *cardinals* are those ordinals κ for which we have that no ordinal $\beta < \kappa$ has the same cardinality as κ. The *finite cardinals* are just the finite ordinals, and ω is the smallest *infinite cardinal*.

The **Well Ordering Theorem** is the statement that every set has the same cardinality as some ordinal.

We take it as an axiom.

The *cardinality* of a set A is the (unique) cardinal κ such that A and κ have the same cardinality. We denote the cardinality of A by $|A|$. We say that A is *finite* iff $|A|$ is a finite cardinal, that is, iff $|A| \in \omega$ that is, iff $|A| < \omega$.

The *power set* $\mathcal{P}(A)$ of a set A is defined as

$$\mathcal{P}(A) \stackrel{\text{def}}{=} \{B \;:\; B \subseteq A\}\,.$$

$\mathcal{P}(A)$ has the same cardinality as ${}^{A}2$ (hence the term "power set").

We do not recall the operations of multiplication, addition and exponentiation of cardinals. Instead, we recall some simple facts concerning these, as follows.

• Addition and multiplication of cardinals are rather trivial when infinite cardinals are involved. For example, if κ, λ are cardinals, $0 < \kappa \leqslant \lambda$ and $\omega \leqslant \lambda$ then $\kappa + \lambda = \kappa \cdot \lambda = \lambda$.

• For any sets A and B, $|A| \cdot |B| = |A \times B|$, $|A| + |B| = |A \cup B|$ if A and B are disjoint, and $|A|^{|B|} = |{}^{B}A|$.

Exercise 1.3.1. Let $R \subseteq A \times A$ for some set A. Prove equivalences (1)–(5) below.

1. R is reflexive iff $x \mathrel{R} x$ for every $x \in A$.
2. R is symmetric iff $x \mathrel{R} y$ implies $y \mathrel{R} x$ for every $x, y \in A$.
3. R is transitive iff $x \mathrel{R} y$ and $y \mathrel{R} z$ imply $x \mathrel{R} z$ for every $x, y, z \in A$.
4. R is connected iff for all $x, y \in A$, $x \mathrel{R} y$ or $y \mathrel{R} x$.
5. R is anti–symmetric iff $x \mathrel{R} y$ implies $y \not\mathrel{R} x$ for all $x, y \in A$.

1.4 Sequences

As we already mentioned, functions are sometimes called *sequence*s. In more detail, if $I = Dom(f)$ then we sometimes call f an *I–sequence*. In particular, if $Dom(f) = \omega$ then we call f an *ω–sequence*; and if $Dom(f) = n$ for some $n \in \omega$ then we call f an *n–sequence* or a *sequence of length n*. A function f is called a *finite sequence* iff it is an n–sequence for some $n \in \omega$. (The latter two examples, ω–sequences and finite sequences, motivate the use of the word "sequence" — an ω–sequence or n–sequence can really be pictured as a concrete sequence or series $\langle f_0, f_1, f_2, \ldots \rangle$ resp. $\langle f_0, f_1, f_2, \ldots, f_{n-1} \rangle$ of values.)

For any set X, X^* denotes the set of all *finite sequences over* X, defined as follows:

$$X^* \stackrel{\text{def}}{=} \{f \ : \ Dom(f) \in \omega \text{ and } Rng(f) \subseteq X\} = \bigcup_{n\in\omega} \left({}^nX\right).$$

Throughout this book, we identify finite sequences of length n with n–tuples discussed in section 1.1. This convention will make our text simpler, and will cause no confusion. Accordingly, we will use the same notation for sequences of length n and n–tuples, e.g. $\langle a_0, \ldots, a_{n-1}\rangle$ may denote an n–tuple as well as an n–sequence. Sometimes we will use the "vector notation" $\bar{a}$ for $\langle a_0, \ldots, a_{n-1}\rangle$.

1.5 Direct product of families of sets

In section 1.2 we introduced the direct product of two sets. Generalizing this concept, we define the *Cartesian* or *direct product* ΠA *of a system of sets* $A = \langle A_i \ : \ i \in I\rangle$ as follows.

A function f with $\big(Dom(f) = I$ and $(\forall i \in I) f(i) \in A_i\big)$ is called a *choice function* for A. Now

$$\Pi A = \Pi\langle A_i \ : \ i \in I\rangle = \Pi_{i\in I} A_i \stackrel{\text{def}}{=} \{f \ : \ f \text{ is a choice function for } A\}.$$

See Figure 1.6.

A variant of Figure 1.6 is Figure 1.7. In the latter, a set is represented by a (vertical) line segment instead of an oval shape (as in Figure 1.6). On Figure 1.7, the index set I is represented by a horizontal line.

Notice that if $(\exists i \in I) A_i = \emptyset$ then $\Pi A = \emptyset$. Also, if $I = \emptyset$ then $|\Pi A| = 1$ (the only element of ΠA being the empty function $\emptyset$).

A köv. ábra balra kilóg az ablakból!

Exercise 1.5.1. Let $\langle A_i \ : \ i \in I\rangle$ be a system of sets. Assume that $(\forall i \in I) A_i = B$ for some set B. Prove that $\Pi\langle A_i \ : \ i \in I\rangle = {}^IB$.

In this context, we call ${}^IB$ the (Cartesian or) *direct power* of the set B. In particular, if $n \in \omega$ and U is any set then $\Pi\langle U \ : \ i < n\rangle = {}^nU$. For $\Pi\langle U \ : \ i < n\rangle$ we sometimes write $\underbrace{U \times \cdots \times U}_{n\text{-times}}$.

Let $A = \langle A_i \ : \ i \in I\rangle$ be a system of sets, and let $j \in I$. Then the j–th *projection function* (or simply, j–th *projection*) $p_j : \Pi A \longrightarrow A_j$ is defined as follows:

$$(\forall f \in \Pi A)\, p_j(f) \stackrel{\text{def}}{=} f(j).$$

The **Axiom of Choice** (AC for short) is the statement that if $\langle A_i \ : \ i \in I\rangle$ is any system of sets with $A_i \neq \emptyset$ for all $i \in I$ then $\Pi_{i\in I} A_i \neq \emptyset$.

We take this as an axiom. We note that AC, Zorn's Lemma and the Well Ordering Theorem are (mutually) equivalent. (The proof is not trivial.)

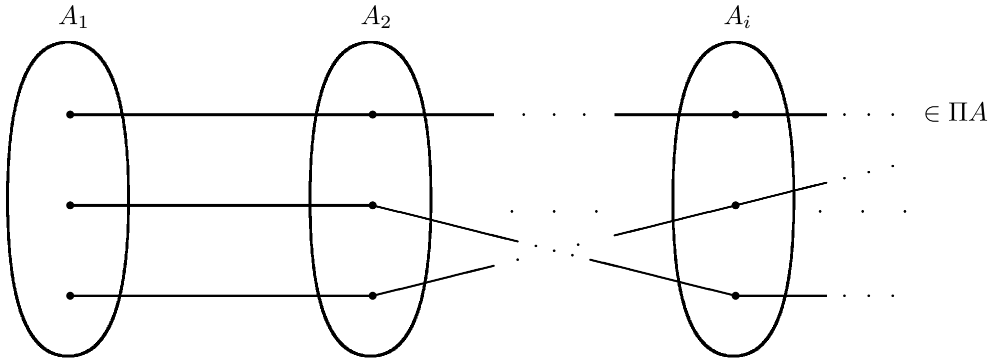


Figure 1.6: Drawing Cartesian (direct) products 1

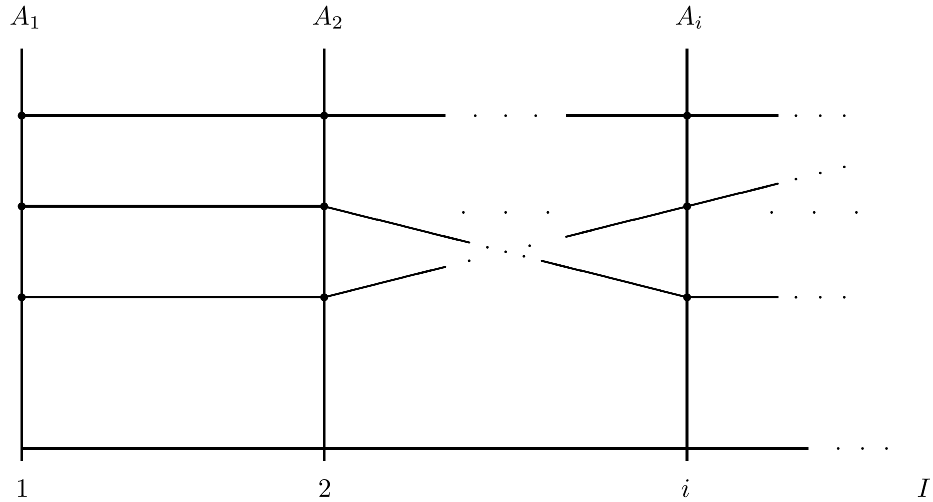


Figure 1.7: Drawing direct products 2

1.6 Relations of higher ranks

In this section we generalize the concept of a binary relation introduced in section 1.2.

If $n \in \omega \smallsetminus \{0\}$ and R is a set of n–sequences then we say that R is an *n–ary relation*. We refer to this n as the *rank* or *arity* of R. Clearly, an n–ary relation R is a subset of the Cartesian product $U_0 \times U_1 \times \cdots \times U_{n-1}$ of the following sets U_i $(i < n)$:

$$\begin{aligned} U_0 &= \{u_0 : \langle u_0, u_1, \ldots, u_{n-1}\rangle \in R \text{ for some } u_1, \ldots, u_{n-1}\}\,, \\ U_1 &= \{u_1 : \langle u_0, u_1, \ldots, u_{n-1}\rangle \in R \text{ for some } u_1, \ldots, u_{n-1}\}\,, \\ &\vdots \\ U_{n-1} &= \{u_{n-1} : \langle u_0, u_1, \ldots, u_{n-1}\rangle \in R \text{ for some } u_1, \ldots, u_{n-1}\}\,. \end{aligned}$$

Also, $R \subseteq {}^nU$ for $U = \bigcup\{U_i : i < n\}$. It is clear that a binary relation as defined previously is a 2–ary relation in the present sense.

If $n \in \omega$ and U is a set then a function f with $Dom(f) = {}^nU$ and $Rng(f) \subseteq U$ is called an *n–ary function* (or *operation*) on U, and we say that the *rank* (or *arity*) *of the function* f is n. We often identify n–ary functions with $n+1$–ary relations, associating to an n–ary function f the $n+1$–ary relation $\{\langle u_0, u_1, \ldots, u_{n-1}, u_n\rangle : \langle\langle u_0, u_1, \ldots, u_{n-1}\rangle, u_n\rangle \in f\}$.

A synonym for "1–ary function (relation)" is "unary function (relation)". Similarly, for "2–ary" and "3–ary" we say "binary" and "ternary", respectively.

If R is a unary relation then we identify R with the set $\{u : \langle u\rangle \in R\}$. Similarly, if f is a 0–ary function on some nonempty set U then $f : {}^0U \longrightarrow U$ (where ${}^0U = \{\emptyset\} = 1$), and we identify f with $f(\emptyset) \in U$. 0–ary functions are called *constants* (or constant functions).

For being able to handle relations of different ranks in a unified framework, we will need so called *ω–ary relations*, too. R is called an ω–ary relation iff $R \subseteq {}^\omega U$ for some set U. Examples for ω–ary relations are:

1. The set of all convergent ω–sequences over $\mathbb{R}$ (in the usual sense of calculus) is clearly an ω–ary relation.
2. The universe of an ω–dimensional vector space is an ω–ary relation. So are the universes of its subspaces.
3. If U is a set then ${}^\omega U$ is called the *Cartesian space* with base U and dimension ω. If $p \in {}^\omega U$, we set

$${}^\omega U^{(p)} \stackrel{\text{def}}{=} \{q \in {}^\omega U : \{i \in \omega : q_i \neq p_i\} \text{ is finite}\}\ ,$$

and we call ${}^{\omega}U^{(p)}$ the *weak Cartesian space* with base U and dimension ω determined by p. Clearly, ${}^{\omega}U$ and ${}^{\omega}U^{(p)}$, for any U and $p \in {}^{\omega}U$, are ω–ary relations.

After having introduced classes at the end of section 1.1, we always spoke about sets and did not mention classes. However, many of the above constructs generalize from sets to proper classes, and we will need some of these in this book. For example, it is meaningful to speak about functions and relations defined on proper classes instead of on sets. Also, sometimes we want to consider ordered tuples $\langle A, B, C, \ldots \rangle$, where some of $A, B, C, \ldots$ may be proper classes. We will use such general constructs the way it is customary in the literature of algebra and model theory. In this book we do not want to go into explaining how to make sure that we cause no set theoretic problems when using such general constructs. (The interested reader can find a compact but thorough exposition to the basics of set theoretic foundations in Chang–Keisler [26, Appendix A].)

1.7 Closure systems

In this section we introduce the following concepts: closure operator, closure system, and Galois connection. We will give a very short summary of these, in particular, we will look into some basic connections between them. These concepts will be very useful as they occur at many places in our study of algebraic logic — as well as in universal algebra.

Definition 1.7.1. (closure operator) Let H be an arbitrary set and $\mathsf{c} : \mathcal{P}(H) \longrightarrow \mathcal{P}(H)$. We call c a *closure operator* (over H) iff conditions (i)–(iii) are satisfied by c, for every $X, Y \subseteq H$.

(i) $X \subseteq \mathsf{c}(X)$ $\quad$ c is *extensive*,

(ii) $X \subseteq Y \Longrightarrow \mathsf{c}(X) \subseteq \mathsf{c}(Y)$ $\quad$ c is *isotone*,

(iii) $\mathsf{cc}(X) = \mathsf{c}(X)$ $\quad$ c is *idempotent*.

We call $\mathsf{c}(X)$ the *closure* (or c-*closure*) of X. If $X = \mathsf{c}(X)$ the we say that X is a *closed set*.

Examples 1.7.2. (closure operators)

(1) The function $\mathsf{c} : \mathcal{P}(H) \longrightarrow \mathcal{P}(H)$ defined by

$$\mathsf{c}(X) \stackrel{\text{def}}{=} H \quad \text{for every} \quad X \subseteq H$$

is clearly a closure operator.

(2) The *ceiling function* $\mathrm{ce} : \mathrm{R} \longrightarrow \mathrm{R}$ (R is the set of all real numbers) is defined as follows. For every $r \in \mathrm{R}$, ce is the smallest integer not smaller than r. This ce function is again a closure operator.

(3) Later we will see a number of examples for closure operators, see **to be filled in**.

Definition 1.7.3. (closure system) By a *closure system* we mean a pair $\langle H, D\rangle$ where H is an arbitrary set and $D \subseteq \mathcal{P}(H)$ is such that

$$(\forall L \subseteq D) \bigcap L \in D\,.$$

The elements of D are called *closed sets*.

If $\langle H, D\rangle$ is a closure system then $H \in D$ because $\emptyset \subseteq D$ and $H = \bigcap \emptyset \in D$.

Lemma 1.7.4. (closure operators and closure systems)

1. *(1) If* c *is a closure operator over* H *then* $\langle H, \{X \subseteq \mathcal{P}(H) : \mathsf{c}(X) = X\}\rangle$ *is a closure system.*
2. *(2) If* $\langle H, D\rangle$ *is a closure system then* $\{\langle X, \bigcap\{Y \in D : X \subseteq Y\}\rangle : X \subseteq H\}$ *is a closure operator.*
3. *(3) Let us denote the closure system associated to* c *in (1) above by* c^*. *Let us denote the closure operator associated to* $\langle H, D\rangle$ *in (2) by* $\langle H, D\rangle^*$. *Then*

$$\mathsf{c}^{**} = \mathsf{c} \qquad \textit{and} \qquad \langle H, D\rangle^{**} = \langle H, D\rangle$$

for every closure operator c *and closure system* $\langle H, D\rangle$.

Proof. Left to the reader. □

The concepts closure operator, closure system and Galois connection can be defined in a more abstract way, where the rôles of powersets used here are taken by (abstract) posets. We will look into this later **reference to be filled in**.

These concepts make sense even when H and D are proper classes instead of sets. Just one has to be careful with the formulation, because $\mathcal{P}(H)$ or $\mathcal{P}(D)$ may not exist. So one has to phrase everything in terms of *subclasses*.

To be continued!!

1.8 First–order logic

In this section we recall the definition of *first–order logic*. We assume that the reader is familiar with this concept, we recall it here only for fixing our notation.

Definition 1.8.1. (first–order logic)

(1) We call a function t a *similarity type* (or *signature* or *ranked alphabet*) iff (i)–(iii) below hold.

(i) $Rng(t) \subseteq \omega$;

(ii) $Dom(t) = Fns_t \mathbin{\dot\cup} Rls_t$ for some sets Fns_t and Rls_t ;

(iii) If $r \in Rls_t$ then $t(r) \neq 0$.

The sets Fns_t and Rls_t are called, respectively, the set of *function symbols* or *operation symbols* and the set of *relation symbols* of t. For any $s \in Dom(t)$, $t(s)$ is called the *rank* or *arity* of s; if $t(s) = 0$ then we call s a *constant symbol*; if $s \in Fns_t$ then we call s a *unary* (*binary*) function symbol iff $t(s) = 1$ ($t(s) = 2$).

From now on until the end of this section, t stands for an arbitrary but fixed similarity type.

(2) Let V be an arbitrary set but such that $V \cap Dom(t) = \emptyset$. We define the set $Trm_t(V)$ of *terms* of similarity type t with *variables* from V and the set $Fml_t(V)$ of *formulas* of similarity type t with variables from V by recursion, as follows. $Trm_t(V)$ is defined to be the smallest set satisfying (i) and (ii) below, and $Fml_t(V)$ is defined to be the smallest set satisfying (iii) and (iv) below.

(i) $V \cup \{c \in Fns_t : t(c) = 0\} \subseteq Trm_t(V)$.

(ii) $\{f(\tau_1, \ldots, \tau_n) : f \in Fns_t, t(f) = n \neq 0 \text{ and } \tau_1, \ldots, \tau_n \in Trm_t(V)\} \subseteq Trm_t(V)$.

(iii) $\{r(\tau_1, \ldots, \tau_n) : r \in Rls_t, t(r) = n \text{ and } \tau_1, \ldots, \tau_n \in Trm_t(V)\} \cup \{\tau = \sigma : \tau, \sigma \in Trm_t(V)\} \subseteq Fml_t(V)$.

The formulas belonging to the left–hand–side set are called *atomic formulas*.

(iv) $\{\neg\varphi : \varphi \in Fml_t(V)\} \cup \{(\varphi \wedge \psi) : \varphi, \psi \in Fml_t(V)\} \cup \{\exists v\varphi : v \in V \text{ and } \varphi \in Fml_t(V)\} \subseteq Fml_t(V)$.

The logical connectives $\neg, \wedge, \exists v$ are called, respectively, *negation, disjunction* and *existential quantifier*, and are read as *not, and, there exists v such that*, respectively.

From $Trm_t(V)$ and $Fml_t(V)$ we will omit t or V and write simply, e.g., Trm_t, $Fml(V)$ or even Trm and Fml when there is no danger of confusion.

(3) By a *model* of similarity type t we mean a pair $\langle M, m\rangle$ satisfying (i)–(ii) below.

(i) M is a non–empty set (called the *universe* of $\langle M, m\rangle$).

(ii) m is a function with $Dom(m) = Dom(t)$ such that

- if $c \in Fns_t$ and $t(c) = 0$ (that is, if c is a constant symbol) then $m(c) \in M$ (that is, $m(c)$ is a constant on M);
- if $f \in Fns_t$ and $t(f) = n \neq 0$ then $m(f) : {}^nM \longrightarrow M$;
- if $r \in Rls_t$ and $t(r) = n$ then $m(r) \subseteq {}^nM$.

The functions $m(c), m(f)$ and the relations $m(r)$ are called the *interpretations* of the symbols c, f and r. They are also referred to as *fundamental operations* or *fundamental relations*, respectively.

Concerning models, we use the following notation. The meta–variables we use for models are the German capitals $\mathfrak{M}, \mathfrak{N}, \mathfrak{W}$, and these with indices like e.g. $\mathfrak{M}_3$. The universe of a model is denoted by the corresponding Roman capital, that is, the universes of $\mathfrak{M}, \mathfrak{N}, \mathfrak{W}, \mathfrak{M}_3$ are respectively M, N, W, M_3. If $\mathfrak{M} = \langle M, m\rangle$, $f \in \mathit{Fns}_t$, $r \in \mathit{Rls}_t$ then $f^{\mathfrak{M}}$ and $r^{\mathfrak{M}}$ stand, respectively, for $m(f)$ and $m(r)$. We may denote $\mathfrak{M}$ as $\langle M, f^{\mathfrak{M}}, r^{\mathfrak{M}}\rangle_{f\in \mathit{Fns}_t, r\in \mathit{Rls}_t}$ as well. If t is finite, for example t consists of a function symbol $+$ and a relation symbol $\leqslant$, then we sometimes denote $\mathfrak{M}$ as $\langle M, +^{\mathfrak{M}}, \leqslant^{\mathfrak{M}}\rangle$. The class of all models of similarity type t is denoted by Mod_t.

(4) Let $\mathfrak{M} \in \mathsf{Mod}_t$ and let V be an arbitrary set of variables (e.g., $V = \{v_i : i \in \omega\}$).

A function $k : V \longrightarrow M$ is called a *valuation* of the variables from V in $\mathfrak{M}$.

Let k be an arbitrary but fixed valuation of the variables in $\mathfrak{M}$. We define when a formula $\varphi \in \mathit{Fml}_t(V)$ is *true in* $\mathfrak{M}$ *at valuation* k *of the variables*, in symbols $\mathfrak{M} \models \varphi[k]$, by recursion, as follows. First we define the value $\tau^{\mathfrak{M}}[k]$ of any term $\tau \in \mathit{Trm}_t(V)$ at k in $\mathfrak{M}$ as:

- $v^{\mathfrak{M}}[k] \stackrel{\text{def}}{=} k(v)$ if $v \in V$,
- $c^{\mathfrak{M}}[k] \stackrel{\text{def}}{=} c^{\mathfrak{M}}$ if $t(c) = 0$,
- $(f(\tau_1, \dots, \tau_n))^{\mathfrak{M}} \stackrel{\text{def}}{=} f^{\mathfrak{M}}\left(\tau_1^{\mathfrak{M}}[k], \dots, \tau_n^{\mathfrak{M}}[k]\right)$ if $f \in \mathit{Fns}_t$, $t(f) = n$, $\tau_1, \dots, \tau_n \in \mathit{Trm}_t$.

Now

- for atomic formulas $r(\tau_1, \dots, \tau_n)$,

$$\mathfrak{M} \models r(\tau_1, \dots, \tau_n)[k] \stackrel{\text{def}}{\iff} \langle \tau_1^{\mathfrak{M}}[k], \dots, \tau_n^{\mathfrak{M}}[k]\rangle \in r^{\mathfrak{M}},$$

  for atomic formulas $\tau = \sigma$,

$$\mathfrak{M} \models \tau = \sigma[k] \stackrel{\text{def}}{\iff} \tau^{\mathfrak{M}}[k] = \sigma^{\mathfrak{M}}[k],$$

- for negated formulas $\neg\varphi$,

$$\mathfrak{M} \models \neg\varphi[k] \stackrel{\text{def}}{\iff} \text{it is not the case that } \mathfrak{M} \models \varphi[k],$$

- for conjunctions $(\varphi \wedge \psi)$,

$$\mathfrak{M} \models (\varphi \wedge \psi)[k] \stackrel{\text{def}}{\iff} \mathfrak{M} \models \varphi[k] \text{ and } \mathfrak{M} \models \psi[k],$$

- for quantified formulas $\exists v\varphi$,

$$\mathfrak{M} \models \exists v\varphi[k] \stackrel{\text{def}}{\iff} \mathfrak{M} \models \varphi[k'] \text{ for some valuation } k' \text{ such that } k\lceil(V \smallsetminus \{v\}) = k'\lceil(V \smallsetminus \{v\})\,.$$

By these, $\mathfrak{M} \models \varphi[k]$ has been defined for any $\varphi \in Fml_t$.

We say that φ is *valid* in $\mathfrak{M}$ or that $\mathfrak{M}$ is a *model of* φ, in symbols $\mathfrak{M} \models \varphi$, iff $\mathfrak{M} \models \varphi[k]$ for every valuation k.

Notation: If $\mathfrak{M} \in \mathsf{Mod}_t$, $k : V \longrightarrow M$, $\varphi \in Fml_t(V)$, and $v_{i_1}, \ldots, v_{i_n}$ are all the variables occurring freely in φ then, instead of $\mathfrak{M} \models \varphi[k]$ we sometimes write $\mathfrak{M} \models \varphi[k(v_{i_1}), \ldots, k(v_{i_n})]$.

(5) Let $\mathsf{K} \subseteq \mathsf{Mod}_t$ and $\Sigma \subseteq Fml_t$.
If $\varphi \in Fml_t$ and $\mathfrak{M} \in \mathsf{Mod}_t$ then

$$\begin{array}{rl} \mathsf{K} \models \varphi & \text{abbreviates that } \mathfrak{M} \models \varphi \text{ for every } \mathfrak{M} \in \mathsf{K}, \text{ and} \\ \mathfrak{M} \models \Sigma & \text{abbreviates that } \mathfrak{M} \models \varphi \text{ for every } \varphi \in \Sigma\,. \end{array}$$

$\mathsf{K} \models \Sigma$ means that $\mathfrak{M} \models \varphi$ for every $\mathfrak{M} \in \mathsf{K}$ and $\varphi \in \Sigma$.

The *first–order theory of* K is defined as:

$$Th(\mathsf{K}) \stackrel{\text{def}}{=} \{\varphi \in Fml_t : \mathsf{K} \models \varphi\}\,.$$

The class of *models of* Σ is defined as:

$$\mathsf{Mod}(\Sigma) \stackrel{\text{def}}{=} \{\mathfrak{M} \in \mathsf{Mod}_t : \mathfrak{M} \models \Sigma\}\,.$$

(6) If $\Sigma \subseteq Fml_t$ and $\varphi \in Fml_t$ then we say that φ is a *semantical consequence of* Σ, in symbols $\Sigma \models \varphi$, iff

$$\text{for every } \mathfrak{M} \in \mathsf{Mod}_t, \qquad \mathfrak{M} \models \Sigma \Longrightarrow \mathfrak{M} \models \varphi\,.$$

If Σ is a singleton, that is if $\Sigma = \{\psi\}$ for some $\psi \in Fml_t$ then we simply write $\psi \models \varphi$ instead of $\{\psi\} \models \varphi$.

(7) We will use the following abbreviations:

$$\begin{array}{rll} (\varphi \vee \psi) & \text{stands for} & \neg(\neg\varphi \wedge \neg\psi), \\ (\varphi \to \psi) & \text{stands for} & \neg\varphi \vee \psi, \\ (\varphi \leftrightarrow \psi) & \text{stands for} & (\varphi \to \psi) \wedge (\psi \to \varphi), \\ \forall v\varphi & \text{stands for} & \neg\exists v(\neg\varphi). \end{array}$$

The derived logical connective $\vee$ is called *conjunction* and is read as *or*; $\rightarrow$ is called *implication* and is read as *implies*; $\leftrightarrow$ is called *equivalence* and is read as *if and only if* or *iff* for short; $\forall v$ is called *universal quantifier* and is read as *for all v such that*.

(8) We define some distinguished subclasses of Fml_t. We assume that the reader is familiar with the concepts of "free occurrence of a variable in a formula", and "a variable occurs under the scope of a quantifier".

- $\varphi \in Fml_t$ is called a *sentence* iff every variable v occurring in it occurs under the scope of a quantifier.
- $\varphi \in Fml_t$ is called a *quantifier–free formula* iff no quantifiers occur in φ. φ is called a *universal formula* iff it is of form $Q_0 \dots Q_n \psi$ where ψ is quantifier–free, $n \in \omega$, $Q_i (i \leqslant n)$ is a universal quantifier ($\forall v$ for some $v \in V$).
- $\varphi \in Fml_t$ is called an *equation* iff it is of the form $\sigma = \tau$, where $\sigma, \tau \in Trm_t$.
- $\varphi \in Fml_t$ is called a *quasi–equation* iff it is of the form

  $$(e_1 \wedge \cdots \wedge e_n) \rightarrow e\,,$$

  where $e, e_1, \dots, e_n$ $(n \in \omega)$ are equations.

If $\mathfrak{M}$ is a model, then sometimes we say that "φ is a formula in the language of $\mathfrak{M}$". By this we mean that φ is a first–order formula of similarity type of that of $\mathfrak{M}$.

A következő változik az új Closure systems részfejezet miatt!

Exercise 1.8.2. Prove that $K \subseteq \mathsf{Mod}(Th(K))$ for any class K of similar models.

Chapter 2

Basics from Universal Algebra

2.1 Examples for algebras

Informally speaking, by an algebra we mean a model in which there are no relations. In more detail, recall from Definition 1.8.1 (1) (in section 1.8) that, in general, the domain of a similarity type consists of both function symbols and relation symbols. By an *algebraic similarity type* we mean a similarity type the domain of which does not contain any relation symbols (i.e., the set of relation symbols is empty), but it may contain function symbols.

To distinguish between algebras and arbitrary models in our notation, for denoting algebras, we use German capitals from the beginning of the alphabet, e.g. $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$; while $\mathfrak{M}, \mathfrak{N}, \mathfrak{W}$ stand for models of a possibly non–algebraic similarity type. When we deal with algebras, we prefer using $X = \{x_i : i \in \omega\}$ for our set of variables of the first–order language of algebras, while in case of general models, we prefer using the set of variables $V = \{v_i : i \in \omega\}$. Otherwise we use the notation introduced for models in case of algebras as well (cf. e.g. our notational conventions in Definition 1.8.1 (3)). Thus if $\mathfrak{A}$ is an algebra of similarity type t then $\mathfrak{A} = \langle A, f^{\mathfrak{A}} \rangle_{f \in Dom(t)}$, further we call A the universe of $\mathfrak{A}$, and $f^{\mathfrak{A}}$ a fundamental operation of $\mathfrak{A}$ or the interpretation of f in $\mathfrak{A}$, etc. In section 2.4, $\mathfrak{F}$ will also denote an algebra (with universe F).

Convention 2.1.1. Throughout the rest of this chapter, unless otherwise specified, by a similarity type we always mean an algebraic one, that is, one the set of relation symbols of which is empty.

Definition 2.1.2. (similarity class etc.)

(i) Let t be an arbitrary (algebraic) similarity type. Then the *similarity class* Alg_t associated to t is defined as follows.

$$\mathsf{Alg}_t \stackrel{\text{def}}{=} \{\mathfrak{A} : \mathfrak{A} \text{ is an algebra of similarity type } t\}.$$

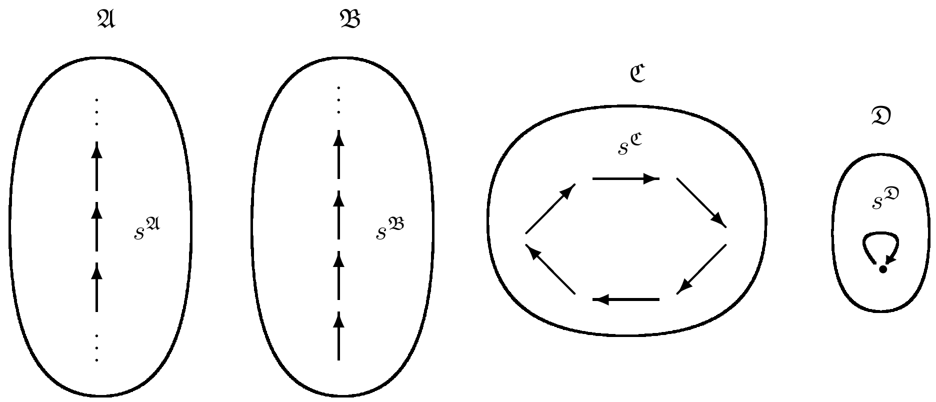


Figure 2.1: Algebras with one unary operation

(ii) We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *similar* iff both $\mathfrak{A}$ and $\mathfrak{B}$ belong to the same similarity class (Alg_t for some t).

(iii) Let $\mathfrak{A}$ be an algebra, with universe A. We say that $\mathfrak{A}$ *is finite* (*infinite*) iff $|A| < \omega$ ($|A| \geqslant \omega$). We say that $\mathfrak{A}$ is *trivial* iff $|A| = 1$.

Example 2.1.3. (unary- and mono-unary algebras) By a *unary algebra* we mean one in the similarity type of which only unary operation symbols occur. An algebra is called *mono-unary* iff its similarity type consists of one unary operation symbol only. Here are four examples for mono-unary algebras.

Let $t = \{\langle s, 1\rangle\}$, that is, t consists of one unary function symbol. The algebras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ illustrated on Figure 2.1 belong to Alg_t.
The definition of $\mathfrak{A} = \langle A, s^{\mathfrak{A}}\rangle$ is this:

$$A \stackrel{\text{def}}{=} \mathbb{Z} \text{ (the set of all integers), and } (\forall z \in \mathbb{Z}) s^{\mathfrak{A}}(z) \stackrel{\text{def}}{=} z + 1\,.$$

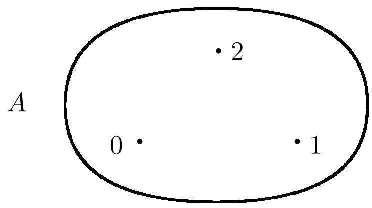
$\mathfrak{B} \stackrel{\text{def}}{=} \langle \omega, s^{\mathfrak{B}}\rangle$, where $s^{\mathfrak{B}}(n) = s^{\mathfrak{A}}(n)$ for every $n \in \omega$. $\mathfrak{C} \stackrel{\text{def}}{=} \langle 6, s^{\mathfrak{C}}\rangle$, where $s^{\mathfrak{C}}(n) = n + 1 \pmod 6$. $\mathfrak{D}$ is a trivial algebra of similarity type t.

Example 2.1.4. (groupoids) Algebras with just one binary fundamental operation are referred to as *groupoids*. Our next example is a groupoid.

Let $p = \{\langle +, 2\rangle\}$. Then $\langle \mathbb{Z}, +^{\mathbb{Z}}\rangle$ with $+^{\mathbb{Z}}$ the usual addition on the integers, and $\mathfrak{A}$ given on Figure 2.2 clearly belong to Alg_p.

Example 2.1.5. (semigroups) Let $+$ be a binary operation symbol (as in Example 2.1.4). The following equation (a) expresses a very common property of binary operations $+$ called *associativity* (we use infix notation).

(a) $(a + b) + c = a + (b + c)$.



$\mathfrak{A} = \langle 3, +(\text{mod } 3)\rangle$

$+^{\mathfrak{A}}$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

addition table for $\mathfrak{A}$

Figure 2.2: An algebra with one binary operation

A groupoid with an associative operation is called a *semigroup*. It is easy to see that $\mathfrak{A} = \langle 3, +(mod 3)\rangle$ on Figure 2.2 is a semigroup. Other simple examples for semigroups are:

- $+^{\mathfrak{A}}$ is a constant operation, that is,

$$a +^{\mathfrak{A}} b = c \;\text{ for some fixed }\; c \in A\,,$$

  for every $a, b \in A$;

- $a +^{\mathfrak{A}} b = a$ for every $a, b \in A$; or similarly, $a +^{\mathfrak{A}} b = b$ for every $a, b \in A$.

Frequently used examples for semigroups are *semigroups of words with concatenation*. Here is the definition:

For a fixed set H, called an *alphabet* in this context, H^* denotes the set of all finite sequences of elements of H, called *words* over H. That is,

$$H^* \overset{\text{def}}{=} \bigcup_{n\in\omega} \left({}^nH\right).$$

If $n = 0$ then ${}^0H = \{\emptyset\}$. In this context, we usually write λ in place of $\emptyset$, and we call λ the *empty word*.

Now, the universe of our semigroup is H^*, and the operation is *concatenation*, usually denoted by $\frown$. If $p, q \in H^*$ then $p = \langle a_1, \ldots a_n\rangle$, $q = \langle b_1, \ldots, b_k\rangle$ for some $n, k \in \omega$ and $a_1, \ldots a_n, b_1, \ldots, b_k \in H$.

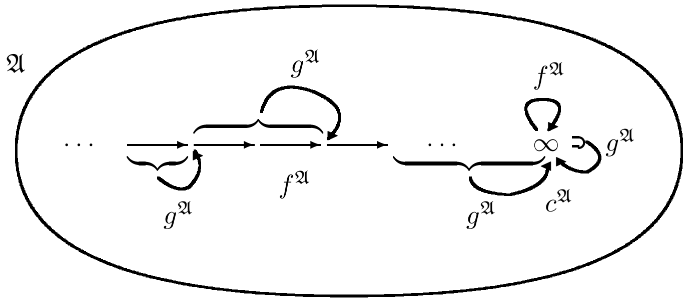
$$p \frown q = \langle a_1, \ldots a_n\rangle \frown \langle b_1, \ldots, b_k\rangle = \langle a_1, \ldots a_n, b_1, \ldots, b_k\rangle \big(\in {}^{n+k}H\big).$$

Clearly, $\langle H^*, \frown\rangle$ is a semigroup. We can get other semigroups of words by considering such subsets of H^* which are closed under $\frown$.

Example 2.1.6. Let $t = \{\langle g, 2\rangle, \langle f, 1\rangle, \langle c, 0\rangle\}$. Then

$$\mathfrak{A} = \langle \mathbb{Z} \cup \{\infty\}, g^{\mathfrak{A}}, f^{\mathfrak{A}}, c^{\mathfrak{A}}\rangle$$

given and illustrated on Figure 2.3 belongs to Alg_t.



$$c^{\mathfrak{A}} = \infty, \qquad f^{\mathfrak{A}}(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{Z} \\ x & \text{if } x = \infty, \end{cases}$$

$$g^{\mathfrak{A}}(x,y) = g^{\mathfrak{A}}(y,x) = \begin{cases} \infty & \text{if } \infty \in \{x,y\} \\ x & \text{if } x \text{ can be reached in finitely many} \\ & \qquad f^{\mathfrak{A}}\text{-steps from } y. \end{cases}$$

Figure 2.3: An algebra with 3 operations

Example 2.1.7. We define $ba \stackrel{\text{def}}{=} \{\langle \vee, 2\rangle, \langle -, 1\rangle\}$. We call ba the *similarity type of Boolean algebras*, we call the function symbols $\vee$ and $-$ *join* and *minus*, respectively. Next we define a distinguished subclass of Alg_{ba} called the class of *powerset Boolean algebras*. Recall from section 1.3 that for a set U, $\mathcal{P}(U)$ denotes the powerset of U. For an arbitrary set U, we define the operation *complementation* ${}_U-$ *relative to* U as follows. For any set X, ${}_U - (X) \stackrel{\text{def}}{=} U \smallsetminus X$. Often we omit the subscript ${}_U$ from ${}_U-$, and thus, ambiguously, we use the same (meta–)symbol $-$ for referring to both the function symbol minus (from ba) and the set theoretic operation complementation. Context will help avoiding confusion.

Now, given a nonempty set U, the powerset Boolean algebra $\mathfrak{P}(U)$ over U is defined to be the algebra with universe $\mathcal{P}(U)$ and fundamental operations $\cup$ and $-$ (interpreting $\vee$ and $-$, respectively). Formally:

$$\mathfrak{P}(U) \stackrel{\text{def}}{=} \langle \mathcal{P}(U), \cup, - \rangle\,,$$

$\mathfrak{P}(U)$ is really an algebra, because $\mathcal{P}(U)$ is closed under both $\cup$ and $-$ (i.e., for any $X, Y \in \mathcal{P}(U)$, $X \cup Y, -X \in \mathcal{P}(U)$).

When working in the similarity class Alg_{ba}, we use the following derived operations:

$$\begin{aligned}
x \wedge y &\stackrel{\text{def}}{=} -(-x \vee -y) && (\wedge \text{ is called } \textit{meet})\\
0 &\stackrel{\text{def}}{=} x \wedge -x && (\textit{zero} \text{ constant})\\
1 &\stackrel{\text{def}}{=} x \vee -x && (\textit{one} \text{ constant})\\
x - y &\stackrel{\text{def}}{=} x \wedge -y && (\textit{binary minus} \text{ operation}).
\end{aligned}$$

It is easy to check that the corresponding “set–operations” in the powerset Boolean algebra over some set U are as follows:

$$\begin{array}{llll}
\vee & \cup & 0 & \emptyset \\
- & {}_U- & 1 & U \\
\wedge & \cap & - & \setminus\,.
\end{array}$$

This checking is left to the reader.

We will often use one more derived operation, called *symmetric difference* and denoted by $\otimes$. In this case, similarly to the case of minus and complementation, we use the same symbol, moreover, the same name for these (corresponding) operations both in the “abstract case” and in the “concrete case” (i.e., powerset Boolean algebra case). This will not cause confusion, because context (and a very strong representation/axiomatizability theorem to be formulated in section 2.7, see Thm.2.7.5), will always help. The definition of symmetric difference is:

$$\begin{aligned}
x \otimes y &\stackrel{\text{def}}{=} (x - y) \vee (y - x) && \text{in the “abstract case”},\\
x \otimes y &\stackrel{\text{def}}{=} (x \setminus y) \cup (y \setminus x) && \text{in powerset Boolean algebras.}
\end{aligned}$$

We refer to the operations $\vee, -, \wedge, 0, 1, \otimes$ (and also to others derivable from these) as *Boolean operations*.

One more kind of examples for algebras are the $\cup$*–reducts* $\langle \mathcal{P}(U), \cup \rangle$ *of powerset Boolean algebras* $\mathfrak{P}(U)$. For any nonempty set U, the algebra $\langle \mathcal{P}(U), \cup \rangle$ and the poset $\langle \mathcal{P}(U), \subseteq \rangle$ are inter–definable in the following sense. In $\langle \mathcal{P}(U), \cup \rangle$, $\subseteq$ can be defined using the fundamental operation $\cup$ as $X \subseteq Y \iff X \cup Y = Y$ for every $X, Y \in \mathcal{P}(U)$. On the other hand, $\cup$ can be captured in $\langle \mathcal{P}(U), \subseteq \rangle$ as $X \cup Y = \sup^{\subseteq}(X, Y)$, where

$$\begin{aligned}
\sup{}^{\subseteq}(X, Y) = Z \stackrel{\text{def}}{\iff} & [Z \in \mathcal{P}(U), X \subseteq Z, Y \subseteq Z, \text{ and}\\
& (\forall W \in \mathcal{P}(U))([X \subseteq W \,\&\, Y \subseteq W] \Rightarrow Z \subseteq W)]\,.
\end{aligned}$$

We showed that the partial order $\langle \mathcal{P}(U), \subseteq \rangle$ is determined by the algebra $\langle \mathcal{P}(U), \cup \rangle$ and vice versa. This is why we use two ways of drawing powerset Boolean algebras:

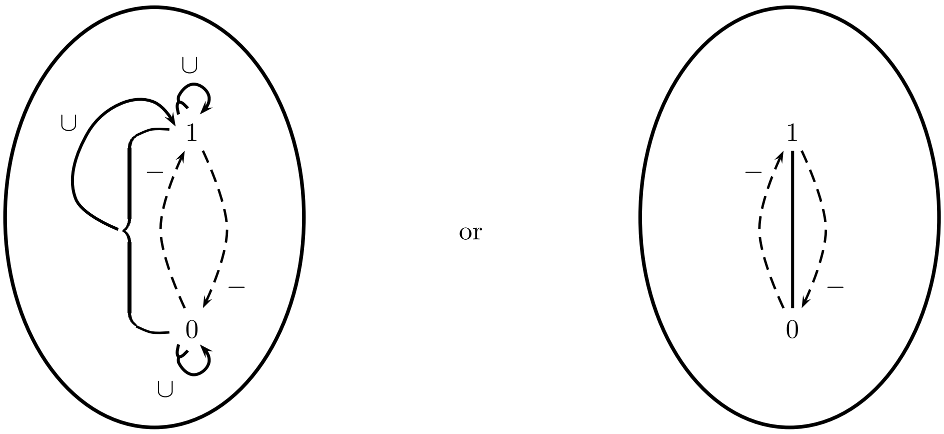


Figure 2.4: $\mathfrak{P}(1)$

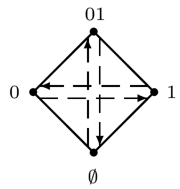


Figure 2.5: $\mathfrak{P}(2)$

one indicating $\cup$, the other indicating $\subseteq$ (the latter are poset diagrams augmented with the function $-$, cf. the beginning of section 1.3 and Figure 1.5).

On Figures 2.4, 2.5, 2.6 and 2.7 you can see the powerset Boolean algebras $\underset{\sim}{\mathbf{2}} \overset{\text{def}}{=} \mathfrak{P}(1)$, $\mathfrak{P}(2)$, $\mathfrak{P}(3)$ and $\mathfrak{P}(4)$, respectively. (In the picture of $\mathfrak{P}(4)$, we draw only the poset diagram of it, omitting complementation $-$ so that the picture be less complicated.)

Remark 2.1.8. Recall that we introduced the logical connectives conjunction and disjunction in Definition 1.8.1 (2) and (7) in section 1.8. Notice that we used the same symbol $\vee$ for denoting both join and conjunction, and we used $\wedge$ for denoting both meet and disjunction. This is not just a coincidence. To the Boolean operations one can naturally associate logical connectives. We will discuss this in Chapters 4, 5.

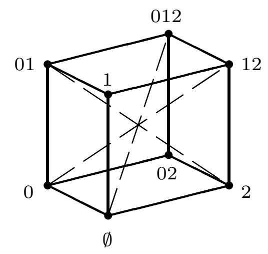


Figure 2.6: $\mathfrak{P}(3)$

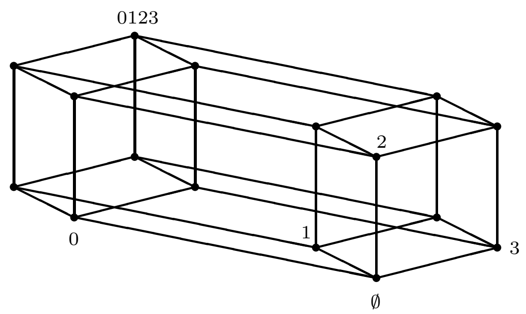


Figure 2.7: $\mathfrak{P}(4)$

2.2 Building new algebras from old ones (operations on algebras)

We want to learn new things from algebras via the old (Greek) method of *analysis and synthesis*. When using this method, one investigates something in such a way that first one takes the thing to (relatively) simple pieces; then one investigates the simple pieces instead of the original thing; finally, one puts the new information obtained about the simple pieces together, gaining this way new knowledge about the whole thing itself. This method, when applied to algebras, assumes the existence of certain ways of obtaining new algebras from already existing ones, namely, two kinds of such ways: ways for gaining simpler algebras from existing ones, and ways of building new, bigger, more complex algebras from existing ones. These ways are performed by introducing *operations on algebras*. Next we will introduce such operations. The ones we will call taking *subalgebras*, *homomorphic images*, forming *direct decompositions* and *subdirect decompositions* give us algebras smaller, simpler than their arguments, while *direct products*, *subdirect products*, *reduced products*, *ultraproducts* produce bigger, more complex, more complicated algebras from a number of simpler, smaller ones.

When decomposing or reducing algebras, it is very natural to ask the question: Can we decompose an algebra to non–decomposable, irreducible, "atomic" fragments? To answer this question, we will introduce *minimal* algebras, *simple* algebras, *directly indecomposable* algebras and *subdirectly irreducible* algebras.

2.2.1 Subalgebra

Let f be an operation of rank n on the nonempty set A, that is, $f : {}^nA \longrightarrow A$, and let $X \subseteq A$. We say that X is *closed w.r.t.* f iff

$$f(a_1, \ldots, a_n) \in X \quad \text{for every } a_1, \ldots, a_n \in X\,.$$

According to this, if f is a constant then X is closed w.r.t. f iff $f \in X$. Thus the empty set $\emptyset$ is closed w.r.t. every operation on A of positive rank, but it is not closed w.r.t. any operation of rank 0.
Example: taking A to be $\mathbb{Z}$, we see that the set of odd integers is closed w.r.t. multiplication but not w.r.t addition.

Definition 2.2.1. (subalgebra, subalgebra generated by a set etc.) Let $\mathfrak{A}$ be an algebra of similarity type t.

(i) Let $X \subseteq A$ be a nonempty set. X is called a *subuniverse* of $\mathfrak{A}$ iff X is closed w.r.t. each fundamental operation of $\mathfrak{A}$.

(ii) An algebra $\mathfrak{B}$ is called a *subalgebra* of $\mathfrak{A}$, in symbols $\mathfrak{B} \subseteq \mathfrak{A}$, iff $\mathfrak{A}$ and $\mathfrak{B}$ are similar, B is a subuniverse of $\mathfrak{A}$, and for each function symbol f of $\mathfrak{A}$, $f^{\mathfrak{B}} = f^{\mathfrak{A}}\lceil B$. We say that $\mathfrak{B}$ is a *proper subalgebra* of $\mathfrak{A}$ iff it is a subalgebra of $\mathfrak{A}$ different from $\mathfrak{A}$, that is, iff $\mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{B} \neq \mathfrak{A}$.

(iii) Let $X \subseteq A$. The *subuniverse of $\mathfrak{A}$ generated by X* is

$$\mathrm{Sg}^{\mathfrak{A}}(X) \stackrel{\mathrm{def}}{=} \bigcap\{B \ : \ X \subseteq B \text{ and } B \text{ is a subuniverse of } \mathfrak{A}\}\,.$$

The *subalgebra of $\mathfrak{A}$ generated by X* is

$$\mathfrak{Sg}^{\mathfrak{A}}(X) \stackrel{\mathrm{def}}{=} \left\langle \mathrm{Sg}^{\mathfrak{A}}(X), f^{\mathfrak{A}}\lceil \mathrm{Sg}^{\mathfrak{A}}(X) \right\rangle_{f \in Dom(t)}.$$

(iv) If K is a class of similar algebras then

$$\mathbf{S}\mathsf{K} \stackrel{\mathrm{def}}{=} \{\mathfrak{B} \ : \ \mathfrak{B} \text{ is a subalgebra of some } \mathfrak{A} \in \mathsf{K}\}\,.$$

Clearly $\mathsf{K} \subseteq \mathbf{S}\mathsf{K}$, because $\mathfrak{A}$ is a subalgebra of itself, for any algebra $\mathfrak{A}$. If $\mathsf{K} = \{\mathfrak{A}\}$ for some algebra $\mathfrak{A}$ then $\mathbf{S}\mathfrak{A} \stackrel{\mathrm{def}}{=} \mathbf{S}\{\mathfrak{A}\}$, that is, $\mathbf{S}\mathfrak{A}$ denotes the set of all subalgebras of $\mathfrak{A}$.

(v) We say that $\mathfrak{A}$ is a *minimal algebra* iff $\mathfrak{A}$ has no proper subalgebras (that is, iff $\mathfrak{B} \subseteq \mathfrak{A} \Longrightarrow \mathfrak{B} = \mathfrak{A}$ for every algebra $\mathfrak{B}$).

Notice that, on Figure 2.1, $\mathfrak{C}$ and $\mathfrak{D}$ are minimal but $\mathfrak{A}$ and $\mathfrak{B}$ are not. We note that $\mathfrak{A}$ is minimal iff $\mathfrak{A}$ is generated by any of its subsets.

Remark 2.2.2. The "act" of taking a subalgebra of an algebra $\mathfrak{A}$ can be visualized as "cutting out" a *coherent* part of A (not just any part!). If we imagine that the operations of $\mathfrak{A}$ are drawn as arrows (like on Figure 2.2.1 below), then we can say that a part of A is coherent if no arrows "go out of" it.

Definition 2.2.3. (Boolean set algebra, SetBA) By a *Boolean set algebra* we mean an element of the class

$$\mathsf{SetBA} \stackrel{\mathrm{def}}{=} \mathbf{S}\{\mathfrak{B} \ : \ \mathfrak{B} \text{ is a powerset Boolean algebra}\} = \mathbf{S}\{\mathfrak{P}(U) \ : \ U \text{ is a nonempty set}\}\,.$$

Exercise 2.2.4. How many subalgebras does $\mathfrak{P}(3)$ have? ($\mathfrak{P}(U)$ was defined in Exercise 2.1.7 (4).) Answer the same question for $\mathfrak{P}(1)$, $\mathfrak{P}(2)$, $\mathfrak{P}(4)$ instead of $\mathfrak{P}(3)$ as well.

Exercise 2.2.5. For any algebra $\mathfrak{A} \in \mathsf{Alg}_t$ and set $X \subseteq A$, we let

$$\begin{aligned} E^{\mathfrak{A}}(X) &\stackrel{\mathrm{def}}{=} X \cup \{f^{\mathfrak{A}}(a_0, \ldots, a_{t(f)-1}) \ : \ f \in Dom(t),\ a_0, \ldots, a_{t(f)-1} \in X\}\,, \\ E^{\mathfrak{A}}_0(X) &\stackrel{\mathrm{def}}{=} X\,, \\ E^{\mathfrak{A}}_{n+1}(X) &\stackrel{\mathrm{def}}{=} E^{\mathfrak{A}}\big(E^{\mathfrak{A}}_n(X)\big) \quad \text{for every } n \in \omega. \end{aligned}$$

Prove that

$$\mathrm{Sg}^{\mathfrak{A}}(X) = \bigcup\{E^{\mathfrak{A}}_n(X) \ : \ n \in \omega\}\,.$$

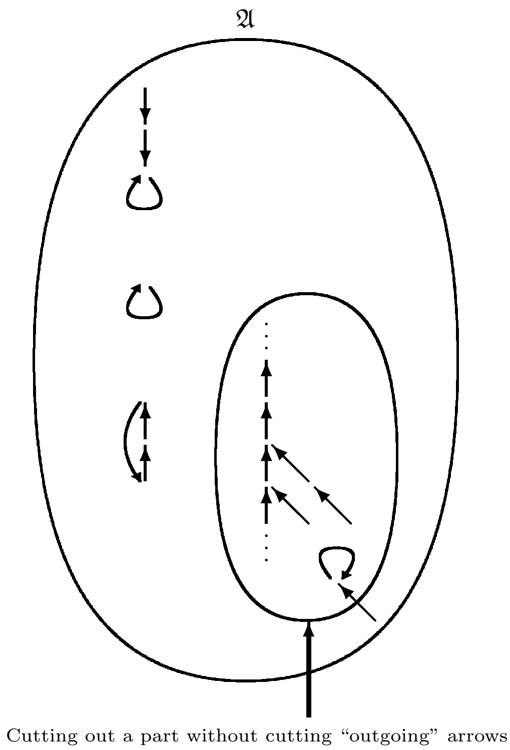


Figure 2.8: Subalgebra

Earlier we said that one possible step in the analytic investigation of algebras is taking subalgebras. It is natural to ask how refined this analysis can be. The following question is a special — and natural — one in this line. Is it true that any algebra $\mathfrak{A}$ has subalgebras of arbitrary sizes (smaller than the size of $\mathfrak{A}$)? The following proposition gives an answer to this question for the case of infinite algebras with finite similarity type.

Proposition 2.2.6. *Let* $\mathfrak{A} \in \mathsf{Alg}_t$, $|A| = \kappa \geqslant \omega$, $|Dom(t)| < \omega$. *Then*

$$\forall\beta\big(\omega \leqslant \beta < \kappa \Longrightarrow (\exists\mathfrak{B} \in \mathbf{S}\mathfrak{A})|B| = \beta\big)\,.$$

Sketch of proof. Let $\mathfrak{A}, t, \kappa$ be as in the formulation of the proposition. Let β be such that $\omega \leqslant \beta < \kappa$, and let $G \subseteq A$, $|G| = \beta$. We let $\mathfrak{B} \stackrel{\text{def}}{=} \mathrm{Sg}^{\mathfrak{A}}(G)$. Then $|B| = \beta$ can be seen as follows.

First one shows that for any set X, $|X| = \beta \Rightarrow |E^{\mathfrak{A}}(X)| = \beta$ ($E^{\mathfrak{A}}(X)$ was defined in Exercise 2.2.5). Here we use the condition $|Dom(t)| < \omega$. From this $(\forall n \in \omega)|E^{\mathfrak{A}}_n(G)| = \beta$ follows. Now, using Exercise 2.2.5,

$$|\mathrm{Sg}^{\mathfrak{A}}(G)| = |\cup\left\{E^{\mathfrak{A}}_n(G) : n \in \omega\right\}| = \cup\{\beta : n \in \omega\} = \beta\,. \qquad \square$$

If $\mathfrak{A}$ is a finite algebra then, in general, it is not true that for any $k < |A|$ there would exist a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ with $|B| = k$. For example, the powerset Boolean algebra $\mathfrak{P}(2)$ has no subalgebra $\mathfrak{B}$ with $|B| = 3$ (cf. Exercise 2.2.4).

2.2.2 Homomorphic image

Definition 2.2.7. (homomorphism) Let t be an arbitrary similarity type, let $\mathfrak{A}, \mathfrak{B} \in \mathsf{Alg}_t$, and let $h : A \longrightarrow B$ be an arbitrary function.

(i) Let $f \in Dom(t)$ (that is, f is a function symbol of t). The function h is said to *respect* f iff

$$h\left(f^{\mathfrak{A}}(a_1, \ldots, a_{t(f)})\right) = f^{\mathfrak{B}}\left(h(a_1), \ldots, h(a_{t(f)})\right)$$

for every $a_1, \ldots, a_{t(f)} \in A$.

(ii) The function h is called a *homomorphism* from $\mathfrak{A}$ into $\mathfrak{B}$ iff h respects every function symbol $f \in Dom(t)$.

(iii) $Hom(\mathfrak{A}, \mathfrak{B})$ denotes the set of all homomorphisms from $\mathfrak{A}$ into $\mathfrak{B}$.

Exercise 2.2.8. Prove that for any set X, if $\mathfrak{A} = \mathfrak{Sg}^{\mathfrak{A}}(X)$, $h, k \in Hom(\mathfrak{A}, \mathfrak{B})$ for some algebra $\mathfrak{B}$, and h, k are such that $h\lceil X = k\lceil X$, then $h = k$.

We distinguish several kinds of homomorphisms and use the following notation for them. Each of

$$h : \mathfrak{A} \longrightarrow \mathfrak{B}, \qquad \mathfrak{A} \stackrel{h}{\longrightarrow} \mathfrak{B}, \qquad h \in Hom(\mathfrak{A}, \mathfrak{B})$$

denotes that h is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. Both

$$h : \mathfrak{A} \rightarrowtail \mathfrak{B} \qquad \text{and} \qquad \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{B}$$

denote that h is a one–one homomorphism (i.e., an injection) from $\mathfrak{A}$ into $\mathfrak{B}$. We call such homomorphisms *embeddings*. If $\mathfrak{A}$ and $\mathfrak{B}$ are such that $\mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{B}$ for some embedding h then we say that $\mathfrak{A}$ is *embeddable* into $\mathfrak{B}$.

Similarly, both

$$h : \mathfrak{A} \twoheadrightarrow \mathfrak{B} \qquad \text{and} \qquad \mathfrak{A} \overset{h}{\twoheadrightarrow} \mathfrak{B}$$

denote that h is a homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ (i.e., a surjection), and in this case we say that $\mathfrak{B}$ is the *homomorphic image* of $\mathfrak{A}$ under h, in symbols, $\mathfrak{B} = h(\mathfrak{A})$. Further, each of

$$h : \mathfrak{A} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{B}, \qquad \mathfrak{A} \overset{h}{\rightarrowtail\!\!\!\!\twoheadrightarrow} \mathfrak{B}, \qquad \mathfrak{A} \overset{h}{\cong} \mathfrak{B}$$

denotes that h is a one-one homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. We call such a homomorphism an *isomorphism*. $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *isomorphic*, which we denote by $\mathfrak{A} \cong \mathfrak{B}$, iff there is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. A homomorphism from $\mathfrak{A}$ into $\mathfrak{A}$ is called an *endomorphism* of $\mathfrak{A}$, and an isomorphism from $\mathfrak{A}$ onto $\mathfrak{A}$ is called an *automorphism* of $\mathfrak{A}$.

Remark 2.2.9. Taking a homomorphic image of an algebra is a new example for an operation on algebras. This operation produces a smaller, simpler algebra from its argument (cf. the introductory paragraphs of the present section 2.2).

Taking a homomorphic image of an algebra $\mathfrak{A}$ involves *identifying* certain elements of $\mathfrak{A}$. In the homomorphic image, we cannot "see" certain details which were still clearly "visible" in $\mathfrak{A}$. See Figure 2.2.2.

If h is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$, then a simplified image of $\mathfrak{A}$ appears in $\mathfrak{B}$. Putting this another way, magnified versions of certain parts of $\mathfrak{B}$ can be found in $\mathfrak{A}$ (moving backwards via h). See Figure 2.2.2.

Definition 2.2.10. $(\mathbf{I}, \mathbf{H})$ For a class K of similar algebras,

$$\mathbf{I}\mathsf{K} \overset{\text{def}}{=} \{\mathfrak{A} \ : \ \mathfrak{A} \text{ is isomorphic to some } \mathfrak{B} \in \mathsf{K}\}, \qquad \text{and}$$
$$\mathbf{H}\mathsf{K} \overset{\text{def}}{=} \{\mathfrak{A} \ : \ \mathfrak{A} \text{ is a homomorphic image of some } \mathfrak{B} \in \mathsf{K}\}\,.$$

Similarly to our convention in case of $\mathbf{S}$, we use the notation

$$\mathbf{H}\mathfrak{A} \overset{\text{def}}{=} \mathbf{H}\{\mathfrak{A}\}, \qquad \mathbf{I}\mathfrak{A} \overset{\text{def}}{=} \mathbf{I}\{\mathfrak{A}\}\,.$$

Principle of identifying isomorphic objects: According to our definition above, an isomorphism is a one–one correspondence between the elements of two algebras that respects the interpretation of each function symbol. Therefore, with regard

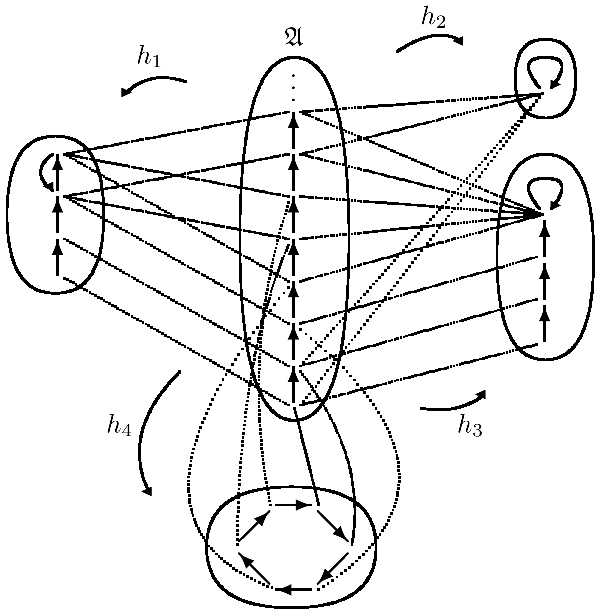


Figure 2.9: Four homomorphic images of $\mathfrak{A}$

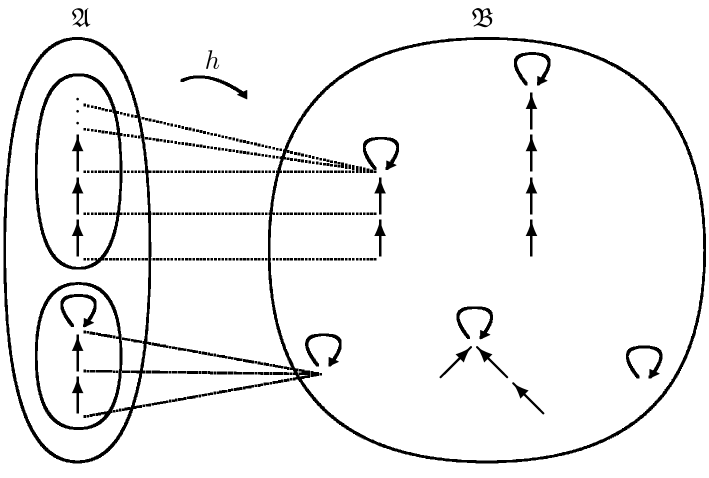


Figure 2.10: $\mathfrak{A}$ is a magnified version of a part of $\mathfrak{B}$

to a host of properties, isomorphic algebras are indistinguishable from each other. This applies to most of the properties with which we shall deal; if they are true in a given algebra, then they are true for all isomorphic images or isomorphic "copies" of that algebra as well. Such properties are called *algebraic* or *abstract* properties.

If φ is a property of algebras and we say that "property φ holds for algebra $\mathfrak{A}$ *up to isomorphism*" then we mean that φ holds for any isomorphic image of $\mathfrak{A}$. For example, $\mathfrak{P}(3)$ has 5 subalgebras but it has only 3 subalgebras up to isomorphism (because some subalgebras of $\mathfrak{P}(3)$ are isomorphic).

Definition 2.2.11. (Boolean algebra, BA) Recall the definition of Boolean set algebras from Definition 2.2.3. The class BA of all *Boolean algebras* is defined as:

$$\mathsf{BA} \stackrel{\text{def}}{=} \mathbf{I}\mathsf{SetBA}\,.$$

2.2.3 A distinguished example: Lattices

To be written later.

2.2.4 Congruence relation

As we have seen, unlike the formation of subalgebras, the formation of homomorphic images involved external considerations. Despite of this, we will show that all the homomorphic images of an algebra can still be captured completely internally, too. To do this, we introduce the concepts of a *congruence relation* and that of a *quotient algebra*.

Definition 2.2.12. (congruence relation, quotient algebra)

(i) Let A be a set and f an n–ary operation on A, that is, $f : {}^nA \longrightarrow A$. Let $\theta \subseteq A \times A$ be an equivalence relation. We say that θ has the *substitution property* w.r.t. f iff for every $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, the following implication holds.

$$\big(\forall i(0 < i \leqslant n \Rightarrow a_i \;\theta\; b_i)\big) \Longrightarrow f(a_1, \ldots, a_n)\;\theta\; f(b_1, \ldots, b_n)\,.$$

(ii) Let $\mathfrak{A}$ be an algebra. A binary relation $\theta \subseteq A \times A$ is called a *congruence relation* (or briefly, *congruence*) *on* $\mathfrak{A}$, iff it is an equivalence relation on A and θ has the substitution property w.r.t. every fundamental operation of $\mathfrak{A}$. $Con(\mathfrak{A})$ denotes the set of all congruence relations on $\mathfrak{A}$.

(iii) If $\mathfrak{A}$ is an algebra then to every $\theta \in Con(\mathfrak{A})$ we associate a new algebra $\mathfrak{A}/\theta$ similar to $\mathfrak{A}$, as follows. The universe of $\mathfrak{A}/\theta$ is the partition A/θ determined by θ (see the definition of a partition after Exercise 1.2.2 in subsection 1.2). (Thus an element of the universe of $\mathfrak{A}/\theta$ is an equivalence class a/θ for some

$a \in A$.) For any function symbol f of $\mathfrak{A}$ of rank n, the fundamental operation $f^{\mathfrak{A}/\theta}$ of $\mathfrak{A}/\theta$ is defined as follows. For any $a_1, \ldots, a_n \in A$,

$$f^{\mathfrak{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = f^{\mathfrak{A}}(a_1, \ldots, a_n)/\theta\,.$$

The algebra $\mathfrak{A}/\theta$ is called the *quotient algebra of* $\mathfrak{A}$ *w.r.t.* θ.

Exercises 2.2.13. Prove that the above definition of a quotient algebra is sound, that is, the definition of its functions does not depend on the choice of the representing elements of the equivalence classes.

Notice that to every algebra $\mathfrak{A}$, $Con(\mathfrak{A})$ has a minimal element and a maximal element w.r.t. inclusion $\subseteq$ as ordering; these are Id_A and $A \times A$. Also notice that $\langle Con(\mathfrak{A}), \subseteq\rangle$ forms a lattice. Therefore, sometimes, we refer to $Con(\mathfrak{A})$ as the *congruence lattice* of $\mathfrak{A}$.

Definition 2.2.14. (kernel) Let $h \in Hom(\mathfrak{A}, \mathfrak{B})$. Then

$$ker(h) \stackrel{\text{def}}{=} \{\langle a, b\rangle \in {}^2A \ : \ h(a) = h(b)\}$$

is called the *kernel* of h.

Exercises 2.2.15. (kernels and homomorphisms) Let $\mathfrak{A}$ be an arbitrary algebra.

(1) Prove that for any homomorphism h of $\mathfrak{A}$, $ker(h) \in Con(\mathfrak{A})$.

(2) Let θ be an equivalence relation on A. We define the function $q : A \longrightarrow A/\theta$ as follows:

$$(\forall a \in A) q(a) \stackrel{\text{def}}{=} a/\theta \quad \text{and} \quad ker(q) = \theta\,.$$

Prove that if $\theta \in Con(\mathfrak{A})$ then $q \in Hom(\mathfrak{A}, \mathfrak{A}/\theta)$.

(3) Let θ be an equivalence relation on A. Prove that $\theta \in Con(\mathfrak{A})$ iff for each function symbol f of $\mathfrak{A}$ of rank n, $\{\langle a_1/\theta, \ldots, a_n/\theta, f^{\mathfrak{A}}(a_1, \ldots, a_n)/\theta\rangle \ : \ a_1, \ldots, a_n \in A\}$ is an operation (an n–ary function) on A/θ.

The homomorphism q defined in Exercise 2.2.15 (2) is called the *quotient map* from $\mathfrak{A}$ onto $\mathfrak{A}/\theta$.

Corollary 2.2.16. $Con(\mathfrak{A}) = \{ker(h) \ : \ h$ *is a homomorphism with* $Dom(h) = A\}$. □

Exercise 2.2.17. Prove that if $\mathfrak{A}$ is an algebra and $\emptyset \neq C \subseteq Con(\mathfrak{A})$ then $\bigcap C \in Con(\mathfrak{A})$.

Theorem 2.2.18. (Homomorphism Theorem) *Let* $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, $\theta = ker(h)$, *and let* $q : \mathfrak{A} \longrightarrow \mathfrak{A}/\theta$ *be the quotient map. Then the unique function* $i : A/\theta \twoheadrightarrow B$ *satisfying* $q \circ i = h$ *is an isomorphism from* $\mathfrak{A}/\theta$ *onto* $\mathfrak{B}$.

Proof. The proof is illustrated on Figure 2.2.4.

We define:

$$i(a/\theta) \stackrel{\text{def}}{=} h(a) \text{ for every } a \in A.$$

- This definition is sound: Suppose $a\theta b$. Then $h(a) = h(b)$ since $\theta = ker(h)$. Thus $i(a/\theta) = h(a) = h(b) = i(b/\theta)$.
- i is one–one:

$$\begin{aligned} i(a/\theta) = i(b/\theta) &\Longrightarrow h(a) = h(b) \\ &\Longrightarrow a/\theta = b/\theta \qquad \text{by } ker(h) = \theta. \end{aligned}$$

- i is onto because h is onto, and $q \circ i = h$.
- i is a homomorphism:

  If f is an n–ary function symbol and $a_1, \dots, a_n \in A$ then

$$\begin{aligned} i\left(f^{\mathfrak{A}/\theta}(a_1/\theta, \dots, a_n/\theta)\right) &= i\left(f^{\mathfrak{A}/\theta}(q(a_1), \dots, q(a_n))\right) && \text{by def. of } q \\ &= i\left(q\left(f^{\mathfrak{A}}(a_1, \dots, a_n)\right)\right) && q \text{ is a homom.} \\ &= h\left(f^{\mathfrak{A}}(a_1, \dots, a_n)\right) && \text{by } q \circ i = h \\ &= f^{\mathfrak{B}}\left(h(a_1), \dots, h(a_n)\right) && h \text{ is a homom.} \\ &= f^{\mathfrak{B}}\left(i(q(a_1)), \dots, i(q(a_n))\right) && \text{by } q \circ i = h \\ &= f^{\mathfrak{B}}\left(i(a_1/\theta), \dots, i(a_n/\theta)\right) && \text{by def. of } q\,. \end{aligned}$$

□

Definition 2.2.19. (simple algebras) An algebra $\mathfrak{A}$ is called *simple* iff $\mathfrak{A}$ is non–trivial and has no nontrivial homomorphic images. More formally: $\mathfrak{A}$ is simple iff

$$|A| > 1 \text{ and } \forall \mathfrak{B}(\forall h \in Hom(\mathfrak{A}, \mathfrak{B}))(h \text{ is one-one} \quad \text{or} \quad |Rng(h)| = 1)\,.$$

Smp denotes the class of all simple algebras. If K is a class of algebras then

$$\mathsf{SmpK} = \mathsf{Smp}(\mathsf{K}) \stackrel{\text{def}}{=} \mathsf{Smp} \cap \mathsf{K}\,.$$

That is, $\mathsf{Smp}(\mathsf{K})$ is the class of all simple algebras of K.

Notice that $\mathfrak{A} \in \mathsf{Smp} \Longleftrightarrow |Con(\mathfrak{A})| = 2$.

Exercises 2.2.20. (concerning simple algebras)

1. Every two–element algebra is simple. Thus so is $\underset{\sim}{\mathbf{2}}(= \mathfrak{P}(1))$, cf. Figure 2.4 in section 2.1.
2. $\mathfrak{A}$ on Figure 2.2 is simple.
3. Let $\mathfrak{A} = \langle \omega, \mathsf{pred} \rangle$, where $(\forall n > 0)\mathsf{pred}(n) = n - 1$ and $\mathsf{pred}(0) = 0$, see Figure 2.12.

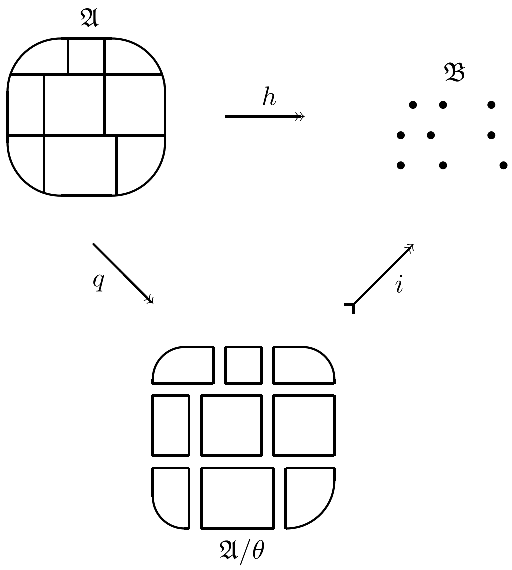


Figure 2.11: Proof of the Homomorphism Theorem

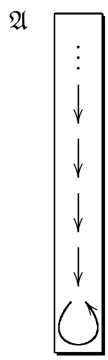


Figure 2.12: "Predecessor algebra"

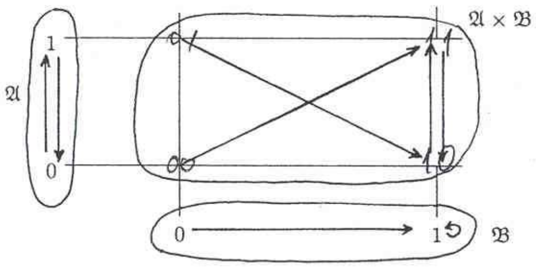


Figure 2.13: Cartesian product of two algebras

Then we have that

$$\forall \mathfrak{B} \left(\forall (\mathfrak{A} \overset{h}{\twoheadrightarrow} \mathfrak{B}) \right) (\mathfrak{A} \cong \mathfrak{B} \quad \text{or} \quad |Rng(h)| = 1)\,. \tag{2.1}$$

But $\mathfrak{A}$ is not simple. Algebras satisfying condition (2.1) are called *pseudo–simple* algebras.

2.2.5 Cartesian product, direct decomposition

Recall the concept of a Cartesian (or direct) product of a system of sets from section 1.5. Now we introduce the direct product of a system of algebras. Before giving the general definition, we describe some special cases.

Let $\mathfrak{A} = \langle A, f^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle B, f^{\mathfrak{B}} \rangle$ be two algebras with f unary (i.e., $f^{\mathfrak{A}} : A \to A$, $f^{\mathfrak{B}} : B \to B$). Their *Cartesian* (or *direct*) *product* $\mathfrak{A} \times \mathfrak{B}$ has universe $A \times B$ and is defined as follows. $\mathfrak{A} \times \mathfrak{B} = \langle A \times B, f \rangle$, where $f(\langle a, b \rangle) = \langle f^{\mathfrak{A}}(a), f^{\mathfrak{B}}(b) \rangle$ for all $\langle a, b \rangle \in A \times B$. See Figure 2.2.5.

Firkás, csúnya ábra!

A straightforward generalization of a product of two algebras is a product of an arbitrary sequence (or family) $\langle \mathfrak{A}_i \ : \ i \in I \rangle$ of algebras, as follows. Let $\mathfrak{A}_i = \langle A_i, f^i \rangle$ for $i \in I$. Then $\Pi_{i \in I} \mathfrak{A}_i = \langle \Pi_{i \in I} A_i, f \rangle$, where $f(\langle a_i \ : \ i \in I \rangle) = \langle f^i(a_i) \ : \ i \in I \rangle$.

The definition for n-ary operations $f : {}^n A \to A$ is completely analogous. So is the definition to arbitrary algebras, as follows.

Definition 2.2.21. (direct [Cartesian] product) Let I be a set, t an arbitrary similarity type and $\{\mathfrak{A}_i \ : \ i \in I\} \subseteq \mathsf{Alg}_t$. The *direct* (or *Cartesian*) *product* of the system $\mathfrak{A} = \langle \mathfrak{A}_i \ : \ i \in I \rangle$ of similar algebras, denoted by $\Pi \langle \mathfrak{A}_i \ : \ i \in I \rangle$ or $\Pi \mathfrak{A}$, is defined to be the algebra of type t with universe $\Pi_{i \in I} A_i$ and fundamental operations defined as follows. If $t(f) = n$ and $a_1, \ldots, a_n \in \Pi_{i \in I} A_i$ then

$$\left(f^{\Pi \mathfrak{A}}(a_1, \ldots, a_n) \right)_i = f^{\mathfrak{A}_i}(a_1(i), \ldots, a_n(i))$$

$$\begin{tikzcd}
 & \mathfrak{B} \arrow[rr, dashed, "k"] \arrow[d, "h_i"'] \arrow[drr, "h_j"] & & \prod_{i\in I}\mathfrak{A}_i \arrow[dll, "p_i"'] \arrow[d, "p_j"] & \\
\cdots & \mathfrak{A}_i & \cdots & \mathfrak{A}_j & \cdots
\end{tikzcd}$$

Figure 2.14: Universal property of direct product

for all $i \in I$.

$\Pi\langle\mathfrak{A}_i \ : \ i \in I\rangle$ is also denoted as $\Pi_{i\in I}\mathfrak{A}_i$ or $\underset{I}{\Pi}\mathfrak{A}$. If, for some algebra $\mathfrak{B}$, $\mathfrak{A}_i = \mathfrak{B}$ for all $i \in I$ then we write ${}^I\mathfrak{B}$ for the direct product, and call it a *direct power* of $\mathfrak{B}$.

Exercise 2.2.22. Prove that if $I = \emptyset$ then $\Pi_{i\in I}\mathfrak{A}_i$ is a trivial algebra.

Lemma 2.2.23. (on projections) *We use the notation introduced in Def.2.2.21 above and the notion of a projection function p_i introduced in section 1.5.*

(i) $(\forall i \in I)(p_i$ *is a homomorphism*$)$.

(ii) $(\forall i \in I)(p_i$ *is onto* $A_i)$.

(iii) $(\forall a, b \in A, a \neq b)(\exists i \in I)p_i(a) \neq p_i(b)$.

The *proof* is left to the reader. $\square$

If I is a set, $\mathfrak{A}_i$ $(i \in I)$ and $\mathfrak{B}$ are similar algebras and $h_i : \mathfrak{B} \longrightarrow \mathfrak{A}_i$ are homomorphisms for every $i \in I$, then we call the system $\langle h_i : i \in I\rangle$ of homomorphisms a *cone*. We also use the notation $\langle h_i\rangle_{i\in I}$ as well as $\langle\mathfrak{B} \xrightarrow{h_i} \mathfrak{A}_i\rangle_{i\in I}$ for denoting this cone. The cone $\langle\Pi_{i\in I}\mathfrak{A}_i \xrightarrow{p_i} \mathfrak{A}_i\rangle_{i\in I}$ of projection homomorphisms is called the *projection cone of the product* $\Pi_{i\in I}\mathfrak{A}_i$.

Lemma 2.2.24. ("universal property" of direct product) *Let* $\langle\mathfrak{A}_i \ : \ i \in I\rangle$ *be an indexed system of similar algebras, and let* $\langle\mathfrak{B} \xrightarrow{h_i} \mathfrak{A}_i\rangle_{i\in I}$ *be a cone, for some algebra* $\mathfrak{B}$ *and homomorphisms* $\mathfrak{B} \xrightarrow{h_i} \mathfrak{A}_i$ $(i \in I)$. *Then there exists a unique homomorphism* $k : \mathfrak{B} \longrightarrow \Pi_{i\in I}\mathfrak{A}_i$ *such that for every* $i \in I$, $k \circ p_i = h_i$, *where* p_i *denotes the i–th projection function. Concisely, our lemma states:*

$$\left(\forall\langle\mathfrak{B} \xrightarrow{h_i} \mathfrak{A}_i\rangle_{i\in I}\right)(\exists! k : \mathfrak{B} \longrightarrow \Pi_{i\in I}\mathfrak{A}_i)(\forall i \in I)k \circ p_i = h_i\,.$$

See Figure 2.2.5.

The *proof* is left to the reader. $\square$

When using Lemma 2.2.24, sometimes we say that the cone $\langle\mathfrak{B} \xrightarrow{h_i} \mathfrak{A}_i\rangle_{i\in I}$ *induces* the homomorphism k. Borrowing the expression from category theory, we call the property described by Lemma 2.2.24 the *universal property of direct products*.

Lemma 2.2.25. (commutativity and associativity of direct product)

(i) $\mathfrak{A}_1 \times \mathfrak{A}_2 \cong \mathfrak{A}_2 \times \mathfrak{A}_1$
(that is, direct product is commutative up to isomorphism).

(ii) $\mathfrak{A}_1 \times (\mathfrak{A}_2 \times \mathfrak{A}_3) \cong (\mathfrak{A}_1 \times \mathfrak{A}_2) \times \mathfrak{A}_3$
(that is, direct product is associative up to isomorphism).

The *proof* is left to the reader. $\square$

Definition 2.2.26. (**P**) Given a class K of similar algebras, we let

$$\mathbf{P}\mathsf{K} \stackrel{\text{def}}{=} \mathbf{I}\{\Pi_{i\in I}\mathfrak{A}_i \ : \ I \text{ is a set, and } (\forall i \in I)\mathfrak{A}_i \in \mathsf{K}\}\,.$$

We often write $\mathbf{P}\mathfrak{A}$ instead of $\mathbf{P}\{\mathfrak{A}\}$.

Ez lehet, hogy majd valtozik, lop alfejezet miatt!!:

Exercises 2.2.27. (1) Prove that $\mathbf{SS}\mathsf{K} = \mathbf{S}\mathsf{K}$.

(2) Prove that $\mathbf{PP}\mathsf{K} = \mathbf{P}\mathsf{K}$.

(3) Prove that $\mathbf{SPSP}\mathsf{K} = \mathbf{SP}\mathsf{K}$.

(4) Prove that there is no family $\langle \mathfrak{A}_i \ : \ i \in I\rangle$ of non–trivial algebras with $|I| \geqslant \omega$ such that $|\Pi_{i\in I}\mathfrak{A}_i| = \omega$.

(5) Let $\mathsf{K} = \{\mathfrak{A}\}$, where $|A| = 2$. Prove that $\mathbf{P}\mathsf{K}$ does not contain any countably infinite algebra.

(6) Let K be as in (5) above. Prove that $\mathbf{SP}\mathsf{K}$ does contain countably infinite algebras.

(4), (5), and (6) above indicate that the operator **SP** is more flexible than **P**! Hence often **SP** is more useful than **P**. Actually, **SP** is one of the most useful operators.

Definition 2.2.28. (directly indecomposable algebras) We say that $\mathfrak{A}$ is *directly indecomposable* iff $\mathfrak{A}$ is not isomorphic to the direct product of two non–trivial algebras.

Clearly, the simple algebras and the algebras of prime cardinality are directly indecomposable.

Theorem 2.2.29. (direct decomposition of finite algebras) *Every finite algebra is isomorphic to the direct product of some directly indecomposable algebras.*

Proof. By induction. Let $\mathfrak{A}$ be finite.
- If $|A| = 1$, then $\mathfrak{A}$ is directly indecomposable.

- Assume $\mathfrak{A}$ is non–trivial. Our *induction hypothesis* is:

$$\forall \mathfrak{B}(|B| < |A| \Longrightarrow \mathfrak{B} \text{ is isomorphic to the direct product of some directly indecomposable algebras}).$$

Case 1: $\mathfrak{A}$ is directly indecomposable. Then we are done.

Case 2: $(\exists \mathfrak{A}_1, \mathfrak{A}_2)(\mathfrak{A} \cong \mathfrak{A}_1 \times \mathfrak{A}_2$ and $1 < |A_1|, 1 < |A_2|)$. Then $|A_1| < |A| > |A_2|$. Then, by the induction hypothesis,

$$\mathfrak{A}_1 \cong \mathfrak{B}_1 \times \cdots \times \mathfrak{B}_m \qquad \text{and} \qquad \mathfrak{A}_2 \cong \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_k\,,$$

for some directly indecomposable $\mathfrak{B}_i$'s and $\mathfrak{C}_i$'s. Thus, using Lemma 2.2.25 (ii) above,

$$\mathfrak{A} \cong \mathfrak{B}_1 \times \cdots \times \mathfrak{B}_m \times \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_k\,.$$

$\square$

Exercise 2.2.30. Give such examples for directly indecomposable algebras which are not mentioned above (that is, they should be non–simple and have non–prime cardinalities).

2.2.6 Subdirect decomposition

Motivation 2.2.31. As we mentioned before starting to introduce operations on algebras, to study a complex system[1] $\mathfrak{A}$, a standard approach is to simplify $\mathfrak{A}$ first, then study the simplified system, and then try to use the so obtained information for a better understanding of the original system $\mathfrak{A}$ itself. One of our tools for simplifying an algebra $\mathfrak{A}$ is taking homomorphic images of it. If $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ is surjective then we can consider $\mathfrak{B}$ as a *simplified version* of $\mathfrak{A}$, and h a "simplification". Such an h is a *real* (or non–trivial) simplification if $h(a) = h(b)$ for some $a, b \in A$, $a \neq b$. If we have only one real simplification $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, then forgetting $\mathfrak{A}$ will cause loss of information, since from $\mathfrak{B}$ alone we cannot reconstruct $\mathfrak{A}$. The reason for this is that the information which was (deliberately) thrown away when a and b were collapsed, cannot be restored from $\mathfrak{B}$ itself. E.g., on Figure 2.2.31, if we look at $\mathfrak{B}$ only (say $\mathfrak{A}$ was forgotten), we will never find out whether "$a \overset{\longleftarrow}{\longrightarrow} b$" or "$\overset{\cap}{a} \qquad \overset{\cap}{b}$" or perhaps "$a \longrightarrow \overset{\cap}{b}$" was the original pattern in $\mathfrak{A}$. However, if we have *another simplification*, say $h_1 : \mathfrak{A} \twoheadrightarrow \mathfrak{B}_1$ such that $h_1(a) \neq h_1(b)$, then the two simplifications h, h_1 *together* might *both* simplify the original picture $\mathfrak{A}$ *and* at the same time, retain all information about $\mathfrak{A}$. Such is the situation, e.g., on Figure 2.2.5, where the algebra $\mathfrak{A} \times \mathfrak{B}$ is simplified into two simple components $\mathfrak{A}$ and $\mathfrak{B}$ without loss of information (the simplifying homomorphisms are the two projections).

[1]Here we use the word "system" in an intuitive way. Our motivation for calling an algebra $\mathfrak{A}$ a system in this motivational text comes from applications of algebra to the theory of (complex) systems.

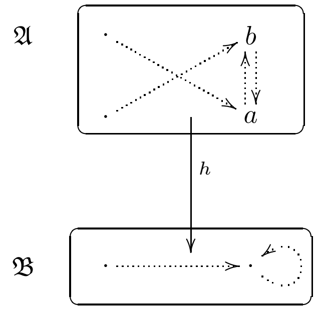


Figure 2.15: Loss of information at taking a homomorphic image

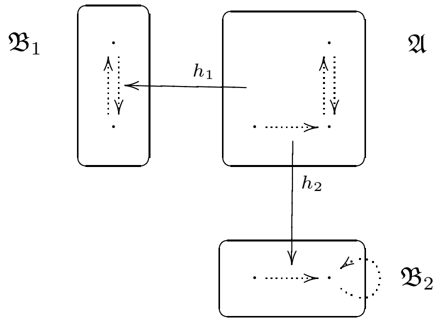


Figure 2.16: Subdirect decomposition

What is the criterion for not loosing information? Well, we need a sequence $h_i : \mathfrak{A} \twoheadrightarrow \mathfrak{B}_i$ $(i \in I)$ of surjective homomorphisms such that

$$(\forall a, b \in A)[a \neq b \Rightarrow (\exists i \in I) h_i(a) \neq h_i(b)]\,. \tag{2.2}$$

This is the case, e.g., when $\mathfrak{A}$ is the direct product of $\langle \mathfrak{B}_i \ : \ i \in I \rangle$ and h_i $(i \in I)$ are the projection functions. Another example is on Figure 2.2.31. (Here $\mathfrak{A}$ is not the direct product of $\mathfrak{B}_1$ and $\mathfrak{B}_2$.) A sequence of homomorphisms satisfying (2.2) is a simplification which certainly does not loose information. In order to make this a *non–trivial* (or real) simplification, we require that $(\forall i \in I)(h_i$ is not injective and $|Rng(h_i)| > 1)$.

Actually, a simplification $\langle h_i \ : \ i \in I \rangle$ of $\mathfrak{A}$ as described above can be considered as a *decomposition* of the complex system $\mathfrak{A}$ into simpler systems $\mathfrak{B}_i$, $i \in I$. For certain reasons, we call such decompositions *subdirect decompositions*[2] . The general schema of subdirect decompositions is like this:

[2]The name "subdirect" comes from a strong connection with the operator **SP**, where **S** is connected with the part "sub" and **P** with the part "direct" of the word.

$$
\begin{tikzcd}
 & & \mathfrak{A} \arrow[dl, "h_i"'] \arrow[dr, "h_j"] & & \\
\cdots & \mathfrak{B}_i & \cdots & \mathfrak{B}_j & \cdots
\end{tikzcd}
$$

Figure 2.17: Subdirect decomposition cone

Definition 2.2.32. (subdirect decomposition) Let $\mathfrak{A}$ be an algebra.

(i) By a *subdirect decomposition* of $\mathfrak{A}$ we understand a system $\langle h_i : i \in I\rangle$ of surjective homomorphisms $h_i : \mathfrak{A} \twoheadrightarrow \mathfrak{B}_i$ such that

$$(\forall a, b \in A, a \neq b)(\exists i \in I) h_i(a) \neq h_i(b)\,.$$

(ii) A subdirect decomposition is called *non–trivial* if

$$(\forall i \in I)(h_i \text{ is not an isomorphism})\,.$$

We call it *trivial* otherwise.

(iii) If $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i\rangle_{i\in I}$ is a subdirect decomposition of $\mathfrak{A}$, then we call $\mathfrak{B}_i$ the i–th component (of the decomposition).

Exercises 2.2.33. (subdirect decomposition)

(1) Give examples of non–trivial subdirect decompositions. Choose first $\mathfrak{A}$ to have three elements and, say, all operations "trivial".

(2) Prove that if $|A| = 2$ then $\mathfrak{A}$ has no non–trivial subdirect decomposition.

(3) Prove that if in $\bar{h} = \langle h_i : i \in I\rangle$ we have $|I| = 1$, then $\bar{h}$ cannot be a non–trivial subdirect decomposition.

(4) Give two different (non–isomorphic) subdirect decompositions of some algebra $\mathfrak{A}$.

Exercises 2.2.34. (subdirect decomposition and direct decomposition)

(1) Assume $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i\rangle_{i<2}$ is a subdirect decomposition of $\mathfrak{A}$. Prove that $\mathfrak{A}$ is embeddable into $\mathfrak{B}_0 \times \mathfrak{B}_1$.

Hint: First check that the statement is true with $\mathfrak{A}$ a small finite algebra of 3 or 4 elements.

(2) Let $\langle \mathfrak{B}_i : i \in I\rangle$ be a system of similar algebras. Recall the projection functions p_i from section 1.5. Prove that $\langle \Pi_{j\in I}\mathfrak{B}_j \xrightarrow{p_i} \mathfrak{B}_i : i \in I\rangle$ is a subdirect decomposition of $\Pi_{j\in I}\mathfrak{B}_j$.

$$\begin{tikzcd}
 & \mathfrak{A} \arrow[rr, dashed, tail, "f"] \arrow[d, two heads, "h_i"'] \arrow[drr, two heads, "h_j"] & & \prod_{i\in I}\mathfrak{B}_i \arrow[dll, "p_i"'] \arrow[d, "p_j"] & \\
\cdots & \mathfrak{B}_i & \cdots & \mathfrak{B}_j & \cdots
\end{tikzcd}$$

Figure 2.18: Diagram for Exercise 2.2.34 (3)

(3) Let $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i\rangle_{i\in I}$ be a subdirect decomposition. Prove that there is an embedding $\mathfrak{A} \overset{f}{\rightarrowtail} \Pi_{i\in I}\mathfrak{B}_i$ such that $(\forall i \in I) f \circ p_i = h_i$, that is, the diagram on Figure 2.2.34 commutes.

(4) Let $\mathfrak{A} \subseteq \Pi_{i\in I}\mathfrak{B}_i$ be an arbitrary subalgebra of an arbitrary direct product. Recall that $p_i \lceil A$ denotes the function p_i restricted to A. Prove that

$$\langle p_i\lceil A \ : \ i \in I\rangle \text{ is a subdirect decomposition of } \mathfrak{A}\,.$$

We will see that subdirect decompositions indeed do not involve loss of information in the following sense. If we are given a subdirect decomposition of an algebra $\mathfrak{A}$ into components $\mathfrak{B}_i$ $(i \in I)$, then $\mathfrak{A}$ can be "built up" from the components $\mathfrak{B}_i$ $(i \in I)$ by the operator $\mathbf{SP}$. That is, $\mathfrak{A} \in \mathbf{SP}\{\mathfrak{B}_i \ : \ i \in I\}$. (Cf. Exercise 2.2.27 (6).)

Theorem 2.2.35. *Let* $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i\rangle_{i\in I}$ *be a subdirect decomposition of the algebra* $\mathfrak{A}$. *Then (i)–(ii) below hold.*

(i) $\mathfrak{A} \in \mathbf{SP}\{\mathfrak{B}_i \ : \ i \in I\}$.

(ii) $f : \mathfrak{A} \rightarrowtail \Pi_{i\in I}\mathfrak{B}_i$, *for some embedding* f *of* $\mathfrak{A}$.

Proof. (i) clearly follows from (ii), thus it is enough to prove (ii).

Let $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i\rangle_{i\in I}$ be a subdirect decomposition of some algebra $\mathfrak{A}$. We define $f : A \longrightarrow \Pi_{i\in I}B_i$ as follows. For each $a \in A$, $f(a) \overset{\text{def}}{=} \langle h_i(a) \ : \ i \in I\rangle$.

Clearly, $f(a) \in \Pi_{i\in I}B_i$ for every $a \in A$. By the definition of a subdirect decomposition, we have

$$(\forall a, b \in A)\big(a \neq b \Rightarrow (\exists i \in I) h_i(a) \neq h_i(b)\big)\,.$$

Therefore f is injective. It remains to check that f is a homomorphism.

To see this, let g be an n–ary operation of $\mathfrak{A}$, and let $\langle a_1, \ldots, a_n\rangle \in {}^nA$. Then

$$\begin{aligned}
f\left(g^{\mathfrak{A}}(a_1,\ldots,a_n)\right) &= \langle h_i\left(g^{\mathfrak{A}}(a_1,\ldots,a_n)\right) : i \in I\rangle && \text{by def. of } f\\
&= \langle g^{\mathfrak{B}_i}\left(h_i(a_1),\ldots,h_i(a_n)\right) : i \in I\rangle && h_i \text{ is a homom.}\\
&= g^{\Pi\mathfrak{B}_i}\left(\langle h_i(a_1)\rangle_{i\in I},\ldots,\langle h_i(a_n)\rangle_{i\in I}\right) && \text{by def. of a product}\\
&= g^{\Pi\mathfrak{B}_i}\left(f(a_1),\ldots,f(a_n)\right)\,.
\end{aligned}$$

This proves that f is a homomorphism, completing the proof of (ii). $\square$

The above theorem shows that subdirect decompositions are useful in the sense that instead of a complicated algebra $\mathfrak{A}$, we may study its "relatively simple" subdirect components $\mathfrak{B}_i$ $(i \in I)$, and after having studied the components, we can reconstruct the original algebra $\mathfrak{A}$ by using **SP**. As we already said in the starting paragraph of section 2.2, this process of decomposing first, then studying the parts, and then recovering the original by putting the parts together is often called the analysis–synthesis method; namely, we analyze complex system $\mathfrak{A}$ into its "simple" parts $\mathfrak{B}_i$, study the parts, and then synthesize the original $\mathfrak{A}$ from parts $\mathfrak{B}_i$.

This analysis–synthesis method suggests the question whether there are some kind of "smallest" i.e. "atomic" building blocks which cannot be further decomposed subdirectly into even "smaller" (or "simpler") components.[3] This thought motivates the definition of *subdirectly indecomposable* algebras. To follow the "official" terminology, we will say "subdirectly irreducible" instead of "subdirectly indecomposable".

Definition 2.2.36. (subdirectly irreducible algebras)

(i) An algebra $\mathfrak{A}$ is called *subdirectly irreducible* iff $\mathfrak{A}$ has no non–trivial subdirect decomposition (in the sense of Definition 2.2.32 (ii) above). That is, $\mathfrak{A}$ is subdirectly irreducible iff for any subdirect decomposition $\langle h_i : i \in I\rangle$ of $\mathfrak{A}$, if $|I| \neq 0$ then
$$(\exists i \in I)(h_i \text{ is an isomorphism}).$$
Putting it another way, $\mathfrak{A}$ is subdirectly irreducible iff for any subdirect decomposition $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i\rangle_{i\in I}$ of $\mathfrak{A}$, we have $\mathfrak{A} \in \mathbf{I}\{\mathfrak{B}_i \ : \ i \in I\}$. That is, for any subdirect decomposition of $\mathfrak{A}$, the original algebra $\mathfrak{A}$ is isomorphic to some of the components $\mathfrak{B}_i$.

(ii) Sir denotes the class of all subdirectly irreducible algebras. If K is a class of algebras then
$$\mathsf{SirK} = \mathsf{Sir}(\mathsf{K}) \stackrel{\text{def}}{=} \mathsf{Sir} \cap \mathsf{K}.$$
That is, SirK is the class of subdirectly irreducible members of K.

Lemma 2.2.37. (on the congruences of subdirectly irreducible algebras) $\mathfrak{A} \in \mathsf{Sir}$ *iff* $Con(\mathfrak{A}) \smallsetminus \{Id_A\}$ *has a smallest element, that is, there is a congruence* $\beta \in Con(\mathfrak{A}) \smallsetminus \{Id_A\}$ *such that*
$$(\forall \alpha \in Con(\mathfrak{A}))(\alpha \neq Id_A \Longrightarrow \beta \subseteq \alpha). \tag{2.3}$$
See Figure 2.2.6.

[3] Of course, this question motivates the related one, of whether every algebra is built up from such "atomic", that is, "indecomposable" building blocks. We will come to this question soon.

Figure 2.19: The congruence lattice of $\mathfrak{A} \in \mathsf{Sir}$

Proof. Assume $\mathfrak{A} \in \mathsf{Sir}$. Let $\beta \stackrel{\text{def}}{=} \bigcap(Con(\mathfrak{A}) \smallsetminus \{Id_A\})$. Then $\beta \in Con(\mathfrak{A})$ by Exercise 2.2.17. Assume $\beta = Id_A$. Then $\langle h \;:\; ker(()h) \in Con(\mathfrak{A}) \smallsetminus \{Id_A\}\rangle$ is a non–trivial subdirect decomposition of $\mathfrak{A}$. By Definition 2.2.36, this contradicts the fact that $\mathfrak{A} \in \mathsf{Sir}$. Thus $\beta \neq Id_A$, and, clearly, β is the smallest element of $Con(\mathfrak{A}) \smallsetminus \{Id_A\}$.

Now assume $\beta \in Con(\mathfrak{A}) \smallsetminus \{Id_A\}$ satisfies (2.3). Let $\langle h_i \;:\; i \in I\rangle$ be a subdirect decomposition of $\mathfrak{A}$. Let $\langle a, b\rangle \in \beta \smallsetminus Id_A$. Then $h_i(a) \neq h_i(b)$ for some $i \in I$. Thus, for such an i, $\langle a, b\rangle \notin ker(()h_i) \in Con(\mathfrak{A})$, therefore $\beta \not\subseteq ker(()h_i)$. This implies $ker(()h_i) = Id_A$ by (2.3). Thus h_i is one–one. But then h_i is an isomorphism because it is onto (by subdirect decomposition). This proves $\mathfrak{A} \in \mathsf{Sir}$. $\square$

Exercises 2.2.38. (subdirectly irreducible algebras)

(1) Prove that for every algebra $\mathfrak{A}$, $|A| = 2 \Rightarrow \mathfrak{A} \in \mathsf{Sir}$.

(2) Prove that $\underset{\sim}{\mathbf{2}} \in \mathsf{Sir}$ and $\mathfrak{P}(2) \notin \mathsf{Sir}$.

(3) Prove that any subalgebra of the algebra $\mathfrak{A}$ given in Example 2.2.20 (3) is subdirectly irreducible.

(4) Prove that every subdirectly irreducible algebra is directly indecomposable.

Exercises 2.2.39. (subdirectly irreducible and simple algebras)

(1) Recall the similarity type *ba* from Example 2.3 **4**. Let $t \stackrel{\text{def}}{=} ba \cup \{\langle \mathsf{c}, 1\rangle\}$. Let the algebras $\mathfrak{A}, \mathfrak{B} \in \mathsf{Alg}_t$ be isomorphic to $\mathfrak{P}(2)$ on the "*ba*–part", and c be as indicated on Figure 2.2.39.

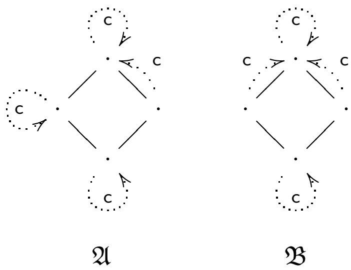


Figure 2.20: $\mathfrak{A} \in \mathsf{Sir} \smallsetminus \mathsf{Smp}$ and $\mathfrak{B} \in \mathsf{Sir}(\mathsf{Smp})$

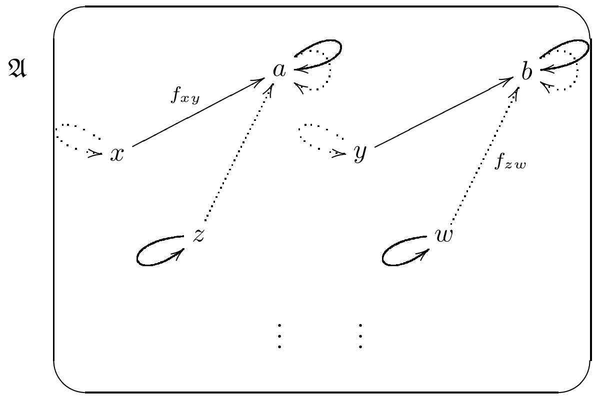


Figure 2.21: Subdirectly irreducible but not simple

Prove that $\mathfrak{A} \in \mathsf{Sir} \smallsetminus \mathsf{Smp}$ and $\mathfrak{B} \in \mathsf{Sir}(\mathsf{Smp})$.

(2) Prove that if $\mathfrak{A}$ satisfies condition $(\star)$ below then $\mathfrak{A}$ is subdirectly irreducible but not simple.

> $(\star)$ Let $a, b \in A$, $a \neq b$. To every $x, y \in A$, if x, y, a, b are all different then there is a fundamental operation $f_{x,y}$ of $\mathfrak{A}$ such that every element of $\mathfrak{A}$ except for x and y are fixed points of f_{xy}, and $f_{xy}(x) = a$ and $f_{xy}(y) = b$. There are no other fundamental operations of $\mathfrak{A}$.

See the illustration on Figure 2.2.39.

(3) Prove that $\mathsf{Smp} \subsetneq \mathsf{Sir}$.

Recall the definitions of Sir and $\mathsf{Sir}(\mathsf{K})$ (for a class K of algebras) from Definition 2.2.36 (ii).

Theorem 2.2.40. (Birkhoff) *Let $\mathfrak{A}$ be an algebra. Then (i)–(ii) below hold.*

(i) $\mathfrak{A} \in \mathbf{SP}\mathsf{Sir}$.

(ii) $\mathfrak{A} \in \mathbf{SP}\mathsf{Sir}(\mathbf{H}\{\mathfrak{A}\})$.

The *proof* will be discussed below the formulation of the following theorem. □

Theorem 2.2.40 above is a powerful tool in simplifying investigations of algebras. Namely, in a sense, it is sufficient to study the subdirectly irreducible algebras only, because the rest is built up from these by **SP**. (Of course, this is not true for *all* kinds of questions, because building up by **SP** might interfere with *certain* kinds of questions. But still, we will experience that it is true for many kinds of questions.) Similarly, if we are investigating a class $\mathsf{K} = \mathbf{H}\mathsf{K}$ of algebras (like Boolean algebras, groups etc.) then it is often sufficient to investigate SirK instead of K, since the rest of K is built up from SirK by **SP**.

Theorem 2.2.41. (Birkhoff's Subdirect Decomposition Theorem) *Every algebra $\mathfrak{A}$ admits a subdirect decomposition into subdirectly irreducible components that are quotient algebras of $\mathfrak{A}$. That is:*

Let $\mathfrak{A}$ be an algebra. Then there is a subdirect decomposition $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i \rangle_{i \in I}$ of $\mathfrak{A}$ such that $\{\mathfrak{B}_i \ : \ i \in I\} \subseteq \mathsf{Sir}(\mathbf{H}\mathfrak{A})$.

Proof. Since every trivial algebra is subdirectly irreducible, we may assume that $\mathfrak{A}$ is not trivial. Let $I = (A \times A) \smallsetminus Id_A$. For each $i = \langle a, b \rangle \in I$, choose a maximal member β_i in the set $\{\theta \in Con(\mathfrak{A}) : \langle a, b \rangle \notin \theta\}$. The existence of such a maximal member follows easily by Zorn's lemma.

Let $i = \langle a, b \rangle \in I$ be arbitrary. We claim that $\mathfrak{A}/\beta_i \in \mathsf{Sir}$. This is so because $(\forall \theta \in Con(\mathfrak{A}/\beta_i)) a/\beta_i \equiv_\theta b/\beta_i$, by the definition of β_i. Thus the smallest congruence containing $\langle a/\beta_i, b/\beta_i \rangle$ is the smallest non–identity element of $Con(\mathfrak{A}/\beta_i)$. Thus $\mathfrak{A}/\beta_i \in \mathsf{Sir}$ by Lemma 2.2.37. We have seen that $(\forall i \in I)\mathfrak{A}/\beta_i \in \mathsf{Sir}$.

Clearly, $\bigcap\{\beta_i : i \in I\} = Id_A$. Let q_i be the quotient map associated to β_i, for every $i \in I$. Then $\langle q_i \ : \ i \in I \rangle$ is a subdirect decomposition of $\mathfrak{A}$, with components the factor algebras $\mathfrak{A}/\beta_i$ of $\mathfrak{A}$, for every $i \in I$. This completes the proof of Theorem 2.2.41. □

Let us return briefly to the analysis–synthesis method outlined above Definition 2.2.36. Theorem 2.2.41 above implies that this method is applicable rather "deterministically" to algebras. Namely, if we want to study a (say, complicated) algebra $\mathfrak{A}$, then we can subdirectly decompose $\mathfrak{A}$ to parts $\mathfrak{B}_i$ $(i \in I)$ which are *not* decomposable subdirectly any further. Since the $\mathfrak{B}_i$'s are not decomposable any further, we have to investigate the $\mathfrak{B}_i$'s, and then "synthesize" the results obtained about the $\mathfrak{B}_i$'s to results about the original $\mathfrak{A}$ along the lines discussed earlier. The important point here is that to any algebra $\mathfrak{A}$, there are "atomic" (i.e. indecomposable) components $\mathfrak{B}_i$ of $\mathfrak{A}$ such that we know when to stop in the "analysis part", that is, in the decomposition process. (Further $\mathfrak{A}$ can be built up from these "atomic" components $\mathfrak{B}_i$ of $\mathfrak{A}$.)

From Theorems 2.2.35 and 2.2.41 we can prove Theorem 2.2.40, as follows.

Proof. Proof of Theorem 2.2.40:
Let $\mathfrak{A}$ be an algebra. By Theorem 2.2.41, there is a subdirect decomposition $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i \rangle_{i \in I}$ with $(\forall i \in I)\mathfrak{B}_i \in \mathsf{Sir}(\mathbf{H}\mathfrak{A})$. Now, Theorem 2.2.35 says that $\mathfrak{A} \in \mathbf{SP}\{\mathfrak{B}_i \ : \ i \in I\} \subseteq \mathbf{SP}\ \mathsf{Sir}(\mathbf{H}\mathfrak{A})$. This completes the proof. $\square$

2.2.7 Ultraproduct, reduced product

Motivation 2.2.42. Like direct product, *ultraproduct* and *reduced product* are such operations on algebras which "build up bigger, more complex algebras from smaller ones" (cf. the introductory paragraphs of section 2.2). To motivate these new operations to be introduced shortly, observe the following. An equation is valid in the direct product $\mathfrak{A}$ of some system $\langle \mathfrak{B}_i \ : \ i \in I \rangle$ of algebras iff it is valid in *each* component $\mathfrak{B}_i$ $(i \in I)$. For example, if the similarity type contains two constant symbols d, e and a unary function symbol f, then $f(d) = e$ is valid in $\mathfrak{A}$ iff it is valid in $\mathfrak{B}_i$ for *every* $i \in I$. Thus $f(d) = e$ is not valid in $\mathfrak{A}$ whenever it is not valid in one of its components, even if $f(d) = e$ is valid in each of the other (possibly infinitely many) components. See Figure 2.22.

Ábrát krumplisan!!

The property of direct product illustrated above shows that this operation is rigid in that it does not "tolerate" any "deviation": an equation cannot be valid in the value of the operation even when it is valid in "almost every" argument (component) of the operation. Now we are aiming at defining operations on algebras which are not so rigid in this sense. We would like to define a "product–like" operation in the value of which a property holds iff it holds in almost every argument (almost everywhere). To have such a concept, we need to clarify first what we want to mean by "almost every" or "almost everywhere".

From a (mathematical) concept of "almost everywhere", we require the following, intuitively natural things.

(i) If something is true everywhere, then it should be true almost everywhere.

(ii) If two properties φ and ψ hold almost everywhere then their conjunction $(\varphi \wedge \psi)$ should hold almost everywhere, too.

(iii) If φ holds almost everywhere and φ (always) implies ψ then ψ should hold almost everywhere, too.

We may also add:

(iv) A property φ holds almost everywhere iff its negation $\neg\varphi$ does not hold almost everywhere.

In the following definition we give two (mathematical) concepts corresponding to the above described intuition of "almost everywhere". These concepts are called *filter* and *ultrafilter*.

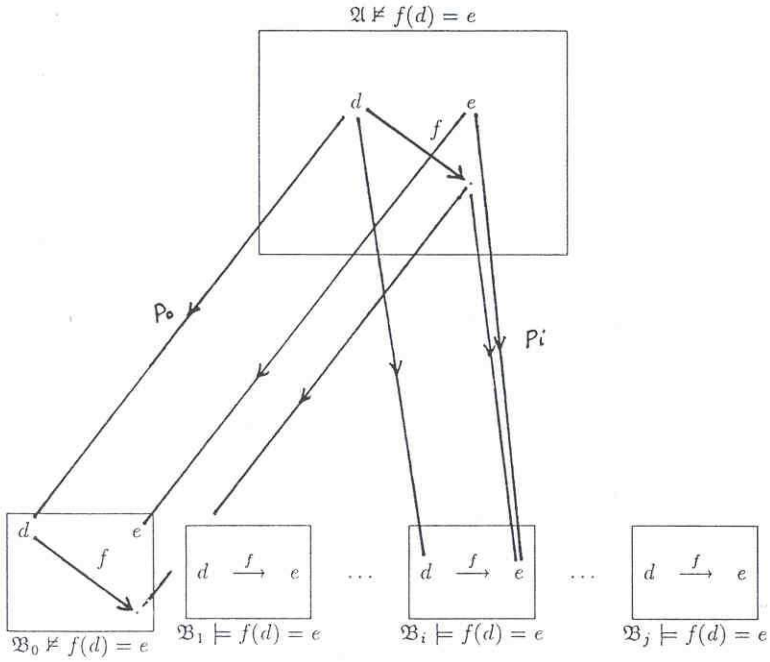


Figure 2.22: Direct product is rigid

Definition 2.2.43. (filter, ultrafilter, special filters) Let I be a nonempty set. Recall that $\mathcal{P}(I)$ denotes the powerset (set of all subsets) of I. Let $\mathcal{F} \subseteq \mathcal{P}(I)$.

1. $\mathcal{F}$ is called a *filter* over I iff conditions (i)–(iii) below are satisfied.

(i) $I \in \mathcal{F}$

(ii) if $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$

(iii) if $X \in \mathcal{F}$ and $X \subseteq Z \subseteq I$ then $Z \in \mathcal{F}$.

2. A filter $\mathcal{F}$ over I is called and *ultrafilter* over I iff for every $X \in \mathcal{P}(I)$, (iv) below holds.

(iv) $X \in \mathcal{F} \iff (I \smallsetminus X \notin \mathcal{F})$.

If we simply say that $\mathcal{F}$ is an ultrafilter then we tacitly assume that $\mathcal{F}$ is an ultrafilter over $\bigcup \mathcal{F}$.

3. $\{I\}$ is called the *trivial filter* (over I), $\mathcal{P}(I)$ the *improper filter*. A filter $\mathcal{F}$ over I is said to be a *proper filter* iff it is not the improper filter $\mathcal{P}(I)$. For each $Y \subseteq I$, the filter $\{X \subseteq I : Y \subseteq X\}$ is called the *principal filter* generated by Y. A filter (ultrafilter) is called *non–principal* iff it is not a principal filter (ultrafilter). If $|I| \geq \omega$ then $\mathcal{F}r(I) \stackrel{\text{def}}{=} \{X \in \mathcal{P}(I) : |I \smallsetminus X| < \omega\}$ is called the *Fréchet filter* (over I).

Exercise 2.2.44. Prove that, for any nonempty set I, the trivial filter over I, any principal filter, and the Fréchet filter over I are all filters.

If $\mathcal{F}$ is a filter over I then we sometimes refer to the elements of $\mathcal{F}$ as to "*big*" or "*$\mathcal{F}$–big*" sets, while we call the rest of the subsets of I "*small*" sets.

Next we investigate filters and ultrafilters.

Definition 2.2.45. (generated filter, finite intersection property) Let E be a subset of $\mathcal{P}(I)$.

(i) By a *filter generated by E* we mean the intersection $\mathcal{F}$ of all filters over I which include E, that is,

$$\mathcal{F} = \bigcap\{\mathcal{D} : E \subseteq \mathcal{D} \text{ and } \mathcal{D} \text{ is a filter over } I\}\,.$$

(ii) E is said to have the *finite intersection property* iff the intersection of any finite number of elements of E is nonempty.

Proposition 2.2.46. *Let E be any subset of $\mathcal{P}(I)$ for some nonempty set I, and let $\mathcal{F}$ be the filter generated by E. Then (i)–(iii) below hold.*

(i) $\mathcal{F}$ is a filter over I.

(ii) $\mathcal{F}$ is the set of all $X \subseteq I$ such that either $X = I$ or for some $Y_1, \ldots, Y_n \in E$,

$$Y_1 \cap \cdots \cap Y_n \subseteq X\,.$$

(iii) $\mathcal{F}$ is a proper filter iff E has the finite intersection property.

Proof. (i) is easy to prove. (iii) follows easily from (ii). To prove (ii), let

$$\mathcal{F}' \stackrel{\text{def}}{=} \{X \subseteq I : X = I \text{ or } Y_1 \cap \cdots \cap Y_n \subseteq X \text{ for some } Y_1, \ldots, Y_n \in E\}.$$

We prove that $\mathcal{F} = \mathcal{F}'$. First we check that $\mathcal{F}'$ is a filter, as follows. $I \in \mathcal{F}'$ by the definition of $\mathcal{F}'$. Let $X, X' \in \mathcal{F}'$, and let $Y_i, Y'_j \in E$ be such that

$$Y_1 \cap \cdots \cap Y_n \subseteq X \quad \text{and} \quad Y'_1 \cap \cdots \cap Y'_k \subseteq X'.$$

If $X \subseteq Z \subseteq I$, then $Y_1 \cap \cdots \cap Y_n \subseteq Z$, thus $Z \in \mathcal{F}'$. Further,

$$Y_1 \cap \cdots \cap Y_n \cap Y'_1 \cap \cdots \cap Y'_k \subseteq X \cap X',$$

thus $X \cap X' \in \mathcal{F}'$. Therefore $\mathcal{F}'$ is a filter over I. Obviously $E \subseteq \mathcal{F}'$. Thus $\mathcal{F} \subseteq \mathcal{F}'$, by the definition of $\mathcal{F}$.

To see $\mathcal{F}' \subseteq \mathcal{F}$, consider any filter $\mathcal{D}$ over I which includes E. Then $\mathcal{F}' \subseteq \mathcal{D}$ can be seen as follows. Clearly $I \in \mathcal{D}$. If $I \neq X \in \mathcal{F}'$ then $X \in \mathcal{D}$ because, for any $Y_1, \ldots, Y_n \in E$, we have $Y_1 \cap \cdots \cap Y_n \in \mathcal{D}$. Thus $\mathcal{F}' \subseteq \mathcal{D}$. This shows that $\mathcal{F}' \subseteq \mathcal{F}$. Thus $\mathcal{F} = \mathcal{F}'$. □

Exercises 2.2.47. (1) Prove (i) and (iii) of Proposition 2.2.46 above.

(2) Prove that for any set I, if C is a nonempty chain of proper filters over I, then $\bigcup C$ is a proper filter over I.

(3) Prove that every proper filter has the finite intersection property.

Proposition 2.2.48. *The following (i) and (ii) are equivalent.*

(i) $\mathcal{F}$ is an ultrafilter over I.

(ii) $\mathcal{F}$ is a maximal proper filter over I. That is, $\mathcal{F}$ is a proper filter over I, and the only proper filter over I which includes $\mathcal{F}$ is $\mathcal{F}$ itself.

Proof. (i)⇒(ii). Assume (i). Then $0 \notin \mathcal{F}$, because $I \in \mathcal{F}$ and $0 = I \smallsetminus I$. Hence $\mathcal{F}$ is a proper filter. Let $\mathcal{D}$ be any proper filter over I which includes $\mathcal{F}$. If $X \in \mathcal{D} \smallsetminus \mathcal{F}$, then $I \smallsetminus X \in \mathcal{F}$ (because $\mathcal{F}$ is an *ultra*filter), thus $I \smallsetminus X \in \mathcal{D}$, and $0 = X \cap (I \smallsetminus X) \in \mathcal{D}$. This contradicts the assumption that $\mathcal{D}$ is proper. Thus $\mathcal{D} \subseteq \mathcal{F}$, so $\mathcal{D} = \mathcal{F}$, and (ii) holds.

(ii)⇒(i). Assume (ii). Consider any $X \subseteq I$. We cannot have both $X \in \mathcal{F}$ and $I \smallsetminus X \in \mathcal{F}$, because then $0 \in \mathcal{F}$, thus $(\forall Y \subseteq I) Y \in \mathcal{F}$, and $\mathcal{F}$ is not proper. It is enough to prove that $I \smallsetminus X \notin \mathcal{F} \Rightarrow X \in \mathcal{F}$. Suppose $I \smallsetminus X \notin \mathcal{F}$. Let $E = \mathcal{F} \cup \{X\}$, and let $\mathcal{D}$ be the filter generated by E. We show that E has the finite intersection property. Consider $Y_1, \ldots, Y_n \in E$, and let $Z = Y_1 \cap \cdots \cap Y_n$. Since $\mathcal{F}$ is closed under finite intersections, we either have $Z \in \mathcal{F}$ or $Z = Y \cap X$ for some $Y \in \mathcal{F}$. In the first case, $Z \neq 0$, because $0 \notin \mathcal{F}$. In the second case, we also have $Z \neq 0$, for otherwise we would have $Y \cap X = 0$, $Y \subseteq I \smallsetminus X$, whence $I \smallsetminus X \in \mathcal{F}$. Thus in any case, $Z \neq 0$. By Proposition 2.2.46, $0 \notin \mathcal{D}$. Thus $\mathcal{D}$ is a proper filter including $\mathcal{F}$, thus by (ii), $\mathcal{D} = \mathcal{F}$. Therefore $E \subseteq \mathcal{F}$ and $X \in \mathcal{F}$. This proves (i). □

Proposition 2.2.49. *If $E \subseteq \mathcal{P}(I)$ and E has the finite intersection property, then there exists an ultrafilter $\mathcal{F}$ over I such that $E \subseteq \mathcal{F}$.*

Proof. By Proposition 2.2.46, the filter $\mathcal{D}$ generated by E does not contain the empty set, thus $\mathcal{D}$ is proper. Moreover, if C is any nonempty chain of proper filters over I, then $\bigcup C$ is a proper filter over I, by Exercise 2.2.47 (2). Furthermore, if $\mathcal{F} \in C$ includes E, then $\bigcup C$ includes E. Then, by Zorn's lemma, the class H of all proper filters over I including E has a maximal element, say $\mathcal{F}$. Thus $E \subseteq \mathcal{F}$. $\mathcal{F}$ is a maximal proper filter over I, because if $\mathcal{F}'$ is a proper filter including $\mathcal{F}$, then $E \subseteq \mathcal{F}'$, and so $\mathcal{F}'$ belongs to H and $\mathcal{F}' = \mathcal{F}$. Thus, by Proposition 2.2.48, $\mathcal{F}$ is an ultrafilter over I. □

Corollary 2.2.50. *Any proper filter over I can be extended to an ultrafilter over I.*

Proof. Every proper filter has the finite intersection property by Exercise 2.2.47 (3). □

Exercises 2.2.51. Below, I is a non–empty set, $\mathcal{U} \subseteq \mathcal{P}(I)$ is an ultrafilter, and $\mathcal{F}r(I)$ denotes the Fréchet filter over I.

(1) (a) Let $X \in \mathcal{U}$, $X = Y_0 \cup \cdots \cup Y_n$ for some $n \in \omega$. Prove that

$$Y_i \in \mathcal{U} \text{ for some } i \leqslant n\,.$$

(b) Prove that $\mathcal{U}$ is principal iff $\mathcal{U} = \{Y \subseteq I \ : \ \{x\} \subseteq Y\}$ for some $x \in I$.

(2) (a) Prove that $\mathcal{F}r(I)$ is not an ultrafilter (but is a filter by Exercises 2.2.44).

(b) Prove that $\mathcal{U}$ is non–principal iff $\mathcal{F}r(I) \subseteq \mathcal{U}$.

(3) Let $E \subseteq \mathcal{P}(I)$ be such that $(\forall \text{ finite } H \subseteq E)|\bigcap H| \geqslant \omega$. Prove that there exists a non–principal ultrafilter containing E.

Definition 2.2.52. (reduced product, ultraproduct) Let I be a nonempty set and $\mathcal{F}$ a proper filter over I. Let $\langle A_i \ : \ i \in I\rangle$ be a system of nonempty sets.

1. Consider the Cartesian product $\Pi_{i\in I}A_i$ of the sets A_i. Let $a, b \in \Pi_{i\in I}A_i$. We say that a and b are *$\mathcal{F}$–equivalent*, in symbols $a \equiv_{\mathcal{F}} b$, iff

$$\{i \in I \ : \ a(i) = b(i)\} \in \mathcal{F}\,.$$

Then $\equiv_{\mathcal{F}}$ is an equivalence relation over $\Pi_{i\in I}A_i$. (Exercise: Prove this!) Let $a/\mathcal{F}$ denote the equivalence class of a:

$$a/\mathcal{F} \stackrel{\text{def}}{=} \{b \in \Pi_{i\in I}A_i : a \equiv_{\mathcal{F}} b\}\,.$$

We define the *reduced product of $\langle A_i \ : \ i \in I\rangle$ modulo $\mathcal{F}$*, in symbols $\Pi\langle A_i : i \in I\rangle/\mathcal{F}$ or $\Pi_{i\in I}A_i/\mathcal{F}$, to be the set of all equivalence classes of $\equiv_{\mathcal{F}}$. That is,

$$\Pi_{i\in I}A_i/\mathcal{F} \stackrel{\text{def}}{=} \{a/\mathcal{F} \ : \ a \in \Pi_{i\in I}A_i\}\,.$$

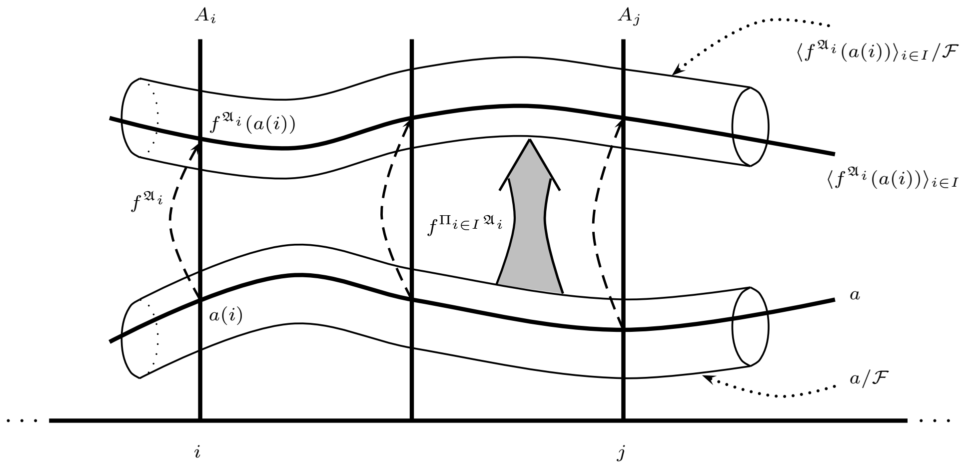


Figure 2.23: Ultraproduct

We call the set I the *index set* for $\Pi_{i\in I}A_i/\mathcal{F}$. In the special case when $\mathcal{F}$ is an ultrafilter, the reduced product $\Pi_{i\in I}A_i/\mathcal{F}$ is called an *ultraproduct*.

2. Now let $\langle \mathfrak{A}_i : i \in I\rangle$ be a system of similar algebras. We define the *reduced product* $\Pi\langle \mathfrak{A}_i \ : \ i \in I\rangle/\mathcal{F} = \Pi_{i\in I}\mathfrak{A}_i/\mathcal{F}$ of this system as follows. $\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F}$ is defined to be an algebra with universe $\Pi_{i\in I}A_i/\mathcal{F}$ (the reduced product of the universes), and if f is an n–ary function symbol of the similarity type of the $\mathfrak{A}_i$'s then f is interpreted on $\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F}$ as follows.

$$f^{\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F}}(a_1/\mathcal{F},\dots,a_n/\mathcal{F}) \stackrel{\text{def}}{=} \langle f^{\mathfrak{A}_i}(a_1(i),\dots,a_n(i)) : i \in I\rangle/\mathcal{F} \tag{2.4}$$

for every $a_1,\dots,a_n \in \Pi_{i\in I}A_i$.

Illustration: Assume that the similarity type of $\langle \mathfrak{A}_i \ : \ i \in I\rangle$ contains a unary function symbol f. Then equation (2.4) above is reduced to

$$f^{\Pi_{i\in I}\mathfrak{A}_i}(a/\mathcal{F}) = \langle f^{\mathfrak{A}_i}(a(i))\rangle_{i\in I}/\mathcal{F}\,.$$

On Figure 2.23, the equivalence class $a/\mathcal{F}$ is drawn as a long "sausage" (containing "line" a).

Nyilak feljebb! Kilóg a jobb oldalon!

The following proposition states that the definition of a reduced product given above is correct.

Proposition 2.2.53. *Let I, $\mathcal{F}$, $\mathfrak{A}_i$ be as in Definition 2.2.52 above. Suppose that $a_1 \equiv_{\mathcal{F}} b_1,\dots,a_n \equiv_{\mathcal{F}} b_n$. Then for every function symbol f of the similarity type of the $\mathfrak{A}_i$'s,*

$$\langle f^{\mathfrak{A}_i}((a_1)_i,\dots,(a_n)_i) \ : \ i \in I\rangle \equiv_{\mathcal{F}} \langle f^{\mathfrak{A}_i}((b_1)_i,\dots,(b_n)_i) \ : \ i \in I\rangle\,.$$

Exercise 2.2.54. Prove Proposition 2.2.53.

Convention 2.2.55. We sometimes omit the subscript $i \in I$ from $\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F}$ when it causes no misunderstanding.

We call the ultraproduct of a system $\langle \mathfrak{A} : i \in I\rangle$ (all the models are the same) an *ultrapower*. An ultrapower is denoted as ${}^I\mathfrak{A}/\mathcal{F}$.

Exercise 2.2.56. Generalize the concept of a reduced product (ultraproduct) of algebras to that of *models*.

The most important theorem we need concerning ultraproducts is the following. Recall our notation for the truth of a formula in an algebra at a valuation from the end of Definition 1.8.1 (4) (in section 1.8).

Theorem 2.2.57. (Łos ultraproduct theorem) *Let $\mathfrak{B}$ be the ultraproduct $\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F}$. Then (i)–(iii) below hold.*

(i) For any term $\tau(x_1, \dots, x_n)$ in the language of $\mathfrak{B}$ and elements $a_1/\mathcal{F}, \dots, a_n/\mathcal{F} \in B$, we have

$$\tau^{\mathfrak{B}}\,(a_1/\mathcal{F}, \dots, a_n/\mathcal{F}) = \langle \tau^{\mathfrak{A}_i}\,(a_1(i), \dots, a_n(i)) \;:\; i \in I\rangle/\mathcal{F}\,.$$

(ii) Given any formula $\varphi(x_1, \dots, x_n)$ of the language of $\mathfrak{B}$ and elements $a_1/\mathcal{F}, \dots, a_n/\mathcal{F} \in B$, we have

$$\mathfrak{B} \models \varphi[a_1/\mathcal{F}, \dots, a_n/\mathcal{F}] \quad \textit{iff} \quad \{i \in I \;:\; \mathfrak{A}_i \models \varphi[a_1(i), \dots, a_n(i)]\} \in \mathcal{F}\,.$$

(iii) For any sentence φ of the language of $\mathfrak{B}$,

$$\mathfrak{B} \models \varphi \qquad \textit{iff} \qquad \{i \in I \;:\; \mathfrak{A}_i \models \varphi\} \in \mathcal{F}\,.$$

Proof. (iii) is an immediate consequence of (i) and (ii). The proofs of (i) and (ii) are by induction on the complexity of terms and formulas, respectively.

Proof of (i): (i) holds whenever $\tau(x_1, \dots, x_n)$ is a variable, a constant, or is of the form $f(x_1, \dots, x_n)$, by the definition of reduced product. Suppose that

$$\tau(x_1, \dots, x_n) = f(\tau_1(x_1, \dots, x_n), \dots, \tau_k(x_1, \dots, x_n)) \;\text{ and }\; \tau_1, \dots, \tau_k \text{ satisfy (i).}$$

Then for any elements $a_1/\mathcal{F}, \dots, a_n/\mathcal{F} \in B$,

$$\tau^{\mathfrak{B}}(a_1/\mathcal{F} \dots a_n/\mathcal{F}) = f^{\mathfrak{B}}(\tau_1^{\mathfrak{B}}(a_1/\mathcal{F} \dots a_n/\mathcal{F}) \dots \tau_k^{\mathfrak{B}}(a_1/\mathcal{F} \dots a_n/\mathcal{F}))\,. \tag{2.5}$$

If $j \in \{1, \dots, k\}$ then $\tau_j^{\mathfrak{B}}(a_1/\mathcal{F}, \dots, a_n/\mathcal{F}) = \langle \tau_j^{\mathfrak{A}_j}(a_1(i), \dots, a_n(i)) \;:\; i \in I\rangle/\mathcal{F}$ by the induction hypothesis. By the definition of reduced product, we have:

$$f^{\mathfrak{B}}(b_1/\mathcal{F}, \dots, b_n/\mathcal{F}) = \langle f^{\mathfrak{A}_i}(b_1(i), \dots, b_n(i)) : i \in I\rangle/\mathcal{F}\,, \tag{2.6}$$

for any $b_1, \ldots, b_n \in \Pi_{i\in I}\mathfrak{A}_i$. For every $i \in I$, there are $b_1, \ldots, b_n$ such that

$$\tau^{\mathfrak{A}_i}(a_1(i), \ldots, a_n(i)) = f^{\mathfrak{A}_i}(b_1(i), \ldots, b_n(i))\,. \tag{2.7}$$

Now

$$\begin{aligned}
\tau^{\mathfrak{B}}(a_1/\mathcal{F} \ldots a_n/\mathcal{F}) &= f^{\mathfrak{B}}\left(\tau_1^{\mathfrak{B}}(a_1/\mathcal{F} \ldots a_n/\mathcal{F}) \ldots \tau_k^{\mathfrak{B}}(a_1/\mathcal{F} \ldots a_n/\mathcal{F})\right) && \text{by (2.5)} \\
&= f^{\mathfrak{B}}(b_1/\mathcal{F} \ldots b_n/\mathcal{F}) && \text{for some } b_1 \ldots b_n \in \Pi_{i\in I}\mathfrak{A}_i \\
&= \langle f^{\mathfrak{A}_i}(b_1(i) \ldots b_n(i)) \;:\; i \in I\rangle/\mathcal{F} && \text{by (2.6)} \\
&= \langle \tau^{\mathfrak{A}_i}(a_1(i) \ldots a_n(i)) \;:\; i \in I\rangle/\mathcal{F} && \text{by (2.7).}
\end{aligned}$$

Proof of (ii): The proof of (ii) for atomic formula (equations) is similar to the above proof of (i); we note that (i) is used in the proof of (ii).

Suppose $\varphi = \neg\psi(x_1, \ldots, x_n)$ and (ii) holds for $\psi(x_1, \ldots, x_n)$. Let $a_1, \ldots, a_n \in \Pi_{i\in I}\mathfrak{A}_i$. Then

$$\begin{aligned}
\mathfrak{B} \models \varphi[a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] &\iff \mathfrak{B} \not\models \psi[a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] \\
&\iff \{i \in I \;:\; \mathfrak{A}_i \models \psi[a_1(i), \ldots, a_n(i)]\} \notin \mathcal{F} \\
&\iff \{i \in I \;:\; \mathfrak{A}_i \not\models \psi[a_1(i), \ldots, a_n(i)]\} \in \mathcal{F} \\
&\iff \{i \in I \;:\; \mathfrak{A}_i \models \varphi[a_1(i), \ldots, a_n(i)]\} \in \mathcal{F}\,.
\end{aligned}$$

The fact that $\mathcal{F}$ is an ultrafilter is used to show that the second "$\iff$" holds. Indeed, this is the only point in the entire proof of the theorem where we need the fact that $\mathcal{F}$ is an ultrafilter, and not merely a proper filter.

The next step is to prove that if ψ and χ satisfy (ii), then so does $(\psi \wedge \chi)$. This is done by writing a string of equivalences like the one we used for $\neg\psi$. This time the crucial fact about $\mathcal{F}$ which we need is that $X \cap Y \in \mathcal{F} \iff X \in \mathcal{F}$ and $Y \in \mathcal{F}$. Every filter has this property. The details of this step in the proof are straightforward, and are left to the reader.

Now suppose that $\varphi(x_1, \ldots, x_n) = \exists x\psi(x, x_1, \ldots, x_n)$, and that (ii) holds for ψ. Let $a_1/\mathcal{F}, \ldots, a_n/\mathcal{F} \in B$. Then

$$\begin{aligned}
\mathfrak{B} \models \varphi[a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] &\iff \mathfrak{B} \models \exists x\psi(x, x_1, \ldots, x_n)[a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] \\
&\iff \mathfrak{B} \models \psi[b/\mathcal{F}, a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] \text{ for some } b \in \Pi_{i\in I}\mathfrak{A}_i \\
&\iff \{i \in I \;:\; \mathfrak{A}_i \models \psi[b(i), a_1(i), \ldots, a_n(i)]\} \in \mathcal{F} \\
&\qquad\qquad \text{for the above } b\text{, by ind. hyp.} \\
&\implies \{i \in I \;:\; \mathfrak{A}_i \models \exists x\psi(x, x_1, \ldots, x_n)[a_1(i), \ldots, a_n(i)]\} \in \mathcal{F} \\
&\iff \{i \in I \;:\; \mathfrak{A}_i \models \varphi[a_1(i), \ldots, a_n(i)]\} \in \mathcal{F}\,.
\end{aligned}$$

To see the other direction, assume $\{i \in I \;:\; \mathfrak{A}_i \models \varphi[a_1, \ldots, a_n]\} \in \mathcal{F}$. Then

$$\{i \in I \;:\; \mathfrak{A}_i \models \psi[d(i), a_1(i), \ldots, a_n(i)]\} \in \mathcal{F} \text{ for some } d \in \Pi_{i\in I}\mathfrak{A}_i$$

$$\begin{aligned}&\iff \mathfrak{B} \models \psi[d/\mathcal{F}, a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}]\\ &\iff \mathfrak{B} \models \varphi[a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}]\,.\end{aligned}$$

By this, our induction for step (ii) is completed. □

Exercise 2.2.58. Prove Theorem 2.2.57 for the case of *models*. That is, consider the theorem we get from Theorem 2.2.57 via substituting models for algebras in it, and using ultraproduct of models (cf. Exercise 2.2.56) instead of that of algebras. Prove this theorem.

Definition 2.2.59. ($\mathbf{P^r}$, **Up**)
Given a class K of similar algebras, we let

$$\mathbf{P^r}\mathsf{K} \stackrel{\text{def}}{=} \mathbf{I}\{\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F} \ : \ I \text{ is a set, } \mathcal{F} \text{ is a filter over the set } I, \text{ and } (\forall i \in I)\mathfrak{A}_i \in \mathsf{K}\}$$

$$\mathbf{Up}\mathsf{K} \stackrel{\text{def}}{=}$$

$$\mathbf{I}\{\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F} \ : \ I \text{ is a set, } \mathcal{F} \text{ is an ultrafilter over the set } I, \text{ and } (\forall i \in I)\mathfrak{A}_i \in \mathsf{K}\}\,.$$

Lemma 2.2.60. $\mathbf{P^r}\mathbf{P^r}\mathsf{K} = \mathbf{P^r}\mathsf{K}$.

Proof. [4] Let K be an arbitrary class of algebras. Then $\mathbf{P}^r\mathsf{K} \subseteq \mathbf{P}^r\mathbf{P}^r\mathsf{K}$ follows from the definition of reduced product. It remains to prove $\mathbf{P}^r\mathbf{P}^r\mathsf{K} \subseteq \mathbf{P}^r\mathsf{K}$.

Let J be a set and $\langle I_j \ : \ j \in J\rangle$ a system of pairwise disjoint sets. For every $j \in J$, let $\langle \mathfrak{A}_i \ : \ i \in I_j\rangle$ be a system of algebras. Let $\mathcal{F}$ be a filter over J, and for every $j \in J$, let $\mathcal{F}_j$ be a filter over I_j. We define

$$I \stackrel{\text{def}}{=} \bigcup_{j\in J} I_j \quad \text{ and } \quad \hat{\mathcal{F}} \stackrel{\text{def}}{=} \{L \subseteq I \ : \ \{j \in J \ : \ L \cap I_j \in \mathcal{F}_j\} \in \mathcal{F}\}\,.$$

($L \in \hat{\mathcal{F}}$ iff L is an "$\mathcal{F}$–big union of $\mathcal{F}_j$–big sets".) It is easy to check that $\hat{\mathcal{F}}$ is a filter over I. It is enough to prove that

$$\prod_{j\in J}\left(\prod_{i\in I_j}\mathfrak{A}_i \Big/ \mathcal{F}_j\right)\Big/ \mathcal{F} \cong \prod_{i\in I}\mathfrak{A}_i/\hat{\mathcal{F}}\,. \tag{2.8}$$

For every $j \in J$, let $f_j : \prod_{i\in I}\mathfrak{A}_i \longrightarrow \prod_{i\in I_j}\mathfrak{A}_i$ be defined as:

$$\left(\forall a \in \prod_{i\in I} A_i\right) f_j(a) \stackrel{\text{def}}{=} a\lceil I_j\,.$$

Clearly f_j is a surjective homomorphism, for every $j \in J$. Let $q_j : \prod_{i\in I_j}\mathfrak{A}_i \longrightarrow \prod_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j$ be the quotient map for every $j \in J$, thus $(\forall b \in \prod_{i\in I_j} A_i) q_j(b) = b/\mathcal{F}_j$. Follow the proof on Figure 2.24.

[4]In Andréka–Németi [11] as well as in Andréka–Burmeister–Németi [3] it is shown that Lemma 2.2.60 cannot be proved without the Axiom of Choice.

$$\begin{tikzcd}
 & & \prod\limits_{i\in I}\mathfrak{A}_i \arrow[r, "q"] \arrow[dll, two heads, "f_{j'}"'] \arrow[drr, two heads, "f_j"] \arrow[dd, dashed, "h"] & \prod\limits_{i\in I}\mathfrak{A}_i/\hat{\mathcal{F}} \arrow[dd, dashed, "g"'] & & \\
\prod\limits_{i\in I_{j'}}\mathfrak{A}_i \arrow[r, "q_{j'}"] & \prod\limits_{i\in I_{j'}}\mathfrak{A}_i/\mathcal{F}_{j'} & & & \prod\limits_{i\in I_j}\mathfrak{A}_i \arrow[r, "g_j"] & \prod\limits_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j \\
 & & \mathfrak{P} \arrow[r, "k"'] \arrow[ul, "p_{j'}"] \arrow[urrr, "p_j"] & \mathfrak{P}/\mathcal{F} & &
\end{tikzcd}$$

Figure 2.24: Proof of Lemma 2.2.60

Let $\mathfrak{P} \stackrel{\text{def}}{=} \prod\limits_{j\in J}\left(\prod\limits_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j\right)$. Consider the cone

$$\langle \prod_{i\in I}\mathfrak{A}_i \stackrel{f_j}{\twoheadrightarrow} \prod_{i\in I_j}\mathfrak{A}_i \stackrel{q_i}{\longrightarrow} \prod_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j \ : \ j \in J\rangle .$$

By Lemma 2.2.23, this cone induces a unique homomorphism

$$h : \prod_{i\in I}\mathfrak{A}_i \longrightarrow \mathfrak{P},$$

thus, denoting the projection homomorphisms of $\mathfrak{P}$ by p_j $(j \in J)$, we have

$$h \circ p_j = f_j \circ q_j, \quad \text{i.e.,}$$
$$(\forall a \in \prod_{i\in I} A_i) h(a)(j) = q_j(f_j(a)) = q_j(a\lceil I_j) = (a\lceil I_j)/\mathcal{F}_j\,, \tag{2.9}$$

for every $j \in J$. Let k denote the quotient map $k : \mathfrak{P} \longrightarrow \mathfrak{P}/\mathcal{F}$. Notice that

$$h \circ k \ \text{ is surjective.} \tag{2.10}$$

The following sequence proves that

$$\theta \stackrel{\text{def}}{=} ker(()h \circ k) = \equiv_{\hat{\mathcal{F}}} \ . \tag{2.11}$$

$$\begin{aligned}
\langle a,b\rangle \in \theta &\iff \langle h(a),h(b)\rangle \in ker(()k) = \equiv_{\mathcal{F}} \\
&\iff \{j\in J : h(a)(j) = h(b)(j)\} \in \mathcal{F} && \text{by def's of } k, \equiv_{\mathcal{F}} \\
&\iff \{j\in J : (a\lceil I_j)/\mathcal{F}_j = (b\lceil I_j)/\mathcal{F}_j\} \in \mathcal{F} && \text{by (2.9)} \\
&\iff \{j\in J : \{i\in I_j : (a\lceil I_j)(i) = (b\lceil I_j)(i)\} \in \mathcal{F}_j\} \in \mathcal{F} \\
&\iff \{j\in J : (\{i\in I : a(i) = b(i)\}\cap I_j) \in \mathcal{F}_j\} \in \mathcal{F} \\
&\iff \{i\in I : a(i) = b(i)\} \in \hat{\mathcal{F}} \\
&\iff \langle a,b\rangle \in \equiv_{\hat{\mathcal{F}}} \ .
\end{aligned}$$

We proved (2.11).

Let q denote the quotient map $q : \prod_{i\in I} \mathfrak{A}_i \longrightarrow \prod_{i\in I} \mathfrak{A}_i/\hat{\mathcal{F}}$. Applying the homomorphism theorem (Theorem 2.2.18), (2.10) and (2.11) together imply the existence of an isomorphism $g : \prod_{i\in I} \mathfrak{A}_i/\hat{\mathcal{F}} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{P}/\mathcal{F}$ such that $q\circ g = h\circ k$. This proves (2.8). $\square$

Exercise 2.2.61. Prove Lemma 2.2.60 for models instead of algebras.

Exercise 2.2.62. Prove that for any class K of similar algebras, (1)–(6) below hold.

(1) $\mathbf{Up}\mathsf{K} \subseteq \mathbf{HP}\mathsf{K}$

(2) $\mathbf{PH}\mathsf{K} \subseteq \mathbf{HP}\mathsf{K}$

(3) $\mathbf{SH}\mathsf{K} \subseteq \mathbf{HS}\mathsf{K}$

(4) $\mathbf{PP}\mathsf{K} = \mathbf{P}\mathsf{K}$

(5) $\mathbf{UpUp}\mathsf{K} = \mathbf{Up}\mathsf{K}$

(6) $\mathbf{UpS}\mathsf{K} \subseteq \mathbf{SUp}\mathsf{K}$, $\mathbf{P}^r\mathbf{S}\mathsf{K} \subseteq \mathbf{SP}^r\mathsf{K}$.

2.3 Categories

Category theory lends us a very useful way of looking at mathematical objects: To find/describe the properties of some object, instead of looking at it as to something consisting of elements which again are bulit up from other entities, forming all kinds of connections among each other, category theory would rather look at the object as a whole, and intends to grasp its properties via its interrelations *with other objects*, being "outside" of the original object. In other words, category theory treats the objects under investigation as "black-box"-s. Sometimes we want to use this way of thinking in this book, therefore, in this section, we introduce some very basic concepts of category theory.

Definition 2.3.1. (category) A category $\mathbb{C}$ consists of

- A (potentially proper) class $\mathbf{Ob}(\mathbb{C})$ the elements of which are called the *objects of* $\mathbb{C}$.

- To any two objects A, B a class $Hom_{\mathbb{C}}(A, B)$. It is called the collection of all morphisms going from A to B. Both

$$A \xrightarrow{f} B \quad \text{and} \quad f : A \longrightarrow B$$

mean that $f \in Hom_{\mathbb{C}}(A, B)$. $Hom_{\mathbb{C}}(A, B)$ is also called the *hom-class* from A to B.

If $A \xrightarrow{f} B$ then we call A the *domain* of f, and denote it by $dom(f)$, while B is called the *codomain* of f and is denoted by $cod(f)$. The class of all *morphisms* $\mathbf{Mor}(\mathbb{C})$ is defined as follows:

$$\begin{aligned}\mathbf{Mor}(\mathbb{C}) &\stackrel{\text{def}}{=} \bigcup\{Hom_{\mathbb{C}}(A, B) : A, B \in \mathbb{C}\} \\ &= \{f : f \in Hom_{\mathbb{C}}(A, B) \text{ for some objects} A \text{ and } B\}\,.\end{aligned}$$

- To any three objects A, B, C a function $\circ$, called *composition*, associating a morphisms $g \circ f$ to any two morphisms $f \in Hom_{\mathbb{C}}(A, B)$ and $g \in Hom_{\mathbb{C}}(B, C)$. That is,

$$\begin{array}{rcl}\circ : Hom_{\mathbb{C}}(A, B) \times Hom_{\mathbb{C}}(B, C) & \longrightarrow & Hom_{\mathbb{C}}(A, C) \\ \langle f, g \rangle & \mapsto & g \circ f\,.\end{array}$$

- To any object A a morphism 1_A, called the *identity morphism at* A, such that $1_A \in Hom_{\mathbb{C}}(A, A)$.

Concerning the above, we postulate the following two axioms:

(i) *Composition is associative*, that is, to any three morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

we have

$$h \circ (g \circ f) = (h \circ g) \circ f\,.$$

(ii) The *identity axiom* says that for any

$$A \xrightarrow{f} B \ \text{ and } \ B \xrightarrow{g} C\,,$$

$$1_B \circ f = f \ \text{ and } \ g \circ 1_B = g\,.$$

It is easy to see that for any $A \in \mathbf{Ob}(\mathbb{C})$, if $e \in Hom_{\mathbb{C}}(A, A)$ satisfies the identity axiom then $e = 1_A$. Namely, $1_A = 1_A \circ e = e$.

A *small category* is a category in which both $\mathbf{Ob}(\mathbb{C})$ and $\mathbf{Mor}(\mathbb{C})$ are actually sets and not proper classes. A category that is not small is said to be large. A *locally small category* is a category such that for all objects A and B, the hom-class $Hom_{\mathbb{C}}(A, B)$ is a set, called a *homset*. Many important categories in mathematics, although not small, are at least locally small.

The morphisms of a category are sometimes called *arrows* due to the influence of drawing them as arrows.

Examples 2.3.2. (Set, $\mathbf{Alg}_t$, $\mathbf{Mod}_t$)

(i) Our first example is the category **Set** of all sets together with functions between sets, where composition is the usual function composition. More precisely, **Ob**(**Set**) is the class of all sets,

$$\mathbf{Mor}(\mathbf{Set}) \stackrel{\text{def}}{=} \{\langle A, f, B\rangle \ : \ A, B \in \mathbf{Ob}(\mathbf{Set}) \text{ and } f \text{ is a function from } A \text{ into } B\},$$

where $dom(\langle A, f, B\rangle) = A$ and $cod(\langle A, f, B\rangle) = B$, further for any $A \in \mathbf{Ob}(\mathbf{Set})$, $1_A \stackrel{\text{def}}{=} Id_A$, and for any functions $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, $g \circ f$ is the usual function composition of f and g with domain A and codomain B. To prove that **Set** is a locally small category is left to the reader.

(ii) $\mathbf{Alg}_t$

(iii) $\mathbf{Mod}_t$

2.4 Free algebras

To be written later.

2.5 Variety characterization, quasi–variety characterization

If we are given a class K of similar algebras, often it is very useful to know what is the *equational theory* (the set of all equations valid in K) and/or the *quasi-equational theory* (the set of all quasi-equations valid in K) of the class K. In this section we give sufficient and necessary conditions for a class K to be definable by equations (Thm.2.5.10); and sufficient and necessary conditions for a class K to be definable by quasi-equations (Thm.2.5.11).

Definition 2.5.1. (term-algebra) Let t be a similarity type and let X be an arbitrary set with $X \cap Dom(t) = \emptyset$. Recall the set $Trm_t(X)$ of terms of similarity type t with variables from X from Definition 1.8.1. The *term-algebra* $\mathfrak{F}_t(X)$ *of similarity type* t *over* X is defined as follows.

(i) The universe of $\mathfrak{F}_t(X)$ is $Trm_t(X)$;

(ii) if $f \in Fns_t,\ t(f) = n$ and $\tau_1, \ldots, \tau_n \in Trm_t(X)$ then

$$f^{\mathfrak{F}_t(X)}(\tau_1, \ldots, \tau_n) \stackrel{\text{def}}{=} f(\tau_1, \ldots, \tau_n).$$

Exercise 2.5.2. Prove that if $|A| = |B|$ for some sets A, B then $\mathfrak{F}_t(A) \cong \mathfrak{F}_t(B)$.

Lemma 2.5.3. *Let t, X be as in Definition 2.5.1 For every algebra $\mathfrak{A} \in \mathsf{Alg}_t$ and function $g : X \longrightarrow A$ there exists a homomorphism $h : \mathfrak{F}_t(X) \longrightarrow \mathfrak{A}$ such that $h \lceil X = g$.*

Proof. We define h by the following recursion:

- For any $x \in X$, $h(x) \stackrel{\text{def}}{=} g(x)$;
- for any $f \in Fns_t$, $t(f) = n$ and $\tau_1, \dots, \tau_n \in Trm_t(X)$,

$$h\big(f(\tau_1, \dots, \tau_n)\big) \stackrel{\text{def}}{=} f^{\mathfrak{A}}\big(h(\tau_1), \dots, h(\tau_n)\big).$$

Then h is a homomorphism with the required properties. □

Definition 2.5.4. We say that a formula φ (φ may be, e.g., an equation or a quasi–equation) is *preserved under an operator* like, e.g., **P** iff for any class K of (similar) algebras,

$$\mathsf{K} \models \varphi \Longrightarrow \mathbf{P}\mathsf{K} \models \varphi\,.$$

Similarly, φ is preserved under **SP** iff

$$\mathsf{K} \models \varphi \Longrightarrow \mathbf{SP}\mathsf{K} \models \varphi\,.$$

The definition for other operations like **H**, $\mathbf{P}^r$ etc. is similar.

By the Łos ultraproduct theorem (Theorem 2.2.57), every first–order formula is preserved under **Up**.

Exercises 2.5.5. (1) Let t be a similarity type containing two constant symbols: 0 and 1. Prove that the formula $(x = 0 \vee x = 1)$ (written in this language) is preserved under **S** but not under **P**.

(2) Prove that equations are preserved under **S** and under **H**.

(3) Prove that equations are preserved under **P**. Use this and (2) above to prove that equations are preserved under **HSP**.

(4) Prove that **S**, **P**, **Up** preserve quasi–equations.

Definition 2.5.6. Let $\mathsf{K} \subseteq \mathsf{Alg}_t$. Then both $Eq_t(\mathsf{K})$ and $Eq_t\mathsf{K}$ denote the set of all equations of similarity type t which are valid in K, that is,

$$Eq_t\mathsf{K} = Eq_t(\mathsf{K}) \stackrel{\text{def}}{=} \{e : \mathsf{K} \models e \text{ and } e \text{ is an equation of similarity type } t\}\,.$$

$Qeq_t(\mathsf{K})$ and $Qeq_t\mathsf{K}$ denote the set of all quasi–equations valid in K, that is,

$$Qeq_t\mathsf{K} = Qeq_t(\mathsf{K}) \stackrel{\text{def}}{=} \{q : \mathsf{K} \models q \text{ and } q \text{ is a quasi–equation of similarity type } t\}\,.$$

Convention 2.5.7. From now on, by a class K of algebras we always mean a class of *similar* algebras.

Definition 2.5.8. (variety, quasi–variety) A class of algebras of similarity type t is called a *variety* (*quasi–variety*) iff it can be defined by a set of equations (quasi–equations) of type t. That is, K is a variety iff

$$(\exists \text{ set } \Sigma \text{ of equations})(\mathfrak{A} \in \mathsf{K} \Longleftrightarrow \mathfrak{A} \models \Sigma)\,;$$

and K is a quasi–variety iff

$$(\exists \text{ set } \Gamma \text{ of quasi–equations})(\mathfrak{A} \in \mathsf{K} \Longleftrightarrow \mathfrak{A} \models \Gamma)\,.$$

We will also use the expressions *equational class* and *quasi–equational class* as synonyms of "variety" and "quasi–variety", respectively.

$\mathsf{Mod}(Eq_t\mathsf{K})$ $(\mathsf{Mod}(Qeq_t\mathsf{K}))$ denotes the class of all models of the equational theory $Eq_t\mathsf{K}$ (quasi–equational theory $Qeq_t\mathsf{K}$, respectively). $\mathsf{Mod}(Eq_t\mathsf{K})$ $(\mathsf{Mod}(Qeq_t\mathsf{K}))$ is called the *variety* (*quasi–variety*) *generated* by K.

Example 2.5.9. It can be proven that the class BA of all Boolean algebras can be defined by finitely many equations, thus it is a variety. Such an equational axiomatization will be given in section 2.7.

Theorem 2.5.10. (Variety characterization) *Let* $\mathsf{K} \subseteq \mathsf{Alg}_t$. *Then*

$$\mathsf{M}od(Eq_t\mathsf{K}) = \mathbf{HSP}\mathsf{K}\,.$$

Thus, V *is a variety iff*

$$\mathsf{V} = \mathbf{HSP}\mathsf{V}\,.$$

Proof. $\mathbf{HSP}\mathsf{K} \subseteq \mathsf{Mod}(Eq_t\mathsf{K})$, because equations are preserved under $\mathbf{HSP}$ (cf. Exercise 2.5.5 (3) above).

To prove $\mathsf{Mod}(Eq_t\mathsf{K}) \subseteq \mathbf{HSP}\mathsf{K}$, let $\mathfrak{A} \in \mathsf{Mod}(Eq_t\mathsf{K})$. Consider the term-algebra $\mathfrak{F}_t(A)$ over the universe A of $\mathfrak{A}$. Then, by Lemma 2.5.3, there is a surjective homomorphism $f : \mathfrak{F}_t(A) \twoheadrightarrow \mathfrak{A}$ which is an extension of the identity map on A.

Now define for all $\sigma, \tau \in Trm_t(A)$

$$\sigma \equiv \tau \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathsf{K} \models (\sigma = \tau).$$

Clearly, $\equiv$ is a congruence relation on $\mathfrak{F}_t(A)$ (check it!). We define a function $h : Trm_t(A)/\equiv \longrightarrow A$ by

$$h(\tau/\equiv) \stackrel{\text{def}}{=} f(\tau) \quad \text{for every } \tau \in Trm_t(A).$$

This definition is sound, since $\equiv \subseteq ker(f)$ by $\mathfrak{A} \in \mathsf{Mod}(Eq_t\mathsf{K})$, and h is a homomorphism from $\mathfrak{F}_t(A)/\equiv$ to $\mathfrak{A}$ (check it, cf. the proof of the Homomorphism Theorem: Theorem 2.2.18). h is surjective, because f is surjective, thus $\mathfrak{A}$ is a homomorphic image of $\mathfrak{F}_t(A)/\equiv$.

$$\begin{tikzcd}
\mathfrak{F}_t(A) \arrow[r, two heads, "f"] \arrow[dr, "q"'] & \mathfrak{A} & \\
& \mathfrak{F}_t(A)/\equiv \arrow[u, "h"'] \arrow[r, tail, "g"] & \prod\limits_{\mathfrak{A}_e \in \mathsf{K}_0} \mathfrak{A}_e \arrow[d] \\
& & \mathfrak{A}_e \quad (\in \mathsf{K}_0)
\end{tikzcd}$$

Figure 2.25: Proof of the Variety Characterization theorem

Next we prove that $\mathfrak{F}_t(A) \in \mathbf{SP}\mathsf{K}$. First, there exists a *set* $\mathsf{K}_0 \subseteq \mathsf{K}$ such that $Eq_t\mathsf{K}_0 = Eq_t\mathsf{K}$, because there are only "set many" equations of similarity type t. Namely, if e is an equation of similarity type t such that $\mathsf{K} \not\models e$ then let $\mathfrak{A}_e \in \mathsf{K}$ and let v_e be a valuation of the variables in $\mathfrak{A}_e$ such that $\mathfrak{A}_e \not\models e[v_e]$. Let

$$\mathsf{K}_0 \stackrel{\text{def}}{=} \{\mathfrak{A}_e \;:\; e \text{ is an equation of similarity type } t \text{ and } \mathsf{K} \not\models e\}.$$

Clearly $Eq_t\mathsf{K}_0 = Eq_t\mathsf{K}$ and $\mathsf{K}_0 \subseteq \mathsf{K}$ is a set, because there are only set-many equations.

We define the function $g : Trm_t(A)/\equiv \;\longrightarrow \Pi_{\mathfrak{A}_e \in \mathsf{K}_0} A_e$ as follows.

$$g(\tau/\equiv) \stackrel{\text{def}}{=} \langle \ldots, \tau^{\mathfrak{A}_e}[v_e], \ldots \rangle_{\mathfrak{A}_e \in \mathsf{K}_0}.$$

This definition is sound, since if $\tau \equiv \sigma$ then $\mathsf{K} \models (\tau = \sigma)$ holds, therefore for all $\mathfrak{A}_e \in \mathsf{K}_0$ we have $\mathfrak{A}_e \models (\tau = \sigma)[v_e]$, that is, $\tau^{\mathfrak{A}} \circ [v_e] = \sigma^{\mathfrak{A}} \circ [v_e]$. We show that g is an injective homomorphism (an embedding) from $\mathfrak{F}_t(A)/\equiv$ into $\Pi_{\mathfrak{A}_e \in \mathsf{K}_0} \mathfrak{A}_e$.

g is a homomorphism: If $k \in Fns_t,\ t(k) = n$ and $\tau_1, \ldots, \tau_n \in Trm_t(A)$ then

$$\begin{aligned} g\big(k(\tau_1/\equiv, \ldots, \tau_n/\equiv)\big)_{\mathfrak{A}_e} &= g\big(k(\tau_1, \ldots, \tau_n)/\equiv\big)_{\mathfrak{A}_e} = k(\tau_1, \ldots, \tau_n)^{\mathfrak{A}_e}[v_e] = \\ &k^{\mathfrak{A}_e}(\tau_1^{\mathfrak{A}_e}[v_e], \ldots, \tau_n^{\mathfrak{A}_e}[v_e]) = k^{\mathfrak{A}_e}\big(g(\tau_1/\equiv)_{\mathfrak{A}_e}, \ldots, g(\tau_n/\equiv)_{\mathfrak{A}_e}\big). \end{aligned}$$

g is injective: Assume $\sigma \not\equiv \tau$. Then $\mathfrak{A}_{(\sigma=\tau)} \not\models (\sigma = \tau)[v_{(\sigma=\tau)}]$ for some algebra $\mathfrak{A}_{(\sigma=\tau)} \in \mathsf{K}_0$, that is, the $\mathfrak{A}_{(\sigma=\tau)}$ coordinates of $g(\tau/\equiv)$ and $g(\sigma/\equiv)$ are different.

Thus $\mathfrak{F}_t(A)/\!\equiv\, \in \mathbf{ISP}\mathsf{K}$ that is, $\mathfrak{A} \in \mathbf{HSP}\mathsf{K}$. ☐

Theorem 2.5.11. (Quasi–variety characterization) *For any class* K *of similar algebras*

$$\mathsf{M}od\,Qeq(\mathsf{K}) = \mathbf{SPUp}\mathsf{K} = \mathbf{SP}^r\mathsf{K}\,.$$

For proving Theorem 2.5.11, we need Lemmas 2.5.12, 2.5.13 below.

Let t be an arbitrary similarity type. Let I be an arbitrary set, and let e, e_i $(i \in I)$ be equations of type t. Then we call the formula $\underset{i\in I}{\wedge} e_i \to e$ an *infinitary quasi–equation*. If $\mathfrak{A} \in \mathsf{Mod}_t$, k is a valuation of the variables into A, then

$$\mathfrak{A} \models \underset{i\in I}{\wedge} e_i \to e[k] \quad \overset{\text{def}}{\Longleftrightarrow} \quad \mathfrak{A} \not\models e_i\,[k] \;\text{ for some } i \in I\text{, or } \mathfrak{A} \models e[k]\,.$$

Notice that if I is finite then $\underset{i\in I}{\wedge} e_i \to e$ is a quasi–equation. Also notice that if $I_0 \subseteq I$ then $\underset{i\in I_0}{\wedge} e_i \to e \models \underset{i\in I}{\wedge} e_i \to e$.

For any class K of similar algebras we define

$$Qeq_\infty(\mathsf{K}) \overset{\text{def}}{=} \{\varphi \;:\; \varphi \text{ is an infinitary quasi–equation and } \mathsf{K} \models \varphi\}\,.$$

Lemma 2.5.12. *For any class* K *of similar algebras,*

$$\mathsf{M}od\,Qeq(\mathsf{K}) \subseteq \mathsf{M}od\,Qeq_\infty(\mathbf{Up}\mathsf{K})\,.$$

Proof. Let $\underset{i\in I}{\wedge} e_i \to e$ be an arbitrary infinitary quasi–equation. We will show that

$$\text{if } \mathbf{Up}\mathsf{K} \models \underset{i\in I}{\wedge} e_i \to e \text{ then there is a finite } I_0 \subseteq I \text{ such that } \mathsf{K} \models \underset{i\in I_0}{\wedge} e_i \to e\,. \tag{2.12}$$

Assume that (2.12) holds, and let $\mathfrak{A} \in \mathsf{Mod}\,Qeq(\mathsf{K})$, $\underset{i\in I}{\wedge} e_i \to e \in Qeq_\infty(\mathbf{Up}\mathsf{K})$. Then, by (2.12), there is a finite $I_0 \subseteq I$ such that $\mathsf{K} \models \underset{i\in I_0}{\wedge} e_i \to e$. Then $\mathfrak{A} \models \underset{i\in I_0}{\wedge} e_i \to e$, and thus $\mathfrak{A} \models \underset{i\in I}{\wedge} e_i \to e$ by $\underset{i\in I_0}{\wedge} e_i \to e \models \underset{i\in I}{\wedge} e_i \to e$. Therefore $\mathfrak{A} \in \mathsf{Mod}\,Qeq_\infty(\mathbf{Up}\mathsf{K})$, since we choose $\underset{i\in I}{\wedge} e_i \to e$ arbitrarily. This proves Lemma 2.5.12. It remains to prove (2.12).

To see (2.12), let $J = \{j \subseteq I \;:\; |j| < \omega\}$, and assume that $\mathsf{K} \not\models \underset{i\in j}{\wedge} e_i \to e$ for every $j \in J$. Then

$$(\forall j \in J)(\exists \mathfrak{A}_j \in \mathsf{K})(\exists \text{ valuation } k_j \text{ of the variables into } A_j)$$
$$\left(\mathfrak{A}_j \models \underset{i\in j}{\wedge} e_i[k_j] \;\text{ but }\; \mathfrak{A}_j \not\models e[k_j]\right).$$

For every $j \in J$, let such an $\mathfrak{A}_j$ be fixed. Let $\mathcal{F}$ be an ultrafilter over J, $\mathfrak{B} \overset{\text{def}}{=} \Pi\langle \mathfrak{A}_j \;:\; j \in J\rangle/\mathcal{F}$, and $k(x) \overset{\text{def}}{=} \langle k_j(x) \;:\; j \in J\rangle/\mathcal{F}$ for every variable x. Then $\mathfrak{B} \models \underset{i\in I}{\wedge} e_i[k]$ but $\mathfrak{B} \not\models e[k]$ by Theorem 2.2.57 (ii). Since $\mathfrak{B} \in \mathbf{Up}\mathsf{K}$, this implies that $\mathbf{Up}\mathsf{K} \models \underset{i\in I}{\wedge} e_i \to e$ does not hold. This proves (2.12). ◻

$$\begin{tikzcd}
& V \arrow[d, hook, "g"] \arrow[ddl, "k"'] & \\
& \mathfrak{A} \arrow[r, "f"] \arrow[dl, bend right, "k'"'] \arrow[dl, bend left, "h_i"] & \mathfrak{B} \arrow[dll, bend left, "p_i"] \\
(\mathsf{K} \ni)\ \mathfrak{C} & &
\end{tikzcd}$$

Figure 2.26: Proof of Claim

Lemma 2.5.13. *For any class of similar algebras* K,

$$\mathsf{M}od\,Qeq_\infty(\mathsf{K}) \subseteq \mathbf{SP}\mathsf{K}\,.$$

Proof. Let $\mathfrak{A} \in \mathsf{Mod}\,Qeq_\infty(\mathsf{K})$. Let

$$\begin{aligned} H \stackrel{\text{def}}{=} \{h : h &\text{ is a homomorphism} \\ &\text{with } Dom(h) = A \text{ and } Rng(h) \subseteq C \text{ for some } \mathfrak{C} \in \mathsf{K}\}\,. \end{aligned}$$

For arbitrary $h_1, h_2 \in H$, we say that h_1 and h_2 are equivalent if $ker(h_1) = ker(h_2)$. There are only set many non–equivalent homomorphisms in H, because $Con(\mathfrak{A})$ is a set and $(\forall h \in H) ker(h) \in Con(\mathfrak{A})$. Thus we can take a system $\langle h_i \ : \ i \in I\rangle$ of homomorphisms from the class H indexed by some *set* I such that $\{ker(h_i) \ : \ i \in I\} = \{ker(h) \ : \ h \in H\}$. For every $i \in I$, let $\mathfrak{A}_i$ be an element of K such that $Rng(h_i) \subseteq A_i$. Let $\mathfrak{B} \stackrel{\text{def}}{=} \Pi_{i\in I}\mathfrak{A}_i$. Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be the homomorphism induced by the product $\mathfrak{B}$. (Cf. Lemma 2.2.23 (iv).) Thus

$$(\forall i \in I) f \circ p_i = h_i\,, \tag{2.13}$$

where p_i is the i–th projection. It is enough to prove that f is injective.

Let V denote a set of variables. Let $g : V \rightarrowtail\!\!\!\!\twoheadrightarrow A$ be a bijection. Let $Diag^+(\mathfrak{A})$ denote the set of all atomic formulas true in $\mathfrak{A}$ at the valuation g of the variables. (We call $Diag^+(\mathfrak{A})$ the *positive diagram* of $\mathfrak{A}$.)

Claim: For any $x, y \in Dom(g)$, if $f(gx) = f(gy)$ then

$$\mathsf{K} \models Diag^+(\mathfrak{A}) \to x = y\,.$$

Proof. The proof of the Claim is illustrated on Figure 2.26.

Let $x, y \in V$ be arbitrary but fixed. Assume

$$f(gx) = f(gy)\,. \tag{2.14}$$

Let $\mathfrak{C} \in \mathsf{K}$ be such that

$$\mathfrak{C} \models Diag^{+}(\mathfrak{A})[k] \tag{2.15}$$

for some valuation k of the variables. Let $k' : A \longrightarrow C$ be the function defined as follows:

$$(\forall a \in A) k'(a) \stackrel{\text{def}}{=} k(g^{-1}a)\,.$$

Because of (2.15), k' is a homomorphism. Then $k' \in H$, and

$$ker(k') = ker(h_i) \tag{2.16}$$

for some $i \in I$. Then $h_i(gx) = h_i(gy)$ by (2.14) and (2.13), therefore $k'(gx) = k'(gy)$ by (2.16). Then $k(x) = k(y)$ be the definition of k'. Thus $\mathfrak{C} \models Diag^{+}(\mathfrak{A}) \rightarrow x = y[k]$, which proves the claim, since $\mathfrak{C}$ and k were chosen arbitrarily. □

Now assume that $a, b \in A$, $f(a) = f(b)$. Then there are variables $x, y \in V$ such that $g(x) = a$ and $g(y) = b$, and by our assumption we have $f(g(x)) = f(g(y))$. Then, by the above Claim, $\mathsf{K} \models Diag^{+}(\mathfrak{A}) \rightarrow x = y$. Notice that $Diag^{+}(\mathfrak{A}) \rightarrow x = y$ is an infinitary quasi–equation. Thus, since $\mathfrak{A} \in \mathsf{Mod}\,Qeq_{\infty}\mathsf{K}$, $\mathfrak{A} \models Diag^{+}(\mathfrak{A}) \rightarrow x = y$ as well. Therefore, in particular, $\mathfrak{A} \models Diag^{+}(\mathfrak{A}) \rightarrow x = y\ [q]$ for any valuation q such that $q(x) = g(x)$ and $q(y) = g(y)$. Then $g(x) = g(y)$, since $\mathfrak{A} \models Diag^{+}(\mathfrak{A})\ [q]$. Thus $a = b$, proving that f is an injection. This completes the proof of Lemma 2.5.13. □

Proof. Proof of Theorem 2.5.11: $\mathbf{SPUp}\mathsf{K} \subseteq \mathsf{Mod}\,Qeq(\mathsf{K})$ holds, because $\mathbf{S}, \mathbf{P}, \mathbf{Up}$ preserve quasi–equations (cf. Exercise 2.5.2 (4)).

To see $\mathsf{Mod}\,Qeq(\mathsf{K}) \subseteq \mathbf{SPUp}\mathsf{K}$, let $\mathfrak{A} \in \mathsf{Mod}\,Qeq(\mathsf{K})$. By Lemmas 2.5.12 and 2.5.13, $\mathfrak{A} \in \mathsf{Mod}\,Qeq_{\infty}(\mathbf{Up}\mathsf{K}) \subseteq \mathbf{SPUp}\mathsf{K}$. □

Corollary 2.5.14. *For any class of similar algebras,*

$$\mathsf{M}od\,Qeq(\mathsf{K}) = \mathbf{SP^{r}}\mathsf{K}\,.$$

Proof. This follows directly from Theorem 2.5.11 and Lemma 2.2.60. □

2.6 Discriminator varieties

⇒ Már ebben a fejezetben nagyon relevánsak a Boole algebrák!! Eltoprengeni, hogyan legyen. Ld. BA-s exerciseket ebben a fejezetben. Ide valo az az exerc. is, hogy BA discr varietas
Bevezető szöveg!

Definition 2.6.1. (discriminator term, discriminator variety)

(i) A class K of algebras is said to *have a discriminator term* iff there is a term $\tau(x, y, z, u)$ in the language of K such that in every member of K we have

$$\tau(x, y, z, u) = \begin{cases} z, & \text{if } x = y, \\ u, & \text{otherwise.} \end{cases}$$

(ii) A variety V is called a *discriminator variety* if the class $\mathsf{Sir}(\mathsf{V})$ of subdirectly irreducible members of V has a discriminator term.

Sometimes, instead of the 4–ary τ, the ternary discriminator term $t(x, y, z) = \tau(x, y, z, x)$ is used. They are interdefinable, since

$$\tau(x, y, z, u) = t(t(x, y, z), t(x, y, u), z).$$

Therefore, it does not matter which one is used.

Exercises 2.6.2. (1) Show that if K has a discriminator term then K consists of simple algebras.

(2) Assume that the Boolean operations $-, \wedge, 0, 1$ are available in K and that they satisfy the Boolean axioms (i.e. every element of K is a Boolean algebra possibly with some further operations). This property will be referred to as 'K *has a Boolean reduct*' or that the elements of K are *Boolean ordered algebras*. Prove that K has a discriminator term iff there is a term $c(x)$ in the language of K such that

$$c(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{otherwise} \end{cases}$$

in every member of K.

(*Hint:* $\tau(x, y, z, u) = \big[c(x \oplus y) \wedge u\big] \vee \big[z \wedge -c(x \oplus y)\big]$, where $\oplus$ denotes symmetric difference.)

Theorem 2.6.3. *Let* K *be a class of similar algebras. Assume that* K *has a discriminator term. Then*

$$\mathbf{HSP}\mathsf{K} = \mathbf{SPUp}\mathsf{K}.$$

The proof we give here is a model theory oriented, more intuitive proof as opposed to the more "computational" proofs in the standard universal algebra books. The idea of giving here such a more intuitive proof was suggested us by Johan van Benthem.

In the proof we will use Proposition 2.6.4 below.

Proposition 2.6.4. *Assume* K *has a discriminator term. Then to every quasi–equation* q *in the language of* K *there is an equation* $e(q)$ *such that (i)–(ii) below hold:*

(i) $\mathsf{K} \models e(q) \leftrightarrow q$,

(ii) $\mathbf{HSP}\mathsf{K} \models e(q) \to q$.

Proof. By the *length of a quasi–equation* we mean the number of the equations on the left hand side of the implication of the quasi–equation. The proof goes by induction on the length of quasi–equations. For any quasi–equation q, $\mathit{length}(q)$ denotes its length.

If $\mathit{length}(q) = 0$ for a quasi–equation q, then it is an equation, and then statements (i) and (ii) are obvious.

The induction step: Assume that for any quasi–equation q' we have:

$$\mathit{length}(q') < n \Longrightarrow \\ (\exists \text{ equation } e(q'))\,(\mathsf{K} \models e(q') \leftrightarrow q' \ \&\ \mathbf{HSP}\mathsf{K} \models e(q') \to q') \ . \tag{2.17}$$

Let q be a quasi–equation of length n, say q is

$$(\sigma_1 = \rho_1 \wedge \cdots \wedge \sigma_n = \rho_n) \to \sigma_0 = \rho_0 \,.$$

Then q is equivalent to

$$\sigma_1 = \rho_1 \to q' \,,$$

where q' is the quasi–equation $(\sigma_2 = \rho_2 \wedge \cdots \wedge \sigma_n = \rho_n) \to \sigma_0 = \rho_0$. Since $\mathit{length}(q') < n$, by the induction hypothesis (IH) above we have

$$\mathsf{K} \models \sigma = \rho \leftrightarrow q' \qquad \text{and} \qquad \mathbf{HSP}\mathsf{K} \models \sigma = \rho \to q' \tag{2.18}$$

for some equation $\sigma = \rho$.

Let τ denote the discriminator term of K, and we define $e(q)$ to be

$$\tau(\sigma_1, \rho_1, \sigma, \rho) = \rho \,.$$

First we prove that

$$\mathsf{K} \models q \to e(q) \,. \tag{2.19}$$

To see this, let $\mathfrak{A} \in \mathsf{K}$ and the valuation $\bar{a} \in {}^{\omega}A$ of the variables into A be arbitrary. It is enough to prove that

$$\mathfrak{A} \models \neg e(q) \to \neg q[\bar{a}] \,. \tag{2.20}$$

Assume $\mathfrak{A} \models \neg e(q)[\bar{a}]$, that is, $\mathfrak{A} \models \tau(\sigma_1, \rho_1, \sigma, \rho) \neq \rho[\bar{a}]$. Since τ is a discriminator term, this implies

$$\mathfrak{A} \models \sigma_1 = \rho_1[\bar{a}] \tag{2.21}$$

and

$$\mathfrak{A} \models \sigma \neq \rho[\bar{a}] \,.$$

By the latter and (2.18),

$$\mathfrak{A} \not\models q'[\bar{a}]\,. \tag{2.22}$$

By (2.21) and (2.22), $\mathfrak{A} \not\models q[\bar{a}]$, proving (2.20). This proves (2.19).

Next we are proving

$$\mathbf{HSP}\mathsf{K} \models \neg q \to \neg e(q)\,. \tag{2.23}$$

To do this, let $\mathfrak{A} \in \mathbf{HSP}\mathsf{K}$ and $\bar{a} \in {}^{\omega}A$, and assume that $\mathfrak{A} \not\models q[\bar{a}]$. Then

$$\mathfrak{A} \models \sigma_1 = \rho_1[\bar{a}] \tag{2.24}$$

and $\mathfrak{A} \not\models q'[\bar{a}]$, thus

$$\mathfrak{A} \models \sigma \neq \rho[\bar{a}] \tag{2.25}$$

by (the second part of) (2.18). Now $\mathfrak{A} \models \tau(\sigma_1, \rho_1, \sigma, \rho) = \sigma \neq \rho[\bar{a}]$ by (2.24) and (2.25). This proves $\mathfrak{A} \not\models e(q)[\bar{a}]$, proving (2.23), since $\mathfrak{A}$ and $\bar{a}$ were chosen arbitrarily. We have proved (2.23).

(2.19) and (2.23) together prove Proposition 2.6.4. □

Proof. **Proof of Theorem 2.6.3:** First we prove that $\mathbf{HSP}\mathsf{K} \models Qeq(\mathsf{K})$. Indeed, let $q \in Qeq(\mathsf{K})$. Then, by Proposition 2.6.4 (i), there is an equation $e(q)$ with $\mathsf{K} \models \big(e(q) \leftrightarrow q\big)$. Since equations are preserved under **HSP**, $\mathbf{HSP}\ \mathsf{K} \models e(q)$ also holds. Now by Proposition 2.6.4 (ii), $\mathbf{HSP}\mathsf{K} \models q$.

Now

$$\mathbf{HSP}\mathsf{K} \subseteq \mathsf{Mod}\,Qeq(\mathsf{K}) = \mathbf{SPUp}\mathsf{K}\,,$$

by Thm.2.5.11, which proves one direction of Theorem 2.6.3.

The other direction, $\mathbf{SPUp}\mathsf{K} \subseteq \mathbf{HSP}\mathsf{K}$ is straightforward, since, by Exercise 2.2.62, $\mathbf{Up} \subseteq \mathbf{HP}$, $\mathbf{PH} \subseteq \mathbf{HP}$, $\mathbf{SH} \subseteq \mathbf{HS}$, and $\mathbf{PP} = \mathbf{P}$. □

Exercise 2.6.5. Check how much one can simplify the proof of Theorem 2.6.3 above if we assume that K has a Boolean reduct (cf. item (2) of Exercises 2.6.2 above).

Corollary 2.6.6. *Assume* K *has a discriminator term. Then*

(i) K *is contained in some discriminator variety.*

(ii) The subdirectly irreducible members of **HSP** K *are exactly the subdirectly irreducibles of* $\mathbf{SUp}\mathsf{K}$.

Proof. (ii): Let $\mathfrak{A}$ be a subdirectly irreducible member of $\mathbf{HSP}\mathsf{K}$. By Theorem 2.6.3, $\mathfrak{A} \in \mathbf{SP}(\mathbf{Up}\mathsf{K})$. Then $\mathfrak{A}$ is a subdirect product of algebras from $\mathbf{SUp}\mathsf{K}$. By irreducibility, then $\mathfrak{A} \in \mathbf{SUp}\mathsf{K}$. This proves (ii).

(i): The discriminator term τ which works for K also works for $\mathbf{SUp}\mathsf{K}$, since the discriminator property

$$\forall x, y, z, u\ \big([x \neq y \Rightarrow \tau(x,y,z,u) = u] \wedge [x = y \Rightarrow \tau(x,y,z,u) = z]\big)$$

is defined by a universal formula, thus is preserved under **SUp**. Thus **SUp**K has a discriminator term. But by (ii) the class $\mathsf{Sir}(\mathbf{HSP}\mathsf{K})$ of subdirectly irreducibles of **HSP**K is in **SUp**K. Then by definition, **HSP**K is a discriminator variety. $\square$

Exercises 2.6.7. (i) Prove that $(d_1 - d_2)$ below are equivalent with saying that τ is a discriminator term for K.

$$\tau(x, x, z, u) = z \tag{2.26}$$

$$\tau(x, y, z, u) = u \vee x = y\,. \tag{2.27}$$

(ii) Prove that if τ is a discriminator term then also $(d_3 - d_5)$ below are true.

$$\tau(x, y, z, u) = z \rightarrow (x = y \vee z = u \tag{2.28}$$

$$\tau(x, y, z, u) = z \vee \tau(x, y, z, u) = u \tag{2.29}$$

$$\tau(x, y, z, z) = z\,. \tag{2.30}$$

Exercises 2.6.8. Assume τ is a discriminator term for K.

(i) Prove that **HSP**$\mathsf{K} \models (d_1 \wedge d_5)$ (for (d_1) and (d_5) see previous exercise).

(ii) Prove that **HS**$\mathsf{K} \models (d_1, d_2)$.

Hint: Basic universal algebra:

(i) Equations are preserved under **H**, **S** and **P**.

(ii) Quasi–equations (implications between equations) are preserved under **S** and **P**.

(iii) Disjunctions of equations are preserved under **H** and **S**.

Exercise 2.6.9. Assume τ is a discriminator term for K. Prove that in **HSP**K (the variety generated by K) d_1 and d_3 are valid.

Hint: Use Theorem 2.5.11 above saying that **SPUp**K = **HSPUp**K if K has a discriminator. Then apply Exercise 2.6.8 (ii) above together with the fact that **Up** preserves all formulas, hence **Up**$\mathsf{K} \models (d_1 - d_3)$.

2.7 Boolean Algebras

Vigyazat, ez a fejezet ideiglenes, at kell gondolni rendesen. Ez egy fontos fejezet.

The basic theory of Boolean algebras (BA's for short) can be found in any textbook on universal algebra, e.g. in Burris et al. [25] or in Halmos's book [36] or in the handbook [44]. For the reader not familiar with BA's, we would like to point out that Halmos's little book [36] is a very easily readable introduction. Since the literature is so rich, we do not reproduce it here, instead we establish only notation and terminology to be used later, and give some basic theorems with outlines of proofs only.

As we already wrote in previous sections, the similarity type of $\mathsf{BA}'s$ is $ba = \{\langle \vee, 2\rangle, \langle -, 1\rangle\}$. We defined BA's in the following steps: the class of *powerset* BA's was defined to be

$$\{\mathfrak{P}(U) : \mathfrak{P}(U) = \langle \mathcal{P}(U), \cup, -\rangle \ \text{ for some set } U\}\,,$$

where set theoretic union $\cup$ is the interpretation of $\vee$, and the set theoretic complementation $-$ is the interpretation of the symbol $-$ (denoted both the same way, ambiguously). (Cf. Examples 2.1.7 4.) Then we defined *Boolean set algebras* in Definition 2.2.3 as

$$\mathsf{SetBA} = \mathbf{S}\{\mathfrak{P}(U) : U \text{ is a set}\}\,.$$

Finally, in Definition 2.2.11, BA's were defined as

$$\mathsf{BA} = \mathbf{I}\mathsf{SetBA}\,.$$

We use the following derived operations:

$$\begin{aligned} x \wedge y &= -(-x \vee -y) \\ 0 &= x \wedge -x \\ 1 &= x \vee -x \\ x - y &= x \wedge -y \\ x \oplus y &= (x - y) \vee (y - x)\,. \end{aligned}$$

As can be seen, we use the complementation symbol $-$ in two different senses, both as a unary operation (this is the basic one), and sometimes also as a binary operation abbreviating $x \wedge -y$. We hope context will help to avoid misunderstanding.

As it is usual, we define the ordering $\leqslant$ of BA's as:

$$x \leqslant y \overset{\text{def}}{\iff} x \vee y = y\,.$$

The "*generic*" BA is the following:

$$\underset{\sim}{\mathbf{2}} = \mathfrak{P}(\{0\})\,.$$

Exercises 2.7.1. Recall that $1 = \{0\}$.

(1) Prove that $\underset{\sim}{\mathbf{2}} = \mathfrak{P}(1)$ has exactly 2 elements.

(2) Prove that $\underset{\sim}{\mathbf{2}}$ has no subalgebras (except itself).

(3) Prove that $\underset{\sim}{\mathbf{2}}$ has no nontrivial homomorphic images.

(4) Prove that there exists a one–element BA.

(5) Prove that there is no 3–element BA.

(6) Prove that in any Alg_t there is only one one–element algebra up to isomorphism. (The expression "up to isomorphism" will be used quite often, and its above use makes the above sentence mean that any two one–element algebras are isomorphic.)

(7) Prove that there is only one two–element BA (up to isomorphism).

(8)* Prove that there is at most one 4–element BA (up to isomorphism).

Definition 2.7.2. Let $\mathfrak{A} \in \mathsf{BA}$. By a *filter* of $\mathfrak{A}$ we understand a congruence class of $1^{\mathfrak{A}}$. By an *ultrafilter* of $\mathfrak{A}$ we understand a maximal filter, that is, the kernel of a homomorphisms of $\mathfrak{A}$ onto $\underset{\sim}{\mathbf{2}}$.

Lemma 2.7.3. characterization of filters and ultrafilters *Let* $\mathfrak{A}$ *be an arbitrary* BA.

(i) $\mathcal{F}$ *is a filter of* $\mathfrak{A}$ *iff the following conditions hold for all* $a, b \in A$.

$$\begin{gathered} 1 \in \mathcal{F} \\ a \in \mathcal{F}, a \leqslant b \Rightarrow b \in \mathcal{F} \\ a, b \in \mathcal{F} \Rightarrow a \wedge b \in \mathcal{F}\,. \end{gathered}$$

(ii) $\mathcal{F}$ *is an ultrafilter of* $\mathfrak{A}$ *iff it is a filter of* $\mathfrak{A}$ *and for all* $a \in A$,

$$a \in \mathcal{F} \Longleftrightarrow -a \in A \smallsetminus \mathcal{F}\,.$$

Proof. The **proof** is left to the reader. □

Definition 2.7.4. Let Bax be the following set of axioms:

$$x \wedge y = -(-x \vee -y) \tag{2.31}$$
$$x \wedge y = y \wedge x \tag{2.32}$$
$$x - (-y - z) = -(-(x \wedge y) - (x \wedge z)) \tag{2.33}$$
$$x - (y - y) = x\,. \tag{2.34}$$
$$\tag{2.35}$$

Theorem 2.7.5. Stone's axiomatizability theorem BA *is a finitely axiomatizable variety. Moreover,*

$$\mathsf{BA} = \mathsf{M}od(Bax)\,,$$

where Bax is the set of axioms given in Definition 2.7.4 above.

Proof. **Outline of proof:** $\mathsf{BA} \subseteq \mathsf{Mod}(Bax)$ is easy to check. To see $\mathsf{Mod}(Bax) \subseteq \mathsf{BA}$, let $\mathfrak{A} \in \mathsf{Mod}(Bax)$ be arbitrary, and we define the following function rep: for every $a \in A$,

$$\mathrm{rep}(a) \stackrel{\mathrm{def}}{=} \{U \;:\; U \text{ is an ultrafilter of } \mathfrak{A} \text{ and } a \in U\}\,.$$

Using that $\mathfrak{A} \models Bax$, it is not hard to check that rep is a Boolean homomorphism into the Boolean set algebra $\langle \{\mathrm{rep}(a) \ : \ a \in A\}, \cup, - \rangle$. Thus

$$\mathsf{Mod}(Bax) \subseteq \mathbf{H}\mathsf{Mod}(Bax) \subseteq \mathsf{SetBA} \subseteq \mathsf{BA} \subseteq \mathsf{Mod}(Bax)$$

completes the proof. $\square$

Lemma 2.7.6. $\mathsf{Sir} \cap \mathsf{BA} = \mathbf{I}\{\underset{\sim}{\mathbf{2}}\}$.

Proof. **Outline of proof:** First one proves that $\underset{\sim}{\mathbf{2}}$ is subdirectly irreducible, i.e. $\underset{\sim}{\mathbf{2}} \in \mathsf{SirBA}$. This was done in Exercise 2.2.33 (2) and also in Exercise 2.2.38 (2); but for completeness we include a proof here. This proof is based on the fact that, by Exercise 2.7.1 (3) above, $\underset{\sim}{\mathbf{2}}$ has no nontrivial homomorphic image, but any subdirect decomposition of an algebra $\mathfrak{A}$ consists of homomorphic images of $\mathfrak{A}$. Therefore any subdirect decomposition of $\underset{\sim}{\mathbf{2}}$ consists of the one element BA and the two element BA and nothing else. But the one element BA is trivial, so should be omitted from the decomposition, while any two element BA is isomorphic to $\underset{\sim}{\mathbf{2}}$, hence any decomposition of $\underset{\sim}{\mathbf{2}}$ consists of the original $\underset{\sim}{\mathbf{2}}$ itself as factors. This means that $\underset{\sim}{\mathbf{2}}$ is subdirectly irreducible.

Next one proves that any subdirectly indecomposable BA is isomorphic to $\underset{\sim}{\mathbf{2}}$. This goes as follows. Assume $\mathfrak{A} \in \mathsf{SirBA}$. Assume $x \in A$ and $x \neq 0$, $x \neq 1$. Then relativising with x as well as relativising with $-x$ are homomorphisms, and they provide a subdirect decomposition of $\mathfrak{A}$; where relativising with x is the function rl_x sending y to $x \cap y$, that is, $\mathrm{rl}_x \stackrel{\text{def}}{=} \langle x \cap y \ : \ y \in A \rangle$. $\square$

Theorem 2.7.7. $\mathsf{BA} = \mathbf{SP}\{\underset{\sim}{\mathbf{2}}\}$.

Proof. By Lemma 2.7.6, $\mathsf{SirBA} = \mathbf{I}\{\underset{\sim}{\mathbf{2}}\}$. Now, by Birkhoff's "subdirect decomposability" theorem (Theorem 2.2.40), we have that $\mathsf{BA} = \mathbf{SP}\mathsf{SirBA} = \mathbf{SPI}\{\underset{\sim}{\mathbf{2}}\} = \mathbf{SP}\{\underset{\sim}{\mathbf{2}}\}$. $\square$

Exercise 2.7.8. Let $\mathfrak{B} \in \mathsf{BA}$. Represent $\mathfrak{B}$ as a set BA $\mathfrak{C}$ such that $\mathfrak{C} \subseteq \mathfrak{P}(U \times U)$ for some set U.

Hint: To any sets U and W, if $|U \times U| \geqslant |W|$ then there is a partition P of $U \times U$ with $|P| = |W|$.

2.8 Boolean Ordered Algebras (Boa's), Boolean Algebras with Operators (BAO's)

By a *Boolean ordered algebra* (Boa) we understand an (arbitrary) universal algebra in which the BA–operations are term definable. A Boolean algebra with operators (BAO) is a special Boa. (A more precise definition follows below.)

More concretely, recall from section 2.1 that by a similarity type we understand a function $t : I \longrightarrow \omega$ for some set I. By a Boa of similarity type t we understand an algebra $\mathfrak{A} = \langle \mathfrak{B}, f_i \rangle_{i \in I}$, where $\mathfrak{B} \in \mathsf{BA}$ and $(\forall i \in I)\, f_i : {}^{t(i)}B \to B$.

By a Boa we understand a Boa of similarity type t for some t. Notice that a Boa of type t is a (special) universal algebra of type $t +$ (Boolean operations).[5]

Convention 2.8.1. Throughout t is fixed to denote the similarity type of the algebra in question. So we might be discussing some algebra $\langle \mathfrak{B}, f_i \rangle_{i \in I}$ without mentioning any "t." Then suddenly we start talking about $t(i)$ (or $t(f_i)$ sometimes). Then $t(i)$ defines (by the force of the present convention) the rank of f_i.

Exercises 2.8.2. (1) $\mathsf{BA} \subseteq \mathsf{Boa}$.

(2) Let $\mathfrak{B} \in \mathsf{BA}, \mathsf{c} : B \longrightarrow B$. We define

$$\mathsf{Cs}_1 \stackrel{\text{def}}{=} \left\{ \langle \mathfrak{B}, \mathsf{c} \rangle : \mathfrak{B} \in \mathsf{BA} \text{ and } \mathsf{c}(x) = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases} \right\}.$$

We call the class Cs_1 the class of *cylindric set algebras of dimension* 1. Clearly $\mathsf{Cs}_1 \subseteq \mathsf{Boa}$.

Claim 2.8.3. $\mathsf{Cs}_1 \subseteq \mathsf{Smp}$.

(3) Let $\mathfrak{B} \in \mathsf{BA}$, $\mathsf{c} : B \longrightarrow B$.

Kapcsolatot teremteni a lop alfejezettel!!:

We call c a *closure operator* iff $x \leqslant \mathsf{c}x = \mathsf{cc}x$ and $x \leqslant y \Rightarrow \mathsf{c}x \leqslant \mathsf{c}y$ for every $x \in B$. We call c a *complemented closure operator* iff c is a closure operator and, in addition, $\mathsf{c}(-\mathsf{c}x) = -\mathsf{c}x$ for every $x \in B$. c is called *additive* iff $\mathsf{c}(x \vee y) = \mathsf{c}x \vee \mathsf{c}y$. Now we define

$$\mathsf{CA}_1 \stackrel{\text{def}}{=} \{ \langle \mathfrak{B}, \mathsf{c} \rangle \ : \ \mathfrak{B} \in \mathsf{BA} \text{ and } \mathsf{c} \text{ is an additive complemented closure operator} \}.$$

CA_1 is called the class of *cylindric algebras of dimension* 1. Clearly $\mathsf{CA}_1 \subseteq \mathsf{Boa}$.

Claim 2.8.4. $\mathsf{Sir}(\mathsf{CA}_1) = \mathsf{Cs}_1$.

Corollary 2.8.5. CA_1 *is a discriminator variety.*

Proof. By Claim 2.8.4, $\mathsf{Sir}(\mathsf{CA}_1) = \mathsf{Cs}_1$. As a consequence of Exercise 2.6.2, Cs_1 has a discriminator term. ☐

Lemma 2.8.6. *Let* $\mathfrak{A} \in \mathsf{Boa}$*,* $\theta \in Con(\mathfrak{A})$*,* $a, b \in A$*. Then*

$$a \mathrel{\theta} b \iff (a \oplus b) \mathrel{\theta} 0 .$$

[5] The careful reader may read "Boa of type t" as an abbreviation for "Boa of extra–Boolean type t". Here we use the former expression for keeping things short.

Proof. Recall that $x = y \iff x \oplus y = 0$. Now

$$\begin{aligned} a\ \theta\ b &\iff a/\theta = b/\theta \\ &\iff a/\theta \oplus b/\theta = 0/\theta \\ &\iff (a \oplus b)/\theta = 0/\theta \\ &\iff (a \oplus b)\ \theta\ 0\,. \end{aligned}$$

□

Definition 2.8.7. By a BAO *(Boolean Algebra with operators) of type t* (a BAO_t) we mean a Boa $\mathfrak{A} = \langle \mathfrak{B}, f_i \rangle_{i \in I}$ of type t such that the set (Dst) of distributivity equations below is valid in $\mathfrak{A}$.

(Dst) each f_i distributes over Boolean "$\vee$", i.e.

$$\begin{aligned} &f_i(x_1, \dots, x_{n-1}, (x_n \vee y_n), x_{n+1}, \dots, x_{t(i)}) = \\ &\quad f_i(x_1, \dots, x_{t(i)}) \vee f_i(x_1, \dots, x_{n-1}, y_n, x_{n+1}, \dots, x_{t(i)}) \end{aligned}$$

for each $i \in I$ and each $n \leqslant t(i)$.

By a *normal* BAO we understand a BAO $\langle \mathfrak{B}, f_i \rangle_{i \in I}$ in which the equations (Nrm) below are valid.

(Nrm) $f_i(x_1, \dots, x_{n-1}, 0, x_{n+1}, \dots, x_{t(i)}) = 0$ for each $n \leqslant t(i)$ and $i \in I$.

BAO_t and $\mathsf{Normal\,BAO_t}$ denote the classes of all BAO's and all normal BAO's of similarity type t, respectively.

Exercises 2.8.8. (1) Recall the definitions of the classes Cs_1 and CA_1 of algebras from Examples 2.8.2. Prove that $\mathsf{Cs}_1 \subseteq \mathsf{BAO}$ and $\mathsf{CA}_1 \subseteq \mathsf{BAO}$.

(2) Give an example for an algebra $\mathfrak{A}$ such that $\mathfrak{A} \in \mathsf{Boa} \smallsetminus \mathsf{BAO}$.

(3) Prove that BAO_t and $\mathsf{normal\,BAO_t}$ are varieties.

(4) When are these two varieties finitely axiomatizable?[6]

Notation 2.8.9. For a BAO $\mathfrak{A} = \langle \mathfrak{B}, f_i \rangle_{i \in I}$ we define $\mathfrak{A}^\circ \stackrel{\text{def}}{=} \mathfrak{B}$. That is, $\mathfrak{A}^\circ$ is the BA–reduct of $\mathfrak{A}$.

Let $\mathfrak{A} \in \mathsf{Boa}$ and $a \in A$. We call a an *atom* iff $0 < a$ and $(\forall b \in A)(0 < b \leqslant a \Rightarrow b = a)$. $\mathfrak{A}$ is called *atomic* iff every element of A is the union of some of the atoms of $\mathfrak{A}$. Notice that every finite Boolean algebra is atomic. A Boa is called *complete* iff its lattice reduct is a complete lattice.

Theorem 2.8.10. (Jónsson–Tarski [42]) [7]

(i) Every BAO $\mathfrak{A}$ *is contained (as a subalgebra) in a complete and atomic* BAO $\mathfrak{A}^+$.

[6] Note that t is a set of pairs hence if $t : I \longrightarrow \omega$ then $|t| = |I|$.

[7] The original theorem says more than what we state here.

(ii) *If* $\mathfrak{A}$ *is normal, then so is* $\mathfrak{A}^+$.

(iii) $\mathfrak{A}^+$ *is completely distributive, i.e. each* f_i *commutes with arbitrary suprema in* $\mathfrak{A}^+$.

(iv) *For every proper filter*[8] $\mathcal{F}$ *of* $\mathfrak{A}^\circ$*, the infimum of* $\mathcal{F}$ *taken in* $\mathfrak{A}^+$ *is nonzero. Formally,* $\inf^{(\mathfrak{A}^+)} \mathcal{F} \neq 0$.

For proving Theorem 2.8.10 we will need the following proposition of BA–theory.

Proposition 2.8.11. *(i) Every* BA $\mathfrak{A}$ *is contained (as a subalgebra) in a complete and atomic* BA $\mathfrak{A}^+$.

(ii) For every proper filter $\mathcal{F}$ *of* $\mathfrak{A}^\circ$, $\inf^{(\mathfrak{A}^+)} \mathcal{F} \neq 0$.

Proof. This is an immediate consequence of the well known (Stone) representation theorem of BA's. It can be found in any textbook on BA's (and most ones on universal algebra).

Hint: let $X_{\mathfrak{A}}$ be the set of ultrafilters of $\mathfrak{A}$, and let $\mathfrak{B}$ be the BA of all subsets of $X_{\mathfrak{A}}$. The embedding of $\mathfrak{A}$ into $\mathfrak{B}$ sends each element of $\mathfrak{A}$ to the set of ultrafilters containing it, i.e. $f(a) = \{x \in X_{\mathfrak{A}} : a \in x\}$ for all $a \in A$. □

Proof. **Proof Theorem 2.8.10:** Let $\mathfrak{A} \in$ BAO. Consider $\mathfrak{A}^\circ \in$ BA. By Proposition 2.8.11, there is a complete and atomic $\mathfrak{B} \in$ BA with $\mathfrak{B} \supseteq \mathfrak{A}^\circ$ satisfying (iv). Let this $\mathfrak{B}$ be fixed, see Figure 2.27.

It is enough to define operations $f_i^+ : {}^{t(i)}B \to B$ (for each i) such that f_i^+ commutes with suprema and agrees with f_i on A. First we define f_i^+ on the atoms At$\mathfrak{B}$ of $\mathfrak{B}$. To make the idea of the proof more visible we write out the rest of the proof for the special case when each f_i is normal and unary. The generalization to n–ary f_i is straightforward.

Definition 2.8.12. We define

(i) $f_i^+(b) \stackrel{\text{def}}{=} \inf^{\mathfrak{B}}\{f_i(x) \ : \ b \leqslant x \in A\}$ for each $b \in \mathrm{At}\mathfrak{B}$, and

(ii) $f_i^+(y) \stackrel{\text{def}}{=} \sup^{\mathfrak{B}}\{f_i^+(b) \ : \ b \in \mathrm{At}\mathfrak{B} \text{ and } b \leqslant y\}$.

Figure 2.28 represents the definition of $f_i^+(b)$ with b an atom.

Both parts of the definition make sense since $\mathfrak{B}$ is complete, and they define a function $f_i^+ : B \to B$. It is useful to notice here that

$$y = \sup^{\mathfrak{B}}\{b \in \mathrm{At}\mathfrak{B} \ : \ b \leqslant y\} \tag{2.36}$$

for any $y \in B$. (This explains how $f_i^+(y)$ is determined by part (i) of the definition.) Clearly, f_i^+ is completely distributive by a bit of Boolean reasoning, using part (ii) of the definition. We let $\mathfrak{A}^+ \stackrel{\text{def}}{=} \langle \mathfrak{B}, f_i^+ \rangle_{i \in I}$. Clearly $\mathfrak{A}^+ \in$ BAO is complete and atomic, etc. The question is how $\mathfrak{A}^+$ is related to $\mathfrak{A}$.

[8] A filter $\mathcal{F}$ is proper if $0 \notin \mathcal{F}$.

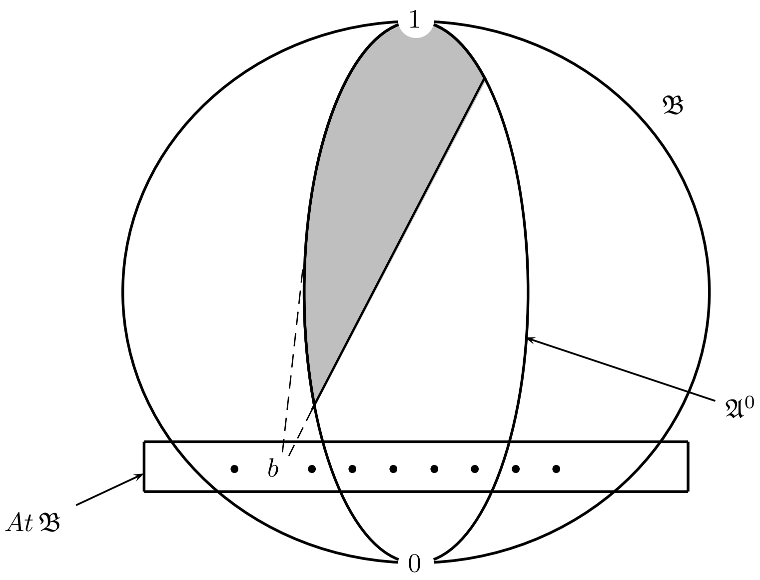


Figure 2.27: Proof of Theorem 2.8.10 1

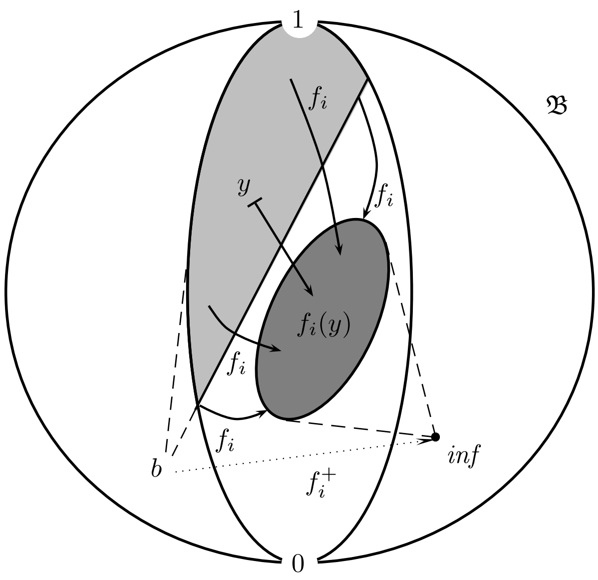


Figure 2.28: Proof of Theorem 2.8.10 2

Claim 2.8.13. *Let $y \in A$. Then $f_i^+(y) \leqslant f_i(y)$.*

Proof. Assume $f_i^+(y) \geq a \in \mathrm{At}\mathfrak{B}$. Then by (ii) $(\exists b \in \mathrm{At}\mathfrak{B})$ $[b \leqslant y$ and $f_i^+(b) \geq a]$. Fix this b. By (i) then a is a lower bound of $\{f_i(x) \ : \ b \leqslant x \in A\}$. Since $b \leqslant y \in A$, this implies $a \leqslant f_i(y)$. We proved that $f_i(y)$ is an upper bound of $\{a \in \mathrm{At}\mathfrak{B} \ : \ a \leqslant f_i^+(y)\}$. By (4.1) then $f_i(y) \geq \sup\{a \in \mathrm{At}\mathfrak{B} \ : \ a \leqslant f_i^+(y)\} = f_i^+(y)$. $\square$

From now on we write f and f^+ for f_i and f_i^+ respectively. Similarly, we write At for At$\mathfrak{B}$.

Claim 2.8.14. *Let $y \in A$, $f(y) \geq b \in \mathrm{At}$. Then there is $a \in \mathrm{At}$ such that $a \leqslant y$ and $b \leqslant f^+(a)$.*

Proof. Let $N \stackrel{\text{def}}{=} \{x \in A : b \nleqslant f(x)\}$. Note that $N \neq \emptyset$ by normality. Let $x_0, \ldots, x_n \in N$. We show that

$$-y \vee x_0 \vee \ldots \vee x_n \neq 1\,. \tag{2.37}$$

Assume the contrary. Then $y \leqslant x_0 \vee \ldots \vee x_n$, thus $b \leqslant f(y) \leqslant f(x_0 \vee \ldots \vee x_n) = f(x_0) \vee \ldots \vee f(x_n)$. Since b is an atom, this means $(\exists i \leqslant n)$ $b \leqslant f(x_i)$ contradicting the definition of N. This proves (2.37).

Let $N^- = \{-x : x \in N\} \cup \{y\}$. Then by (4.2) for any finite $H \subseteq N^-$, $\inf(H) \neq 0$. Hence N^- is contained in a proper filter of $\mathfrak{A}$ whence, by (iv), there is an atom $a \leqslant \inf(N^-)$. Assume now $a \leqslant x \in A$, for some x. Then

$$a \leqslant x \underset{\substack{\uparrow \\ a \neq 0}}{\Rightarrow} a \nleqslant -x \quad \Rightarrow \quad -x \notin N^- \Rightarrow x \notin N \Rightarrow b \leqslant f(x).$$

This proves $b \leqslant \inf\{f(x) \ : \ a \leqslant x \in A\} = f^+(a)$. By $a \leqslant \inf(N^-) \leqslant y$ we are done. $\square$

Let $y \in A$ and $f(y) \geq b \in \mathrm{At}$ be arbitrary. By Claim 2.8.14, there is an atom $a \leqslant y$ with $f^+(a) \geq b$. Now,

$$f^+(y) = \sup\{f^+(a) \ : \ y \geq a \in \mathrm{At}\} \geq f^+(a) \geq b.$$

So for any atom $b \leqslant f(y)$ we proved $f^+(y) \geq b$. Then $f(y) = \sup\{b \in \mathrm{At} \ : \ f(y) \geq b\} \leqslant f^+(y)$. By the choice of y we proved $(\forall y \in A)$ $f^+(y) \geq f(y)$. Together with Claim 2.8.13 this implies $(\forall y \in A)$ $f^+(y) = f(y)$. That is,

$$f_i^+ \lceil A = f_i \quad \text{for all } i \in I.$$

But this means exactly that $\mathfrak{A} \subseteq \mathfrak{A}^+$. The rest is immediate from the properties of $\mathfrak{B}$ and its relationship with $\mathfrak{A}^\circ$ both of which we inherited from Proposition 2.8.11. We have proved Theorem 2.8.10. $\square$

Definition 2.8.15. An equation e in the language of BAO's is called *positive* if "$-$" does not occur in e except perhaps in front of constant symbols. I.e. if $t(i) = 0$ then $-f_i$ may occur in e.

Theorem 2.8.16. (Jónsson–Tarski) *Let* $\mathfrak{A} \in \mathsf{BAO}$*. Then there is a complete and atomic* BAO $\mathfrak{A}^+ \supseteq \mathfrak{A}$ *as in Theorem 2.8.10, such that the following holds. For any positive equation* e*, we have* $(\mathfrak{A} \models e) \Rightarrow (\mathfrak{A}^+ \models e)$*.*

Proof. It can be found, e.g. in Jónsson–Tarski [43], [37, §2.7 (Theorems 2.7.13, 2.7.14, 1.7.16)]. □

To be filled in later. Puska: Venema diszi BAO appendix

Valahol legyenek lezarasi operatorok is, meg felcsoportok is. Lehet mondani, hogy BA a logika absztrakciojakent jott letre; felcsoportok meg szamok absztrakciojakent.

Chapter 3

General framework for studying logics

3.1 Defining the framework

Defining a logic is an experience similar to defining a language. (This is no coincidence if you think about the applications of logic in e.g. theoretical linguistics.) So how do we define a language, say a programming language like C^{++}. First one defines the *syntax* of C^{++}. This amounts to defining the set of all C^{++} programs. This definition tells us which strings of symbols count as C^{++} programs and which do not. But this information in itself is not very useful, because having only this information enables the user to write programs but the user will have no idea what his programs will do. Indeed, the second, and more important step in defining C^{++} amounts to describing what the various C^{++} programs will do when executed. In other words, we have to define the meaning, or *semantics* of the language, e.g. of C^{++}. Defining semantics can be done in two steps, (i) we define the class M of *possible machines* that understand C^{++}, and then (ii) to each machine $\mathfrak{M}$ and each string φ of symbols that counts as a C^{++} program we tell what $\mathfrak{M}$ will do if we "ask" it to execute φ. In other words we define the *meaning* $mng(\varphi, \mathfrak{M})$ of program φ in machine $\mathfrak{M}$.

The procedure remains basically the same if the language in question is not a programming language but something like a natural language or a simple declarative language like first-order logic. When teaching a foreign language, e.g. German, one has to explain which strings of symbols are German sentences and which are not (e.g. "Der Tisch ist rot" is a German sentence while "Das Tisch ist rot" is not). This is called explaining the syntax of German. Besides this, one has to explain what the German sentences mean. This amounts to defining the semantics of German. If we want to formalize the definition of semantics (for, say, a fragment of German) then one again defines a class M of possible situations or with other words, "possible worlds" in which our German sentences are interpreted, and then to each situation $\mathfrak{M}$ and each sentence φ we define the *meaning* or *denotation* $mng(\varphi, \mathfrak{M})$ of φ in situation (or possible world) $\mathfrak{M}$.

At this point we could discuss the difference between a language and a logic, but we do not do that. For our present purposes it is enough to say that the two things are very-very similar.[1]

Soon (in Definition 3.1.3 below) we will define what we mean by a logic. (A more carefully chosen expression would be "logical system".) Roughly speaking, a *logic* $\mathcal{L}$ is a five-tuple

$$\mathcal{L} = \langle F_{\mathcal{L}}, \vdash_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}} \rangle ,$$

where

- $F_{\mathcal{L}}$ is a set, called the set of all *formulas* of $\mathcal{L}$;
- $\vdash_{\mathcal{L}}$ is a binary relation between sets of formulas and individual formulas, that is, $\vdash_{\mathcal{L}} \subseteq \mathcal{P}(F_{\mathcal{L}}) \times F_{\mathcal{L}}$ (for any set X, $\mathcal{P}(X)$ denotes the powerset of X); $\vdash_{\mathcal{L}}$ is called the *provability relation* of $\mathcal{L}$;
- $M_{\mathcal{L}}$ is a class, called the class of all *models* (or *possible worlds*) of $\mathcal{L}$;
- $mng_{\mathcal{L}}$ is a function with domain $F_{\mathcal{L}} \times M_{\mathcal{L}}$, called the *meaning function* of $\mathcal{L}$, hence, by the usual convention of set theory,

  $$mng_{\mathcal{L}} : F_{\mathcal{L}} \times M_{\mathcal{L}} \longrightarrow V$$

  where V is the class of all sets (cf. the end of subsection 1.1).
- $\models_{\mathcal{L}}$ is a binary relation, $\models_{\mathcal{L}} \subseteq M_{\mathcal{L}} \times F_{\mathcal{L}}$, called the *validity relation* of $\mathcal{L}$;
- there is some connection between $\models_{\mathcal{L}}$ and $mng_{\mathcal{L}}$, namely for all $\varphi, \psi \in F_{\mathcal{L}}$ and $\mathfrak{M} \in M_{\mathcal{L}}$ we have

  $$\big(mng_{\mathcal{L}}(\varphi, \mathfrak{M}) = mng_{\mathcal{L}}(\psi, \mathfrak{M}) \text{ and } \mathfrak{M} \models_{\mathcal{L}} \varphi\big) \Longrightarrow \mathfrak{M} \models_{\mathcal{L}} \psi\,. \tag{3.1}$$

Intuitively, $F_{\mathcal{L}}$ is the collection of "texts" or "sentences" or "formulas" that can be "said" or "expressed" in the language $\mathcal{L}$. For $\Gamma \subseteq F_{\mathcal{L}}$ and $\varphi \in F_{\mathcal{L}}$, the intuitive meaning of $\Gamma \vdash_{\mathcal{L}} \varphi$ is that φ is provable (or derivable) from Γ with the syntactic inference system (or deductive mechanism) of $\mathcal{L}$. In all important cases, $\vdash_{\mathcal{L}}$ is subject to certain (well-known) conditions like $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Gamma \cup \{\varphi\} \vdash_{\mathcal{L}} \psi$ imply $\Gamma \vdash_{\mathcal{L}} \psi$ for any $\Gamma \subseteq F_{\mathcal{L}}$ and $\varphi, \psi \in F_{\mathcal{L}}$. The meaning function tells us what the texts belonging to $F_{\mathcal{L}}$ mean in the possible worlds from $M_{\mathcal{L}}$. For fixed $\varphi \in F_{\mathcal{L}}$ and $\mathfrak{M} \in M_{\mathcal{L}}$, $mng_{\mathcal{L}}(\varphi, \mathfrak{M})$ is called the *meaning* or *denotation* or *intension* of the expression φ in the model (or "possible environment" or "possible interpretation")

[1] The philosophical minded reader might enjoy looking into the book [2], cf. e.g. B. Partee's paper therein.

$\mathfrak{M}$.[2] The validity relation tells us which texts are "true" in which possible worlds (or models) under what conditions. In all the interesting cases from $mng_{\mathcal{L}}$ the relation $\models_{\mathcal{L}}$ is definable. A typical possible definition of $\models_{\mathcal{L}}$ from $mng_{\mathcal{L}}$ is the following.

$$\mathfrak{M} \models_{\mathcal{L}} \varphi \quad \text{iff} \quad (\forall \psi \in F_{\mathcal{L}})\big(mng_{\mathcal{L}}(\psi, \mathfrak{M}) \subseteq mng_{\mathcal{L}}(\varphi, \mathfrak{M})\big),$$

for all $\varphi \in F_{\mathcal{L}}$, $\mathfrak{M} \in M_{\mathcal{L}}$. However, in general, definability of $\models_{\mathcal{L}}$ from $mng_{\mathcal{L}}$ is *not* required (condition (3.1) above is *not* a definition).

When no confusion is likely, we omit the subscripts $\mathcal{L}$ from $F_{\mathcal{L}}$, $\vdash_{\mathcal{L}}$ etc.

Usually $F_{\mathcal{L}}$ and $\vdash_{\mathcal{L}}$ are defined by what is called a grammar in mathematical linguistics. $\langle F_{\mathcal{L}}, \vdash_{\mathcal{L}}\rangle$ together with the grammar defining them is called the *syntactical part* of $\mathcal{L}$, while $\langle M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}}\rangle$ is the *semantical part* of $\mathcal{L}$.

When defining a logic, a typical definition of F has the following recursive form. Two sets, P and $Cn(\mathcal{L})$ are given; P is called the set of primitive or *atomic formulas* and $Cn(\mathcal{L})$ is called the set of *logical connectives* of $\mathcal{L}$ (these are operation symbols with finite or infinite ranks). Then we require F to be the smallest set H satisfying

(1) $P \subseteq H$, and

(2) for every $\varphi_1, \ldots, \varphi_n \in H$ and $f \in Cn(\mathcal{L})$ of rank n, $f(\varphi_1, \ldots, \varphi_n) \in H$.

For example, in propositional logic, if p is some propositional variable (atomic formula according to our terminology), then $(\neg p)$ is defined to be a formula (where $\neg$ is a logical connective of rank 1).

For formulas $\varphi \in F$ and models $\mathfrak{M} \in M$, $mng(\varphi, \mathfrak{M})$ and $\mathfrak{M} \models \varphi$ are defined in uniform ways (by some finite "schemas").

Given a logic $\mathcal{L}$, for $\varphi \in F_{\mathcal{L}}$ we say that φ is *valid* (in $\mathcal{L}$), in symbols $\models_{\mathcal{L}} \varphi$, iff $(\forall \mathfrak{M} \in M_{\mathcal{L}})\mathfrak{M} \models \varphi$. For φ as above and $\Gamma \subseteq F_{\mathcal{L}}$ we say that φ is a *semantical consequence* of Γ, in symbols $\Gamma \models_{\mathcal{L}} \varphi$, iff $(\forall \mathfrak{M} \in M_{\mathcal{L}})\big((\forall \psi \in \Gamma)\mathfrak{M} \models_{\mathcal{L}} \psi \Longrightarrow \mathfrak{M} \models_{\mathcal{L}} \varphi\big)$. (We hope that the traditional double use of symbol $\models$ does not cause real ambiguity.) One of the important topics of Logic is the study of the connection between semantic consequence $\Gamma \models_{\mathcal{L}} \varphi$ and the syntactic consequence $\Gamma \vdash_{\mathcal{L}} \varphi$. If the two coincide then $\vdash_{\mathcal{L}}$ is said to be strongly complete and sound (for $\mathcal{L}$).

Figure 3.1 below illustrates the general pattern of a logic.

Exercises 3.1.1 below are designed to illuminate the intuitive content of the concept of a logic as outlined above, and to show how familiar logics are special cases of our general concept.

Ábra kilóg jobbra!

Exercises 3.1.1.

[2]The literature makes subtle distinctions between these words. We deliberately ignore these distinctions, because on the present level of abstraction they are not relevant yet.

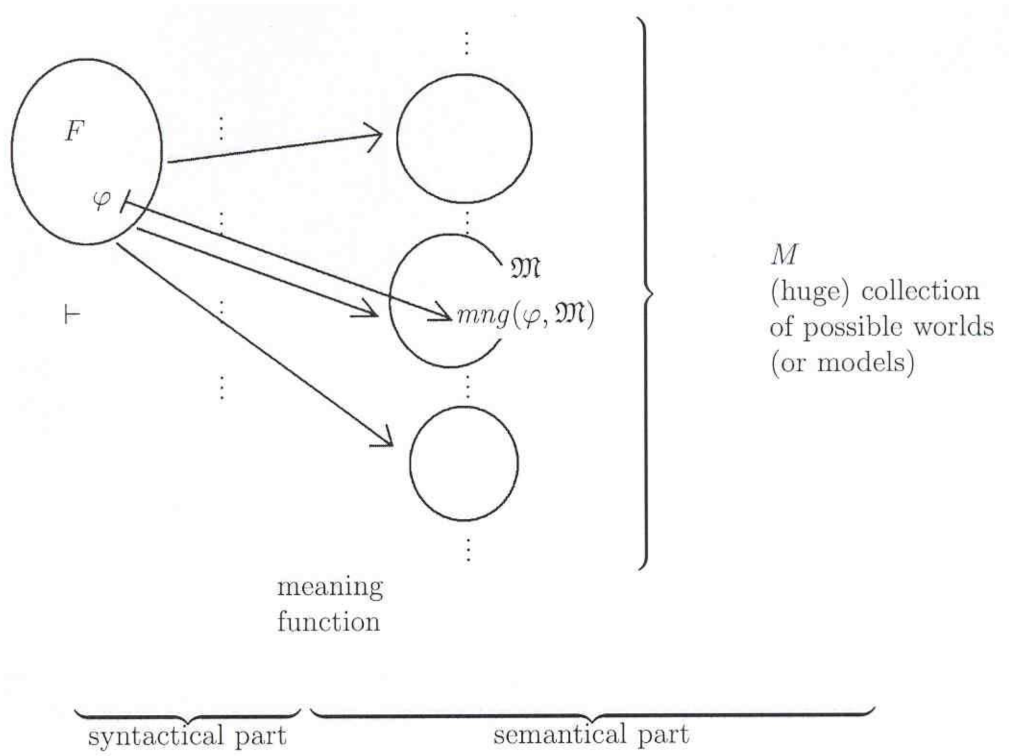


Figure 3.1: The “fan-structure” of a language with semantics

(1) Create an illustration for the above outlined concept of a logic, that is, for $\mathcal{L} = \langle F_\mathcal{L}, \vdash_\mathcal{L}, M_\mathcal{L}, mng_\mathcal{L}, \models_\mathcal{L}\rangle$, by formalizing *classical sentential logic* in this spirit; and do this in the following way. Let P be a set, called the set of *atomic formulas* of $\mathcal{L}_S$. Let $\{\wedge, \neg\} = Cn(\mathcal{L}_S)$ be a set disjoint from P, called the set of *logical connectives* of $\mathcal{L}_S$ (usually called *Boolean connectives*). Define the set F_S (of formulas) to be the smallest set H satisfying the two conditions: $P \subseteq H$ and $\big(\varphi, \psi \in H \Longrightarrow (\varphi \wedge \psi), \neg\varphi \in H\big)$. Further, define the class M_S (of models) as $M_S \stackrel{\text{def}}{=} \{\langle W, v\rangle : W$ is a non-empty set, and $v : P \longrightarrow \mathcal{P}(W)\}$. Now, you want to recast sentential logic $\mathcal{L}_S$ in the form $\mathcal{L}_S^0 = \langle F_S, \vdash_S^0, M_S, mng_S^0, \models_S^0\rangle$ such that it could be a concrete example of our general ideas outlined above. For this, F_S and M_S are already defined. We leave $\vdash_S^0$ to the end. Let *Sets* denote the class of all sets. Define $mng_S^0 : F_S \times M_S \to Sets$ in the following way. Let $\mathfrak{M} = \langle W, v\rangle \in M_S$ be arbitrary but fixed. For any $p \in P$ define $mng_S^0(p, \mathfrak{M}) \stackrel{\text{def}}{=} v(p)$. For any $\varphi, \psi \in F_S$ define $mng_S^0\big((\varphi \wedge \psi), \mathfrak{M}\big) \stackrel{\text{def}}{=} mng_S^0(\varphi, \mathfrak{M}) \cap mng_S^0(\psi, \mathfrak{M})$ and $mng_S^0(\neg\varphi, \mathfrak{M}) \stackrel{\text{def}}{=} W \smallsetminus mng_S^0(\varphi, \mathfrak{M})$. For any $\mathfrak{M} = \langle W, v\rangle \in M_S$, $\varphi \in F_S$ let $\mathfrak{M} \models_S^0 \varphi$ iff $mng_S^0(\varphi, \mathfrak{M}) = W$. Check that you indeed defined (the set of formulas together with) the "semantical part" $\langle F_S, M_S, mng_S^0, \models_S^0\rangle$ of a logic in the sense outlined above these exercises.

Let us turn to defining a possible choice of $\vdash_S^0$.

Throughout, we use $(\varphi \to \psi)$ as an abbreviation for $\neg(\varphi \wedge \neg\psi)$ and $(\varphi \leftrightarrow \psi)$ as that for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. List a set Ax of valid formulas of $\mathcal{L}_S$ and call these *logical axioms*.[3] Possible elements of this list are $(\varphi \to \varphi)$ for all $\varphi \in F_S$, $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$, $(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \neg\varphi) \to (\psi \wedge \neg\psi)$, $\varphi \to (\psi \to \varphi)$, for all $\varphi, \psi \in F_S$. Having defined your set Ax of logical axioms, add the inference rule $\{\varphi, (\varphi \to \psi)\} \vdash \psi$ (for all $\varphi, \psi \in F_S$) which is called Modus Ponens. If you wish, you may add similar rules like $\{\varphi, \psi\} \vdash (\varphi \wedge \psi)$ (but they are not really needed). For $\Gamma \subseteq F_S$, define $\Gamma \vdash_S^0 \varphi$ to hold iff $\varphi \in H$ for the smallest set $H \subseteq F_S$ such that $\Gamma \cup Ax \subseteq H$ and H is closed under your inference rules, e.g. whenever $\psi, (\psi \to \rho) \in H$ then also $\rho \in H$. With this, you defined your choice of $\vdash_S^0$ for $\mathcal{L}_S^0$. If $\big(\Gamma \vdash_S^0 \varphi \Longrightarrow \Gamma \models_S^0 \varphi\big)$ for all Γ, φ then $\vdash_S^0$ is called *sound*. If the other direction "$\Longleftarrow$" holds, then $\vdash_S^0$ is called *strongly complete*. Spend a little time with trying to guess whether your $\vdash_S^0$ has one of these properties. Now, check that you indeed defined a logic

$$\mathcal{L}_S^0 = \langle F_S, \vdash_S^0, M_S, mng_S^0, \models_S^0\rangle$$

in the sense outlined above the present exercises.

[3] If the instructions below would be too vague for the non-logician reader then s/he has three options: (i) Consult Definitions 4.1.19–4.1.22 together with the 11 lines preceding Definition 4.1.19 in section 4.1 herein. There we define and discuss inference systems $\vdash_\mathcal{L}$ in detail, so that should suffice. (ii) Recall any of the known inference systems for propositional logic from the literature. (iii) Ignore this "$\vdash$-part" of this exercise, since we will not rely on it later.

(2) Compare the just defined version $\mathcal{L}^0_S$ of sentential logic with the ideas outlined above.

(3) Compare $\mathcal{L}^0_S$ with your own previous concept of sentential logic, and try to prove that they are the same thing (perhaps in different forms).

(4) Change the logic $\mathcal{L}^0_S$ obtaining $\mathcal{L}^1_S$ in the following way. Leave F_S and $\vdash^0_S$ unchanged. Define the new M^1_S by postulating that its elements are functions $\mathfrak{M} : P \to \{0,1\}$. (Identify 0 with *False* and 1 with *True*.) Define $mng^1_S : F_S \times M^1_S \to \{0,1\}$ and $\models^1_S$ in the natural way. (Hint: If $p \in P$ then $mng^1_S(p,\mathfrak{M}) \stackrel{\text{def}}{=} \mathfrak{M}(p)$, and $mng^1_S(\neg\varphi,\mathfrak{M}) \stackrel{\text{def}}{=} 1 - mng^1_S(\varphi,\mathfrak{M})$, etc.) Check that what you obtained, $\mathcal{L}^1_S = \langle F_S, \vdash^0_S, M^1_S, mng^1_S, \models^1_S\rangle$, is again an example of our general concept of a logic.

(5) Try to compare logics $\mathcal{L}^0_S$ and $\mathcal{L}^1_S$. Try to find ways in which they could be called equivalent. (Hint: Prove e.g. that they have the same semantic consequence relation, i.e. $(\forall \Gamma \cup \{\varphi\} \subseteq F_S)\ \Gamma \models^0_S \varphi \Leftrightarrow \Gamma \models^1_S \varphi$.)

(6) Let $\varphi \in F_S$ be arbitrary. Prove that φ is valid in every model of $\mathcal{L}^0_S$ iff it is valid in every model of $\mathcal{L}^1_S$. That is, the validities of $\mathcal{L}^0_S$ and $\mathcal{L}^1_S$ coincide. Try to find further similar "equivalence properties".

??? FELADAT ???: A kovetkezo reszt (Remark 3.1.2-val bezarolag) 2 reszre kell majd bontani. A tenyleg ervelo, $mng_\vdash$-t megszerkeszto reszt kicsit kesobbre kene halasztani.

Instead of the general concept of a logic outlined above, in many cases we will consider only four of its five components: $F_\mathcal{L}$, $M_\mathcal{L}$, $mng_\mathcal{L}$ and $\models_\mathcal{L}$. Namely, we found that we can simplify the theory *without loss of generality* by not dragging $\vdash_\mathcal{L}$ along with us for the following reason.[4]

The validity relation $\models_\mathcal{L}$ (or the function $mng_\mathcal{L}$ if you like) induces the *semantical consequence relation* $\models_\mathcal{L}\ \subseteq \mathcal{P}(F_\mathcal{L}) \times F_\mathcal{L}$, given above Exercises 3.1.1. There is a natural temptation to try to replace $\vdash_\mathcal{L}$ with $\models_\mathcal{L}$ in the theory, though at several places (e.g. at completeness theorems) this would be a grave oversimplification. Surprisingly enough, we found that all the theorems we prove for $\models_\mathcal{L}$ carry over to $\vdash_\mathcal{L}$, whenever the theorems are not about connections between $\models_\mathcal{L}$ and $\vdash_\mathcal{L}$ (see explanation below). Therefore we decided to drop $\vdash_\mathcal{L}$ for the time being and introduce it only where we must say something about $\vdash_\mathcal{L}$ which cannot be said about $\models_\mathcal{L}$ in itself.

The *reader interested in logics in the purely syntactical sense* $\langle F_\mathcal{L}, \vdash_\mathcal{L}\rangle$ is invited to read our paper in the way described as follows.

[4]The following considerations, together with Remark 3.1.2, grew out from discussions with Wim Blok, Joseph M. Font, Ramon Jansana and Don Pigozzi. In particular, Remark 3.1.2 is due to Font and Jansana.

Let $\mathcal{L}_{syn} = \langle F, \vdash\rangle$ be a logic in the syntactical sense. To simplify the arguments below, we assume that $\mathcal{L}_{syn}$ has a derived logical connective "$\leftrightarrow$" just as classical logics do, see Ex. 3.1.1 (1) above. Of course, we assume the usual properties of "$\leftrightarrow$", e.g. $\{\varphi, (\varphi \leftrightarrow \psi)\} \vdash \psi$ etc. (cf. the $\vdash^0_S$ part of Ex. 3.1.1 (1)). Intuitively, $(\varphi \leftrightarrow \psi)$ expresses that φ and ψ are equivalent. In Remark 3.1.2 below the present discussion, we discuss how to eliminate the assumption of the expressibility of "$\leftrightarrow$". (However, the reader may safely skip Remark 3.1.2, since we will not rely on it later.)

Assume we want to study the "syntactical logic" $\mathcal{L}_{syn} = \langle F, \vdash\rangle$. To be able to apply the theorems of the present paper, we will associate a class $M_\vdash$ of pseudo-models, a $mng_\vdash$ etc. to $\mathcal{L}_{syn}$. The class of *pseudo-models* is

$$M_\vdash \stackrel{\text{def}}{=} \{T \subseteq F \ : \ T \text{ is closed under } \vdash\}.$$

For any pseudo-model $T \in M_\vdash$ and formula $\varphi \in F$,

$$mng_\vdash(\varphi, T) \stackrel{\text{def}}{=} \{\psi \in F : T \vdash (\varphi \leftrightarrow \psi)\}\ .$$

Further, validity in pseudo-models $T \in M_\vdash$ is defined as

$$T \models_\vdash \varphi \quad \stackrel{\text{def}}{\iff} \quad \varphi \in T\,.$$

Now, if we want to investigate the "syntactic logic" $\langle F, \vdash\rangle$, we apply our theorems to the logic

$$\mathcal{L}_\vdash \stackrel{\text{def}}{=} \langle F, M_\vdash, mng_\vdash, \models_\vdash\rangle\,.$$

Then condition (3.1) above holds for $\mathcal{L}_\vdash$ and the semantical consequence relation induced by $\models_\vdash$ coincides with the original syntactical one $\vdash$. (These are easy to check.) Hence, applying the theorems to the logic $\mathcal{L}_\vdash$ yields results about $\langle F, \vdash\rangle$ as was desired. In other words, $\mathcal{L}_\vdash$ is an equivalent reformulation of the "syntactic logic" $\langle F, \vdash\rangle$, hence studying $\mathcal{L}_\vdash$ is the same as studying $\langle F, \vdash\rangle$.

Remark 3.1.2. (Eliminating the assumption of expressibility of "$\leftrightarrow$") Here we show that in the above argument showing that our results can be applied to a wider class of syntactical logics $\mathcal{L}_{syn} = \langle F, \vdash\rangle$, the assumption of expressibility of "$\leftrightarrow$" in $\mathcal{L}_{syn}$ is not needed. It will turn out in Definition 4.1.8 in section 4.1 that for any logic $\mathcal{L}$, the set F of formulas has an algebraic structure, that is F is the universe of an algebra $\mathfrak{F}$. (The operations of $\mathfrak{F}$ are the logical connectives of $\mathcal{L}$ collected in $Cn(\mathcal{L})$.) Let

$$\begin{aligned} M_\vdash \stackrel{\text{def}}{=} \{\langle T, h\rangle \ : \ & T \subseteq F, \\ & T \text{ is closed under } \vdash, h \text{ is a homomorphism from } \mathfrak{F} \text{ into } \mathfrak{F}\}. \end{aligned}$$

For any $\varphi \in F,\ \langle T, h\rangle \in M_\vdash$, let

$$\begin{aligned} mng_\vdash(\varphi, \langle T, h\rangle) &\stackrel{\text{def}}{=} h(\varphi) \\ \langle T, h\rangle \models_\vdash \varphi \ &\stackrel{\text{def}}{\iff} \ h(\varphi) \in T. \end{aligned}$$

Then $\mathcal{L}_\vdash \stackrel{\text{def}}{=} \langle F, M_\vdash, mng_\vdash, \models_\vdash \rangle$ is a logic such that for all $\Gamma \cup \{\varphi\} \subseteq F$, $(\Gamma \models_\vdash \varphi$ iff $\Gamma \vdash \varphi)$ holds. Moreover, if $\vdash$ satisfies some natural conditions then $\mathcal{L}_\vdash$ is a "structural" logic (cf. Def. 4.1.8), therefore all the theorems of this paper can be applied to it. For more information in this line see [30].

Summing up, for a while we will concentrate our attention on the simplified form

$$\mathcal{L} = \langle F_\mathcal{L}, M_\mathcal{L}, mng_\mathcal{L}, \models_\mathcal{L} \rangle$$

of a logic. For the reasons outlined above, this temporary restriction of attention will not result in any loss of generality.

To conclude this subsection, we turn to nailing down our definitions formally in the form we will use them.

For any set X, we let X^* denote the set of all finite sequences ("words") over X. That is, $X^* \stackrel{\text{def}}{=} \bigcup_{n\in\omega} ({}^nX)$ (cf. [71]).

Definition 3.1.3. (logic) By a *logic* $\mathcal{L}$ we mean an ordered quadruple

$$\mathcal{L} \stackrel{\text{def}}{=} \langle F_\mathcal{L}, M_\mathcal{L}, mng_\mathcal{L}, \models_\mathcal{L} \rangle,$$

where (i)–(v) below hold.

(i) $F_\mathcal{L}$ (called the set of *formulas*) is a set of finite sequences (called *words*) over some set X (called the *alphabet* of $\mathcal{L}$) that is, $F_\mathcal{L} \subseteq X^*$.

(ii) $M_\mathcal{L}$ is a class (called the class of *models*).

(iii) $mng_\mathcal{L}$ is a function with domain $F_\mathcal{L} \times M_\mathcal{L}$ (called the *meaning function*).

(iv) $\models_\mathcal{L}$ (called the *validity relation*) is a relation between $M_\mathcal{L}$ and $F_\mathcal{L}$ that is, $\models_\mathcal{L} \subseteq M_\mathcal{L} \times F_\mathcal{L}$. (According to the tradition, instead of "$\langle \mathfrak{M}, \varphi \rangle \in \models_\mathcal{L}$" we write "$\mathfrak{M} \models_\mathcal{L} \varphi$".)

(v) For all $\varphi, \psi \in F_\mathcal{L}$ and $\mathfrak{M} \in M_\mathcal{L}$ we assume (3.1), that is,

$$\big(mng_\mathcal{L}(\varphi, \mathfrak{M}) = mng_\mathcal{L}(\psi, \mathfrak{M}) \text{ and } \mathfrak{M} \models_\mathcal{L} \varphi\big) \Longrightarrow \mathfrak{M} \models_\mathcal{L} \psi\,.$$

Remark 3.1.4. (i) In the above definition, we nailed down the expression "model of $\mathcal{L}$" instead of the more suggestive one "possible world of $\mathcal{L}$" only for purely technical reasons, namely, to avoid a danger of potential ambiguity with the literature.

(ii) By requiring (i) of Def.3.1.3 above we exclude infinitary languages like $\mathcal{L}_{\kappa,\lambda}$ or $\mathcal{L}^n_{\infty,\omega}$. This exclusion is not necessary, all the methods go through with some modifications. **Igaz-e? Konkrét pointer?:** Actually, occasionally we will look into properties of the finite variable fragment $\mathcal{L}^n_{\infty,\omega}$ of infinitary logic, because it naturally admits applications of our methods and plays an essential rôle in finite model theory and in theoretical computer science.

(iii) Although it is not automatically permitted in ZFC, we *may* assume that for any five classes $C_1, \ldots, C_5$ the tuple $\langle C_1, \ldots, C_5 \rangle$ exists and is again a class, cf. the relevant text at the end of subsection 1.6.

Definition 3.1.5. (semantical consequence, valid formulas)
Let $\mathcal{L} = \langle F_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}} \rangle$ be a logic. For every $\mathfrak{M} \in M_{\mathcal{L}}$ and $\Sigma \subseteq F_{\mathcal{L}}$,

$$\mathfrak{M} \models_{\mathcal{L}} \Sigma \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall \varphi \in \Sigma)\, \mathfrak{M} \models_{\mathcal{L}} \varphi,$$
$$Mod_{\mathcal{L}}(\Sigma) \stackrel{\text{def}}{=} \{\mathfrak{M} \in M_{\mathcal{L}} : \mathfrak{M} \models_{\mathcal{L}} \Sigma\}.$$

$Mod_{\mathcal{L}}(\Sigma)$ is called the class of models of Σ.

A formula φ is said to be *valid*, in symbols $\models_{\mathcal{L}} \varphi$, iff $Mod_{\mathcal{L}}(\{\varphi\}) = M_{\mathcal{L}}$.

For any $\Sigma \cup \{\varphi\} \subseteq F_{\mathcal{L}}$,

$$\Sigma \models_{\mathcal{L}} \varphi \stackrel{\text{def}}{\Longleftrightarrow} Mod_{\mathcal{L}}(\Sigma) \subseteq Mod_{\mathcal{L}}(\{\varphi\}),$$
$$Csq_{\mathcal{L}}(\Sigma) \stackrel{\text{def}}{=} \{\varphi \in F_{\mathcal{L}} \ : \ \Sigma \models_{\mathcal{L}} \varphi\}\,.$$

If $\varphi \in Csq_{\mathcal{L}}(\Sigma)$ then we say that φ is a *semantical consequence* of Σ (in logic $\mathcal{L}$). Csq abbreviates "consequence".

Definition 3.1.6. (theory, set of validities) Let $\mathcal{L} = \langle F_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}} \rangle$ be any logic. For any $K \subseteq M_{\mathcal{L}}$ let the *theory of K in $\mathcal{L}$* be defined as

$$Th_{\mathcal{L}}(K) \stackrel{\text{def}}{=} \{\varphi \in F_{\mathcal{L}} \ : \ (\forall \mathfrak{M} \in K)\ \mathfrak{M} \models_{\mathcal{L}} \varphi\}.$$

If $K = \{\mathfrak{M}\}$ for some $\mathfrak{M} \in M_{\mathcal{L}}$ then instead of $Th_{\mathcal{L}}(\{\mathfrak{M}\})$ we write $Th_{\mathcal{L}}(\mathfrak{M})$.

The set $Th_{\mathcal{L}}(M_{\mathcal{L}})$ is called the *set of validities* of $\mathcal{L}$.

For any set X^* of "strings of symbols", the notion of a *decidable* subset $H \subseteq X^*$ is introduced in almost any introductory book on logic or on the theory of computation (see e.g. [56]). The same applies to $H \subseteq X^*$ being *recursively enumerable* (*r.e.*).

Az ELTE-s diákok (tömegesen!) nem tudják, mia az, hogy eld"onthetőség, pedig tanulták. (Emlékszenek viszont, mi az hogy rek.fels.) Ezért itt kéne egy mondat emlékeztető erre.

Definition 3.1.7. (decidability of logics) We say that

a logic $\mathcal{L} = \langle F_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}} \rangle$ is *decidable*

iff the set $Th_{\mathcal{L}}(M_{\mathcal{L}})$ of validities of $\mathcal{L}$ is a decidable subset of the set $F_{\mathcal{L}}$ of formulas.

3.2 Concrete logics in the new framework

3.2.1 Distinguished logics

Now we define some basic logics. Some of them are well-known, but we recall their definitions for illustrating that they are special cases of the concept defined in Definition 3.1.3 above, and also for fixing our notation.

Definition 3.2.1. (Propositional or sentential logic $\mathcal{L}_S$) Let P be a set, called the set of *atomic formulas* of $\mathcal{L}_S$. Let $\{\land, \neg\}$ be a set disjoint from P, called the set of *logical connectives* of $\mathcal{L}_S$ (usually called *Boolean connectives*).

Propositional (or *sentential*) *logic* (corresponding to P) is defined to be a quadruple

$$\mathcal{L}_S \stackrel{\text{def}}{=} \langle F_S, M_S, mng_S, \models_S \rangle,$$

for which conditions (i)–(iii) below hold.

(i) The set F_S of formulas is the smallest set H satisfying

- $P \subseteq H$
- $\varphi, \psi \in H \Longrightarrow (\varphi \land \psi) \in H$ and $(\neg\varphi) \in H$.

(That is, the alphabet of this logic is $\{\land, \neg\} \cup P$.)

(ii) The class M_S of *models* of $\mathcal{L}_S$ is defined by

$$M_S \stackrel{\text{def}}{=} \{\langle W, v\rangle : W \text{ is a non-empty set and } v : P \to \mathcal{P}(W)\}\,.$$

If $\mathfrak{M} = \langle W, v\rangle \in M_S$ then

W is called the set of *possible states* (or *worlds*[5] or *situations*) of $\mathfrak{M}$.

(iii) Let $\langle W, v\rangle \in M_S$, $w \in W$, and $\varphi \in F_S$. We define the binary relation $w \Vdash_v \varphi$ by recursion on the complexity of the formulas:

- if $p \in P$ then $\big(w \Vdash_v p \stackrel{\text{def}}{\Longleftrightarrow} w \in v(p)\big)$
- if $\psi_1, \psi_2 \in F_S$, then

$$w \Vdash_v \neg\psi_1 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad w \nVdash_v \psi_1$$
$$w \Vdash_v (\psi_1 \land \psi_2) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad w \Vdash_v \psi_1 \text{ and } w \Vdash_v \psi_2.$$

If $w \Vdash_v \varphi$ then we say that φ is *true in* w, or w *forces* φ.

Now $mng_S(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$.

$\langle W, v\rangle \models_S \varphi$ (φ is *valid in* $\langle W, v\rangle$), iff for every $w \in W$, $w \Vdash_v \varphi$.

[5] It is *important* to keep the two senses in which "possible world" can be used separate. The elements $\langle W, v\rangle$ of M_S can be called possible worlds since we inherit this usage from the general concept of a logic. At the same time, the elements $w \in W$ can be called "possible states or worlds" as a technical expression of modal logic. So there is a potential confusion here, which has to be kept in mind.

It is important to note that the set P of atomic formulas is a parameter in the definition of $\mathcal{L}_S$. Namely, in the definition above, P is a fixed but *arbitrary* set. So in a sense $\mathcal{L}_S$ is a function of P, and we could write $\mathcal{L}_S(P)$ to make this explicit. However, the choice of P has only limited influence on the behaviour of $\mathcal{L}_S$, therefore, following the literature we write simply $\mathcal{L}_S$ instead of $\mathcal{L}_S(P)$. From time to time, however, we will have to remember that P is a freely chosen parameter because in certain investigations the choice of P does influence the behaviour of $\mathcal{L}_S = \mathcal{L}_S(P)$.

Exercises 3.2.2.

(1) Think of $P = \emptyset$, of $P = \{p\}$ a singleton, or of infinite P. Write up explicitly what $\mathcal{L}_S$ is like in each of these three cases. What is the cardinality $|F_S|$ of the formulas in each case? What is the cardinality $|\{Mod_{\mathcal{L}_S}(\Sigma) : \Sigma \subseteq F_S\}|$ of axiomatizable model classes in each case?

(2) Let $(\varphi \to \psi) \overset{\text{def}}{\iff} \neg(\varphi \wedge \neg\psi)$ and $(\varphi \leftrightarrow \psi) \overset{\text{def}}{\iff} \big((\varphi \to \psi) \wedge (\psi \to \varphi)\big)$. Prove that

- $\{\varphi\} \models_S \psi \iff \models_S (\varphi \to \psi)$
- $\big(\{\varphi\} \models_S \psi$ and $\{\psi\} \models_S \varphi\big) \iff \models_S (\varphi \leftrightarrow \psi)$.

Exercises 3.2.3. (decidability issues)

(1) Prove that $\mathcal{L}_S$ is a decidable logic (cf. Def. 3.2.1).

(2) (Important!) Let $Ax \subseteq F_S$ be an arbitrary but finite set of formulas. Prove that the set $Csq_{\mathcal{L}_S}(Ax)$ of consequences of Ax (cf. Def. 3.1.7) is decidable.

(3) (This might be too hard. Then ignore it.) Show that (2) becomes false if we generalize it to all decidable sets Ax. (Hint: Use an infinite set P.)

(4) Assume that P is finite. Prove that then (2) becomes true for any set Ax. (Might be too hard; then come back to this after doing the next Ex.(5).)

(5) (Important!) Assume P is finite. Let $\mathfrak{M} \in M_S$ be arbitrary. Prove that $Th_{\mathcal{L}_S}(\mathfrak{M})$ is decidable. (Hint: Let $\varphi \equiv \psi$ iff $\mathfrak{M} \models_S (\varphi \leftrightarrow \psi)$. Prove that $F_S/{\equiv}$ is finite (use that P is finite). But then $F_S/{\equiv}$ together with the logical connectives is a finite algebra. Show that in such a finite algebra we can always compute the "meaning" of any formula.)

As Exercises 3.2.3 show, logic $\mathcal{L}_S$ has a lot of "nice" properties. On the other hand, $\mathcal{L}_S$ is a very "weak" logic. It is well-known that e.g. first-order logic $\mathcal{L}_{\text{FOL}}$ (cf. Def. 3.2.23 below) is much stronger than $\mathcal{L}_S$. However, to build up $\mathcal{L}_{\text{FOL}}$ from $\mathcal{L}_S$ we have to modify the notion of a model, of an atomic formula, etc. in the usual way. We do not want to "throw out" $\mathcal{L}_S$ so drastically, we want to increase the expressive power without changing the class of models or without any other "major surgery". Is it possible to leave M_S unchanged *and* to obtain some significantly stronger (and more interesting) logic (e.g. by adding some new connectives)? The answer is affirmative according to Def. 3.2.4 and Exercises 3.2.6 below. However, we are also interested in how far we can push this procedure of obtaining stronger and stronger logics without changing the models (or other parts) of $\mathcal{L}_S$. What is the price of this increasing expressive power? How far do the nice properties of $\mathcal{L}_S$ remain true?

Definition 3.2.4. (Modal logic $S5$) The set of connectives of *modal logic* $S5$ is $\{\land, \neg, \Diamond\}$.

The set of formulas (denoted as F_{S5}) of $S5$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi \in F_{S5} \implies \Diamond\varphi \in F_{S5}.$$

Let $M_{S5} \stackrel{\text{def}}{=} M_S$. The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of $\Diamond$:

$$w \Vdash_v \Diamond\varphi \quad \stackrel{\text{def}}{\iff} \quad (\exists w' \in W)\ w' \Vdash_v \varphi.$$

Then $mng_{S5}(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{S5}$ is defined as follows.

$$\langle W, v\rangle \models_{S5} \varphi \qquad \stackrel{\text{def}}{\iff} \qquad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, modal logic $S5$ is $S5 \stackrel{\text{def}}{=} \langle F_{S5}, M_{S5}, mng_{S5}, \models_{S5}\rangle$.

Remark 3.2.5. According to a rather respectable (and useful) tradition, an extra-Boolean connective is called a *modality* iff it distributes over disjunction. This will not be true for all of our connectives that we will call modalities. (Exercise: check for which ones it is true). Thus, regrettably, we sometimes ignore this useful tradition. For this tradition cf. e.g. Venema [81, Appendix A (pp. 143–152)].

Exercises 3.2.6. (decidability issues)

(1) Prove that $S5$ is a decidable logic. (Hint: Prove that if $\langle W, v\rangle \not\models_{S5} \varphi$ then $\langle W_0, v\rangle \not\models_{S5} \varphi$ for some finite $W_0 \subseteq W$ in the following way. Let P_0 be the set of atomic formulas occurring in φ. Define an equivalence relation $\sim$ on W by stipulating that $w_1 \sim w_2$ iff they agree on every element of P_0. Then from each equivalence class of $W/{\sim}$ keep only one element in W_0.)

Note that this amounts to repeating Exercises 3.2.3 (1) above for $S5$ in place of $\mathcal{L}_S$.

(2) Repeat Exercises 3.2.3 (2) above for $S5$ in place of $\mathcal{L}_S$.

(3) (Important!) Repeat Exercises 3.2.3 (5) above for $S5$ in place of $\mathcal{L}_S$.

(4) Try doing Exercises 3.2.3 (4) for $S5$.

The following logic is discussed e.g. in Sain [69, 70], Venema [81], Roorda [66], but see also Segerberg [74] who traces this logic back to von Wright.

Definition 3.2.7. (Difference logic $\mathcal{L}_D$) The set of connectives of *difference logic* $\mathcal{L}_D$ is $\{\wedge, \neg, D\}$. The set of formulas (denoted as F_D) of $\mathcal{L}_D$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi \in F_D \implies D\varphi \in F_D.$$

Let $M_D \stackrel{\text{def}}{=} M_{S5} (= M_S)$. The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of D:

$$w \Vdash_v D\varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists w' \in W \smallsetminus \{w\})\ w' \Vdash_v \varphi.$$

Then $mng_D(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_D$ is defined as follows.

$$\langle W, v\rangle \models_D \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, difference logic $\mathcal{L}_D$ is $\mathcal{L}_D \stackrel{\text{def}}{=} \langle F_D, M_D, mng_D, \models_D\rangle$. We note that $\mathcal{L}_D$ is also called *"Some-other-time logic"* (cf. Sain [70], Segerberg [74]).

Exercises 3.2.8.

(1) (Important!) Try to guess whether Exercises 3.2.3 (1), (4), (5) extend to $\mathcal{L}_D$. Try hard, do not give up too soon and remember that you are required to *guess* only. Try to formulate some reasons why you are guessing the outcome you do. Try to guess the same for Exercises 3.2.3 (2) and (3).

(2) Prove that Exercises 3.2.3 (1), (4), (5) do generalize to $\mathcal{L}_D$! (Hint: Use the same equivalence relation $\sim$ defined on W as in Exercises 3.2.6 (1). But now, from each equivalence class keep two elements (if there are more than one there) in W_0.)

(3) Prove that the connective $\Diamond$ of $S5$ is expressible in $\mathcal{L}_D$. Prove that D is not expressible in $S5$. (Hint: If the second one is too hard, postpone it to the end of this subsection.)

The logics $\mathcal{L}_{\kappa\text{-times}}$ to be introduced below play quite an essential rôle in Artificial Intelligence in the theory what is called there "stratified logic", cf. e.g. works of H. J. Ohlbach, see e.g. [33].

Definition 3.2.9. (κ-times logic $\mathcal{L}_{\kappa\text{-times}}$, twice logic Tw) Let κ be any cardinal. The set of connectives of *κ-times logic* $\mathcal{L}_{\kappa\text{-times}}$ is $\{\wedge, \neg, \Diamond_\kappa\}$. The set of formulas (denoted as $F_{\Diamond_\kappa}$) of $\mathcal{L}_{\kappa\text{-times}}$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi \in F_{\Diamond_\kappa} \implies \Diamond_\kappa\varphi \in F_{\Diamond_\kappa}.$$

Let $M_{\Diamond_\kappa} \stackrel{\text{def}}{=} M_{S5} (= M_S)$. The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of $\Diamond_\kappa$:

$$w \Vdash_v \Diamond_\kappa\varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists H \subseteq W)\big(|H| = \kappa \text{ and } (\forall w' \in H)\ w' \Vdash_v \varphi\big)\,.$$

Then $mng_{\Diamond_\kappa}(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\Diamond_\kappa}$ is defined as follows.

$$\langle W, v\rangle \models_{\Diamond_\kappa} \varphi \quad \stackrel{\text{def}}{\iff} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, κ-times logic $\mathcal{L}_{\kappa\text{-times}}$ is $\mathcal{L}_{\kappa\text{-times}} \stackrel{\text{def}}{=} \langle F_{\Diamond_\kappa}, M_{\Diamond_\kappa}, mng_{\Diamond_\kappa}, \models_{\Diamond_\kappa}\rangle$. We note that if $\kappa = 2$ then logic $\mathcal{L}_{2\text{-times}}$ is also called *Twice logic* and is denoted as Tw.

Exercises 3.2.10.

(1) Prove that $\mathcal{L}_{0\text{-times}}$ is equivalent to $\mathcal{L}_S$ and that $\mathcal{L}_{1\text{-times}}$ is equivalent to $S5$. Prove that $\Diamond_2$ is expressible in $\mathcal{L}_D$. (What do you think of the other direction of expressing D in $\mathcal{L}_{n\text{-times}}$, for some $n \in \omega$?)

(2) Try to guess whether Exercises 3.2.3 (1), (4), (5) extend to $\mathcal{L}_{n\text{-times}}$ for finite n (that is, for $\kappa = n \in \omega$). How about $n = 0$?? How about $n = 1$?

(3) (Probably too hard. May be ignored.) Try to guess how the logics introduced so far, especially the various $\mathcal{L}_{\kappa\text{-times}}$ logics for different cardinals κ, relate to each other in terms of expressive power. (Do *not* spend all your time on this!) Is the connective $\Diamond$ of $S5$ expressible in $\mathcal{L}_{2\text{-times}}$?

(4) Prove that Exercises 3.2.3 (1), (4), (5) generalize to $\mathcal{L}_{2\text{-times}}$. (Hint: The same as given for $\mathcal{L}_D$ in (the hints of) Exercises 3.2.8 (3), 3.2.6 (1).)

(5) Can you generalize Exercises 3.2.3 (1), (4), (5) to $\mathcal{L}_{3\text{-times}}$? If yes, how about $\mathcal{L}_{n\text{-times}}$, for finite n? (Hint: Keep n elements from each equivalence class of $\sim$.)

(6) What do you think, does the method of Exercises 3.2.6 (1), 3.2.8 (3) and 3.2.10 (5), (6) above generalize to $\mathcal{L}_{\kappa\text{-times}}$ when κ is infinite? (Hint: Look at the hint of Exercise 3.2.14 below. Do not spend all your time with this exercise.)

(7) Think about the logic with extra-Boolean logical connectives $\Diamond_2$ *and* $\Diamond_3$. Is it equivalent to $\mathcal{L}_{2\text{-times}}$ or to $\mathcal{L}_{3\text{-times}}$? (Hint: No.) Is it decidable?

(8) Think about the logic $\mathcal{L}_{\text{count}}$ with extra-Boolean connectives $\{\Diamond_n : n \in \omega\}$. It can "count" up to any natural number. Is it decidable? (Hint: Yes.)

So far the *extra-Boolean connectives* $\Diamond$, D, $\Diamond_\kappa$ were all unary ones. Next we will see examples in which the extra-Booleans are binary.

Definition 3.2.11. ($\mathcal{L}_{\mathsf{bin}}$) The set of connectives of $\mathcal{L}_{\text{bin}}$ is $\{\wedge, \neg, \blacklozenge\}$, where $\blacklozenge$ is a new *binary* modality. The set of formulas (denoted as F_{bin}) of $\mathcal{L}_{\text{bin}}$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi, \psi \in F_{\text{bin}} \implies \blacklozenge(\varphi, \psi) \in F_{\text{bin}}.$$

Let $M_{\text{bin}} \stackrel{\text{def}}{=} M_S$. The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of $\blacklozenge$:

$$w \Vdash_v \blacklozenge(\varphi, \psi) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists u, z \in W)\big[w \neq u \neq z \neq w \text{ and } u \Vdash_v \varphi \text{ and } z \Vdash_v \psi\big]\,.$$

As usual, $mng_{\text{bin}}(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{bin}$ is defined as follows.

$$\langle W, v\rangle \models_{\text{bin}} \varphi \qquad \stackrel{\text{def}}{\Longleftrightarrow} \qquad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, let $\mathcal{L}_{\text{bin}} \stackrel{\text{def}}{=} \langle F_{\text{bin}}, M_{\text{bin}}, mng_{\text{bin}}, \models_{\text{bin}}\rangle$.

Exercises 3.2.12.

(1) Compare $\mathcal{L}_{\text{bin}}$ with the previous logics. E.g. show that $\Diamond$ and D are expressible in $\mathcal{L}_{\text{bin}}$. Is $\Diamond_3$ expressible in $\mathcal{L}_{\text{bin}}$? (Hint: $\blacklozenge\big(\varphi \wedge \blacklozenge(\varphi, \varphi)\big)$.)

(2) Try to guess whether Exercises 3.2.3 (1), (4), (5) extend to $\mathcal{L}_{\text{bin}}$. (Hint: The method of extending Exercises 3.2.3 (1) to $\mathcal{L}_D$ should be adaptable to the present case, cf. hint of Exercises 3.2.8 (3). So validity in $\mathcal{L}_{\text{bin}}$ should be decidable. To attack Exercises 3.2.3 (5) in this case, recall the equivalence $\equiv$ on formulas in the hint for Exercises 3.2.3 (5). Check whether $F_{\text{bin}}/\!\equiv$ is still finite!)

Definition 3.2.13. ($\mathcal{L}_{\mathsf{more}}$) The set of connectives of $\mathcal{L}_{\text{more}}$ is $\{\wedge, \neg, \blacklozenge_{\text{more}}\}$, where $\blacklozenge_{\text{more}}$ is a new *binary* modality. The set of formulas (denoted as F_{more}) of $\mathcal{L}_{\text{more}}$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi, \psi \in F_{\text{more}} \implies \blacklozenge_{\text{more}}(\varphi, \psi) \in F_{\text{more}}.$$

Let $M_{\text{more}} \stackrel{\text{def}}{=} M_S$. The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of $\blacklozenge_{\text{more}}$:

$$w \Vdash_v \blacklozenge_{\text{more}}(\varphi, \psi) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad |\,\{u \in W : u \Vdash_v \varphi\}\,| \geq |\,\{u \in W : u \Vdash_v \psi\}\,|.$$

As usual, $mng_{\text{more}}(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\text{more}}$ is defined as follows.

$$\langle W, v\rangle \models_{\text{more}} \varphi \qquad \stackrel{\text{def}}{\Longleftrightarrow} \qquad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, $\mathcal{L}_{\text{more}} \stackrel{\text{def}}{=} \langle F_{\text{more}}, M_{\text{more}}, mng_{\text{more}}, \models_{\text{more}}\rangle$.

Exercises 3.2.14.

(1) Show that the connective $\Diamond$ of $S5$ is expressible in $\mathcal{L}_{\text{more}}$.

(2) Compare $\mathcal{L}_{\text{more}}$ with the previous logics (concerning their expressive power).

(3) Try to guess whether Exercises 3.2.3 (1) or (5) extend to $\mathcal{L}_{\mathrm{more}}$. (Hint: Recall the hint given for Exercises 3.2.3 (5). Try to prove that for any fixed $\mathfrak{M}$, assuming that P is finite, the set $F_{\mathrm{more}}/\equiv$ is still finite.)

(4) (If too hard, might be postponed to the end of this paper, but give it a few hours first, and then look at the detailed hints in subsection 3.2.3.) Prove that Exercises 3.2.3 (1) does extend to $\mathcal{L}_{\mathrm{more}}$ (i.e. $\mathcal{L}_{\mathrm{more}}$ is decidable). (Hint: If you followed the hints given for Exercises 3.2.6 (1), 3.2.8 (3), etc. then you proved for those logics the so called *finite model property (fmp)*. ("fmp" says that a formula is valid [in $\mathcal{L}$] iff it is valid in all finite models [of $\mathcal{L}$]. The cardinality of a model $\langle W, v\rangle$ is that of W.) Decide whether $\mathcal{L}_{\mathrm{more}}$ has the fmp. You will see, it does not. Thus the hint given for Exercises 3.2.6 (1), 3.2.8 (3), etc. has to be refined in order to make it applicable here. See subsection 3.2.3 for a detailed hint.)

(5) Define $\Diamond_{\mathrm{max}}$ to be $\blacklozenge(\varphi, \mathit{True})$, where *True* abbreviates $(\varphi \vee \neg\varphi)$. Define $\mathcal{L}_{\mathrm{max}}$ by replacing $\Diamond_\kappa$ with $\Diamond_{\mathrm{max}}$ in $\mathcal{L}_{\kappa\text{-times}}$. What are the basic properties of $\mathcal{L}_{\mathrm{max}}$? Write up an explicit definition for $\mathcal{L}_{\mathrm{max}}$ without referring to $\mathcal{L}_{\mathrm{more}}$. Is $\Diamond_{\mathrm{max}}$ expressible in one of the logics in Def's. 3.2.1–3.2.13?

Beginning with Definition 3.2.15 below, we start discussing various Arrow Logics. The field of Arrow Logics grew out of application areas in Logic, Language and Computation, and plays an important rôle there, cf. e.g. van Benthem [78, 79], [54], and the proceedings of the Arrow Logic day at the conference "Logic at Work" (December 1992, Amsterdam [CCSOM of Univ. of Amsterdam]).

So far we strengthened $\mathcal{L}_S$ without modifying the class M_S of models. The mildest way of modifying M_S is to take a subclass (i.e. the models themselves do not change, only some of them are excluded).

Definition 3.2.15. (Arrow logic $\mathcal{L}_{\mathrm{PAIR}}$) The set of connectives of $\mathcal{L}_{\mathrm{PAIR}}$ is $\{\wedge, \neg, \circ\}$, where $\circ$ is a binary connective. The set of formulas (denoted as F_{PAIR}) of $\mathcal{L}_{\mathrm{PAIR}}$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi, \psi \in F_{\mathrm{PAIR}} \implies \varphi \circ \psi \in F_{\mathrm{PAIR}}.$$

Let $M_{\mathrm{PAIR}} \stackrel{\mathrm{def}}{=} \{\langle W, v\rangle \in M_S : W \subseteq U \times U \text{ for some set } U\}$.

The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of $\circ$:

$$\langle a, b\rangle \Vdash_v \varphi \circ \psi \iff \exists c\big[\langle a, c\rangle, \langle c, b\rangle \in W \text{ and } \langle a, c\rangle \Vdash_v \varphi \text{ and } \langle c, b\rangle \Vdash_v \psi\big].$$

As usual, $\mathit{mng}_{\mathrm{PAIR}}(\varphi, \langle W, v\rangle) \stackrel{\mathrm{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\mathrm{PAIR}}$ is defined as follows.

$$\langle W, v\rangle \models_{\mathrm{PAIR}} \varphi \quad \stackrel{\mathrm{def}}{\iff} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, arrow logic $\mathcal{L}_{\mathrm{PAIR}}$ is $\mathcal{L}_{\mathrm{PAIR}} \stackrel{\mathrm{def}}{=} \langle F_{\mathrm{PAIR}}, M_{\mathrm{PAIR}}, \mathit{mng}_{\mathrm{PAIR}}, \models_{\mathrm{PAIR}}\rangle$.

Exercises 3.2.16.

(1) (Important!) Try to guess whether Exercises 3.2.3 (1), (4), (5) extend to $\mathcal{L}_{\mathrm{PAIR}}$. Guess separately (the answers need not be uniform). Concentrate first only on Exercises 3.2.3 (1). This will be very hard but spend some considerable time with guessing each of the exercises. Do not spend all your time on this, but 8 hours is reasonable. Do not worry if you cannot prove anything in this connection, the insight gained by trying is enough. The solutions will be given in subsection 3.2.3, *but* wait one week at least before reading them!!

(2) Assume that the set P of atomic formulas is *finite*. Is there a model $\mathfrak{M}$ of $\mathcal{L}_{\mathrm{PAIR}}$ such that $Th_{\mathcal{L}_{\mathrm{PAIR}}}(\mathfrak{M})$ is not even recursively enumerable? Note that this is a generalization of Exercises 3.2.3 (5). (Why?) (Hint: A set X is called *transitive* if $(\forall y \in X)\ y \subseteq X$. A set Y is called *hereditarily finite* if $Y \subseteq X$ for some finite transitive set X. Let $\mathfrak{M} = \langle W, v\rangle$ be defined as follows.

$$\begin{aligned}
W &\stackrel{\text{def}}{=} \text{“all hereditarily finite sets”} \\
P &\stackrel{\text{def}}{=} \{p_0, p_1, p_2\} \\
v(p_0) &\stackrel{\text{def}}{=} \{\langle a, b\rangle \in W \ : \ a \in b\} \\
v(p_1) &\stackrel{\text{def}}{=} \{\langle a, b\rangle \in W \ : \ b \in a\} \\
v(p_2) &\stackrel{\text{def}}{=} \{\langle a, b\rangle \in W \ : \ a = b\} \ .
\end{aligned}$$

Show first that many relations definable in the model $\mathfrak{W} = \langle W, \in\rangle$ of Finite Set Theory (using first-order logic) are also definable in $\mathfrak{M}$ using $\mathcal{L}_{\mathrm{PAIR}}$. Define first the relation $\{\langle\emptyset, \emptyset\rangle\}$. (Hint: $p_2 \wedge \neg(\mathit{True} \circ p_0)$.) Then the relation $\{\langle X, Y\rangle : Y \subseteq X \in W\}$. Next try to define the relations $\{\langle X, \cup X\rangle : X \in W\}$, and $\{\langle X, \mathcal{P}(X)\rangle : X \in W\}$. Eventually you will have to use the well known fact that the set of first-order formulas involving only 3 variables (free or bound) and valid in $\mathfrak{W}$ is not recursively enumerable.

(This exercise is not easy if you are not experienced with first-order logic and Gödel's incompleteness theorem, so you may postpone doing it after having spent about 7 hours with it.)

(3) Compare the answer to the previous exercise with the fact that $Th(\mathfrak{M})$ is decidable for all the logics discussed so far. Observe the contrast! Try to find a reason for the sudden change of behaviour (of the logics we are looking at)!

(4) Try to guess the answer (yes or no) to Exercises 3.2.3 (2), (3) when applied to $\mathcal{L}_{\mathrm{PAIR}}$. Is there e.g. a finite set $Ax \subseteq F_{\mathrm{PAIR}}$ such that $Csq_{\mathcal{L}_{\mathrm{PAIR}}}(Ax)$ would not be decidable? (Do not spend all your time here. But spend a few hours.)

Definition 3.2.17. (Arrow logic $\mathcal{L}_{\mathrm{REL}}$) The set of connectives of $\mathcal{L}_{\mathrm{REL}}$ is $\{\wedge, \neg, \circ\}$.

We let $F_{\mathrm{REL}} \stackrel{\mathrm{def}}{=} F_{\mathrm{PAIR}}$ and

$$M_{\mathrm{REL}} \stackrel{\mathrm{def}}{=} \{\langle W, v\rangle \in M_S : W = U \times U \text{ for some set } U\}\,.$$

The definition of $w \Vdash_v \varphi$ is the same as in the case of $\mathcal{L}_{\mathrm{PAIR}}$.

As usual, $mng_{\mathrm{REL}}(\varphi, \langle W, v\rangle) \stackrel{\mathrm{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\mathrm{REL}}$ is defined as follows.

$$\langle W, v\rangle \models_{\mathrm{REL}} \varphi \quad \stackrel{\mathrm{def}}{\Longleftrightarrow} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, arrow logic $\mathcal{L}_{\mathrm{REL}}$ is $\mathcal{L}_{\mathrm{REL}} \stackrel{\mathrm{def}}{=} \langle F_{\mathrm{REL}}, M_{\mathrm{REL}}, mng_{\mathrm{REL}}, \models_{\mathrm{REL}}\rangle$.

Exercises 3.2.18.

(1) The logics $\mathcal{L}_{\mathrm{REL}}$ and $\mathcal{L}_{\mathrm{PAIR}}$ are among the most important ones discussed in the whole material. So think about $\mathcal{L}_{\mathrm{REL}}$ and compare it with the previous ones!

(2) Show that the connective $\Diamond$ of $S5$ is expressible in $\mathcal{L}_{\mathrm{REL}}$.

(Hint: $\Diamond\varphi$ is $(\mathit{True} \circ \varphi) \circ \mathit{True}$.) Show that "$\circ$" is associative in $\mathcal{L}_{\mathrm{REL}}$ (i.e.

$$\models_{\mathrm{REL}} \big((\varphi_1 \circ \varphi_2) \circ \varphi_3\big) \leftrightarrow \big(\varphi_1 \circ (\varphi_2 \circ \varphi_3)\big).$$

(Hence omitting brackets and writing "$\mathit{True} \circ \varphi \circ \mathit{True}$" [for $\Diamond\varphi$] is justified.)

(3) (Important!) Try to guess whether some of Exercises 3.2.3 (1)–(5) generalize to $\mathcal{L}_{\mathrm{REL}}$ (give yes or no answers). (This is very hard, so concentrate on only one item for a while. Do not spend all your time, but spend 6–8 hours. Solutions will be in subsection 3.2.3, but wait a few weeks before looking at them.)

(4) Try to prove that the set $Th_{\mathcal{L}_{\mathrm{REL}}}(M_{\mathrm{REL}})$ of validities of $\mathcal{L}_{\mathrm{REL}}$ is recursively enumerable. (Hint: To $\varphi \in F_{\mathrm{REL}}$ associate a first-order formula $f(\varphi)$ such that

$$\models_{\mathrm{REL}} \varphi \quad \Longleftrightarrow \quad \models f(\varphi).$$

Then use the recursive enumerability of the validities of first-order logic (e.g. via Gödel's completeness theorem). If this would be too hard, you may postpone it to the end of the subsection, but do not postpone it forever.)

Definition 3.2.19. (Arrow logics $\mathcal{L}_{\mathrm{ARW0}}$, $\mathcal{L}_{\mathrm{ARROW}}$, $\mathcal{L}_{\mathrm{RA}}$) The set of connectives of *arrow logics* $\mathcal{L}_{\mathrm{ARW0}}$, $\mathcal{L}_{\mathrm{ARROW}}$, $\mathcal{L}_{\mathrm{RA}}$ is $\{\wedge, \neg, \circ, {}^{\smile}, \mathit{Id}\}$, where $\circ$ is a binary, ${}^{\smile}$ is a unary, and Id is a zero-ary modality.

- The set of formulas (denoted as F_{ARW0}) of $\mathcal{L}_{\mathrm{ARW0}}$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clauses:

$$\begin{aligned}&\varphi, \psi \in F_{\mathrm{ARW0}} \implies (\varphi \circ \psi),\ \varphi^{\smile} \in F_{\mathrm{ARW0}}\\ &\mathit{Id} \in F_{\mathrm{ARW0}}.\end{aligned}$$

The models are those of propositional logic $\mathcal{L}_S$ enriched with three relations, called *accessibility relations*. That is,

$$M_{\text{ARW0}} \stackrel{\text{def}}{=} \{\big\langle\langle W,v\rangle, C_1, C_2, C_3\big\rangle \ : \ \langle W,v\rangle \in M_S,\ C_1 \subseteq W\times W\times W, \\ C_2 \subseteq W\times W,\ C_3 \subseteq W\}.$$

For propositional connectives $\neg$ and $\wedge$ the definition of $w \Vdash_v \varphi$ is the same as in the propositional case. For the new connectives we have:

$$\begin{aligned}
w \Vdash_v (\varphi\circ\psi) \ &\stackrel{\text{def}}{\Longleftrightarrow}\ (\exists w_1, w_2 \in W) \\
&\qquad\qquad \big(C_1(w,w_1,w_2) \text{ and } w_1 \Vdash_v \varphi \text{ and } w_2 \Vdash_v \psi\big) \\
w \Vdash_v \varphi^{\smile} \ &\stackrel{\text{def}}{\Longleftrightarrow}\ (\exists w' \in W)\big(C_2(w,w') \text{ and } w' \Vdash_v \varphi\big) \\
w \Vdash_v \mathit{Id} \ &\stackrel{\text{def}}{\Longleftrightarrow}\ C_3(w).
\end{aligned}$$

As usual, $mng_{\text{ARW0}}(\varphi, \langle W,v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\text{ARW0}}$ is defined as follows.

$$\langle W,v\rangle \models_{\text{ARW0}} \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Then arrow logic $\mathcal{L}_{\text{ARW0}}$ is

$$\mathcal{L}_{\text{ARW0}} \stackrel{\text{def}}{=} \langle F_{\text{ARW0}}, M_{\text{ARW0}}, mng_{\text{ARW0}}, \models_{\text{ARW0}}\rangle.$$

- $F_{\text{ARROW}} \stackrel{\text{def}}{=} F_{\text{ARW0}}$. $M_{\text{ARROW}} \stackrel{\text{def}}{=} M_{\text{PAIR}}$. For connectives $\neg$, $\wedge$ and $\circ$ the definition of $w \Vdash_v \varphi$ is the same as in the case of $\mathcal{L}_{\text{PAIR}}$. For the new connectives we have:

$$\begin{aligned}
\langle a,b\rangle \Vdash_v \varphi^{\smile} \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad \big[\langle b,a\rangle \in W \text{ and } \langle b,a\rangle \Vdash_v \varphi\big], \\
\langle a,b\rangle \Vdash_v \mathit{Id} \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad a = b.
\end{aligned}$$

As usual, $mng_{\text{ARROW}}(\varphi, \langle W,v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\text{ARROW}}$ is defined by

$$\langle W,v\rangle \models_{\text{ARROW}} \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Arrow logic $\mathcal{L}_{\text{ARROW}}$ is defined by

$$\mathcal{L}_{\text{ARROW}} \stackrel{\text{def}}{=} \langle F_{\text{ARROW}}, M_{\text{ARROW}}, mng_{\text{ARROW}}, \models_{\text{ARROW}}\rangle.$$

- $F_{\text{RA}} \stackrel{\text{def}}{=} F_{\text{ARROW}}$. $M_{\text{RA}} \stackrel{\text{def}}{=} M_{\text{REL}}$. The definitions of $w \Vdash_v \varphi$, mng_{RA} and $\models_{\text{RA}}$ are the same as in the case of $\mathcal{L}_{\text{ARROW}}$.

  Arrow logic $\mathcal{L}_{\text{RA}}$ is $\mathcal{L}_{\text{RA}} \stackrel{\text{def}}{=} \langle F_{\text{RA}}, M_{\text{RA}}, mng_{\text{RA}}, mng_{\text{RA}}\rangle$. $\mathcal{L}_{\text{RA}}$ is also called the *logic of relation algebras*.

Exercise 3.2.20. Consider the fragment

$$\mathcal{L}^0_{\mathrm{ARW0}} = \langle F^0_{\mathrm{ARW0}}, M^0_{\mathrm{ARW0}}, mng^0_{\mathrm{ARW0}} \models^0_{\mathrm{ARW0}}\rangle$$

of arrow logic $\mathcal{L}_{\mathrm{ARW0}}$ defined above which differ from the original version only in that it does not contain the logical connectives $\smile$ and Id. Prove that $\mathcal{L}_{\mathrm{ARW0}}$ is equivalent to $\mathcal{L}_{\mathrm{PAIR}}$ in the sense that they have the same semantical consequence relation that is, for all $\Sigma \cup \{\varphi\} \subseteq F^0_{\mathrm{ARW0}} = F_{\mathrm{PAIR}}$

$$\Sigma \models^0_{\mathrm{ARW0}} \varphi \quad \iff \quad \Sigma \models_{\mathrm{PAIR}} \varphi.$$

Prove that $\mathcal{L}_{\mathrm{ARW0}}$ is not equivalent, in the above sense, to $\mathcal{L}_{\mathrm{ARROW}}$.

Definition 3.2.21. (First-order logic with n variables $\mathcal{L}_n$) Let $V \stackrel{\text{def}}{=} \{v_0, \ldots, v_{n-1}\}$ be a set, called the set of *variables* of $\mathcal{L}_n$. Let the set P of *atomic formulas* of $\mathcal{L}_n$ be defined as $P \stackrel{\text{def}}{=} \{r_i(v_0 \ldots v_{n-1}) : i \in I\}$ for some set I.

(i) The set F_n of formulas is the smallest set H satisfying

- $P \subseteq H$
- $(v_i = v_j) \in H$ for each $i, j < n$
- $\varphi, \psi \in H,\ v_i \in V \implies (\varphi \wedge \psi),\ \neg\varphi,\ \exists v_i\varphi \in H.$

(ii) The class M_n of *models* of $\mathcal{L}_n$ is defined by

$$M_n \stackrel{\text{def}}{=} \{\langle M, R_i\rangle_{i\in I} \ :\ M \text{ is a non-empty set and for all } i \in I,\ R_i \subseteq {}^nM\}.$$

If $\mathfrak{M} = \langle M, R_i\rangle_{i\in I} \in M_n$ then M is called the *universe* (or *carrier*) of $\mathfrak{M}$.

(iii) Let $\mathfrak{M} = \langle M, R_i\rangle_{i\in I} \in M_n$, $q \in {}^nM$ and $\varphi \in F_n$. We define the ternary relation $\mathfrak{M} \models \varphi[q]$ by recursion on the complexity of φ as follows.

- $\mathfrak{M} \models r_i(v_0 \ldots v_{n-1})[q] \quad \stackrel{\text{def}}{\iff} \quad q \in R_i \quad (i \in I)$
- $\mathfrak{M} \models (v_i = v_j)[q] \quad \stackrel{\text{def}}{\iff} \quad q_i = q_j \quad (i, j < n)$
- if $\psi_1, \psi_2 \in F_n$, then

$$\begin{aligned} \mathfrak{M} \models \neg\psi_1[q] &\quad \stackrel{\text{def}}{\iff} \quad \text{not } \mathfrak{M} \models \psi_1[q] \\ \mathfrak{M} \models (\psi_1 \wedge \psi_2)[q] &\quad \stackrel{\text{def}}{\iff} \quad \mathfrak{M} \models \psi_1[q] \text{ and } \mathfrak{M} \models \psi_2[q] \\ \mathfrak{M} \models \exists v_i\psi_1[q] &\quad \stackrel{\text{def}}{\iff} \quad (\exists q' \in {}^nM)(\forall j < n)\big(j \neq i \ \Rightarrow \\ &\qquad\qquad (q'_j = q_j \text{ and } \mathfrak{M} \models \psi_1[q'])\big). \end{aligned}$$

If $\mathfrak{M} \models \varphi[q]$ then we say that the *evaluation* q *satisfies* φ *in the model* $\mathfrak{M}$.

Now we define mng_n as follows.

$$mng_n(\varphi, \mathfrak{M}) \stackrel{\text{def}}{=} \{q \in {}^nM \ :\ \mathfrak{M} \models \varphi[q]\}.$$

(iv) Validity is defined by

$$\mathfrak{M} \models_n \varphi \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\forall q \in {}^n M) \quad \mathfrak{M} \models \varphi[q]\,.$$

First-order logic with n variables

$$\mathcal{L}_n \overset{\text{def}}{=} \langle F_n, M_n, mng_n, \models_n \rangle$$

has been defined.

Intuitive explanation

Our $\mathcal{L}_n$ might look somewhat unusual because we do not allow substitution of variables in atomic formulas $r_i(v_0 \dots)$. This does not restrict generality, because substitution is expressible by using quantifiers and equality. This is explained in more detail in Remark 3.2.24 (2) below.

Exercises 3.2.22.

(1) Write up a detailed definition of $\mathcal{L}_n$ as a modal logic. (Hint: Define the class of models by

$$M_n \overset{\text{def}}{=} \{\langle W, v\rangle \in M_S \ : \ W = {}^n U \text{ for some set } U\}\ .$$

The extra-Boolean connectives are "$\exists v_i$" and "$v_i = v_j$" for $i, j < n$. Here $(\exists v_i)$ is a unary modality while $(v_i = v_j)$ is a zero-ary modality.)

(2) Show that in some sense $\mathcal{L}_1$ is equivalent to modal logic S5. (In what sense? Try to define!)

(3) Show that in some sense $\mathcal{L}_D$ and $\mathcal{L}_{\text{2-times}}$ are comparable with $\mathcal{L}_2$. Show that $\mathcal{L}_D$ and $\mathcal{L}_{\text{2-times}}$ are strictly weaker than $\mathcal{L}_2$.

Next we define first-order logic in a non-traditional form. Therefore, below the definition, we will give intuitive explanations for our present definition.

Definition 3.2.23. (First-order logic $\mathcal{L}_{\text{FOL}}$, rank-free formulation) Recall that ω is the set of natural numbers. Let $V \overset{\text{def}}{=} \{v_i : i \in \omega\}$ be a set, called the set of *variables* of $\mathcal{L}_{\text{FOL}}$. As before, let P be an arbitrary set, called the set of *atomic formulas* of $\mathcal{L}_{\text{FOL}}$. (Now, we will think of atomic formulas as relation symbols, hence we will use the letter R for elements of P rather than p as in case of $\mathcal{L}_S$.)

(i) The set F_{FOL} is the smallest set H satisfying

- $P \subseteq H$
- $(v_i = v_j) \in H$ for each $i, j \in \omega$
- $\varphi, \psi \in H,\ i \in \omega \implies (\varphi \wedge \psi),\ \neg\varphi,\ \exists v_i \varphi \in H.$

(ii) The class M_{FOL} of *models* of $\mathcal{L}_{\mathrm{FOL}}$ is

$$M_{\mathrm{FOL}} \stackrel{\mathrm{def}}{=} \{ \mathfrak{M} : \mathfrak{M} = \langle M, R^{\mathfrak{M}} \rangle_{R \in P}, M \text{ is a non-empty set and } \text{for all } R \in P, R^{\mathfrak{M}} \subseteq {}^{n}M \text{ for some } n \in \omega \big\} .$$

If $\mathfrak{M} \in M_{FOL}$ then M denotes the universe of $\mathfrak{M}$. Further, for $R \in P$, $R^{\mathfrak{M}}$ denotes the meaning of R in $\mathfrak{M}$.

(iii) Validity relation $\models_{FOL}$.

In $\mathcal{L}_{S5}$ the "basic semantical units" were the possible situations $w \in W$. In FOL the basic semantical units are the evaluations of individual variables into models $\mathfrak{M}$, where $q \in {}^{\omega}M$ and q evaluates variables v_i as element $q_i \in M$ in the model $\mathfrak{M}$. To follow model theoretic tradition, instead of $\mathfrak{M}, q \Vdash \varphi$ we will write $\mathfrak{M} \models \varphi[q]$ (though the former would be more in the line with our definitions of $\mathcal{L}_{S5}$ etc.).

Let $\mathfrak{M} = \langle M, R^{\mathfrak{M}} \rangle_{R \in P} \in M_{\mathrm{FOL}}$, $q \in {}^{\omega}M$ and $\varphi \in F_{\mathrm{FOL}}$. We define the ternary relation "$\mathfrak{M} \models \varphi[q]$" by recursion on the complexity of φ as follows:

- $\mathfrak{M} \models R[q] \quad \stackrel{\mathrm{def}}{\Longleftrightarrow} \quad \langle q_0, \ldots, q_{n-1} \rangle \in R^{\mathfrak{M}} \quad$ for some $n \in \omega \quad (R \in P)$
- $\mathfrak{M} \models (v_i = v_j)[q] \quad \stackrel{\mathrm{def}}{\Longleftrightarrow} \quad q_i = q_j \quad (i, j \in \omega)$
- if $\psi_1, \psi_2 \in F_{\mathrm{FOL}}$, then

$$\begin{aligned}
\mathfrak{M} \models \neg\psi_1[q] &\stackrel{\mathrm{def}}{\Longleftrightarrow} \text{not } \mathfrak{M} \models \psi_1[q] \\
\mathfrak{M} \models (\psi_1 \land \psi_2)[q] &\stackrel{\mathrm{def}}{\Longleftrightarrow} \mathfrak{M} \models \psi_1[q] \text{ and } \mathfrak{M} \models \psi_2[q] \\
\mathfrak{M} \models \exists v_i \psi_1[q] &\stackrel{\mathrm{def}}{\Longleftrightarrow} (\exists q' \in {}^{\omega}M)(\forall j \in \omega) \\
&\qquad \big(j \neq i \Rightarrow (q'_j = q_j \text{ and } \mathfrak{M} \models \psi_1[q'])\big).
\end{aligned}$$

If $\mathfrak{M} \models \varphi[q]$ holds then we say that q *satisfies* φ in $\mathfrak{M}$.

Now we define mng_{FOL} as follows.

$$\models_{\mathrm{FOL}} (\varphi, \mathfrak{M}) \stackrel{\mathrm{def}}{=} \{q \in {}^{\omega}M : \mathfrak{M} \models \varphi[q]\}.$$

(iv) Validity is defined by

$$\mathfrak{M} \models_{\mathrm{FOL}} \varphi \quad \stackrel{\mathrm{def}}{\Longleftrightarrow} \quad (\forall q \in {}^{\omega}M)\ \mathfrak{M} \models \varphi[q].$$

(v) *First-order logic* (in rank-free form) is

$$\mathcal{L}_{\mathrm{FOL}} \stackrel{\mathrm{def}}{=} \langle F_{\mathrm{FOL}}, M_{\mathrm{FOL}}, mng_{\mathrm{FOL}}, \models_{\mathrm{FOL}} \rangle .$$

For more on $\mathcal{L}_{\mathrm{FOL}}$ see e.g. Henkin–Tarski [39], Simon [75], Venema [81], Henkin–Monk–Tarski [37, §4.3].

Intuitive explanations for $\mathcal{L}_{\mathrm{FOL}}$

There are two kinds of explanations needed. Namely,

(i) Why does the definition go as it does?

and

(ii) Why do we say that $\mathcal{L}_{\mathrm{FOL}}$ is first-order logic? That is, what are the connections between $\mathcal{L}_{\mathrm{FOL}}$ and the more traditional formulations of first-order logic?

We discuss (ii) in Remark 3.2.24 below. Let us first turn to (i).

Let R be a relation symbol, that is $R \in P$. Then instead of the traditional formula $R(v_0, v_1, v_2, \dots)$ we simply write R. That is, we treat R as a shorthand for $R(v_0, v_1, v_2, \dots)$.

So this is why R is an atomic formula. The next part of the definition which may need intuitive explanation is the definition of the satisfaction relation's behaviour on R. That is, the definition of $\mathfrak{M} \models R[q]$. So let $R^{\mathfrak{M}} \subseteq {}^nM$ be given. Recall that R abbreviates $R(v_0, v_1, v_2, \dots)$ here. Clearly we want $\mathfrak{M} \models R[q]$ to hold if in the traditional sense $\mathfrak{M} \models R(v_0, v_1, v_2, \dots)[q]$ holds. But by the traditional definition this holds iff $\langle q_0, \dots, q_{n-1}\rangle \in R^{\mathfrak{M}}$. Which agrees with our definition. The rest of the definition of $\mathcal{L}_{\mathrm{FOL}}$ coincides with the definition of the most traditional version of first-order logic.

Remark 3.2.24. (Connections between $\mathcal{L}_{\mathrm{FOL}}$ and the more traditional form of first-order logic)

Szerintem kellene valahol altalanosan foglalkozni azzal, hogy ket logika hanyfele es milyen ertelemben lehet ekvivalens es akkor itt lehetne majd precizen beszelni. Akkor nem kene ez az egesz ceco, itt lehetne fol-t tradicionalisan is defni [egyebkent is, hiszen az is logic Def. 3.1.3 ertelmeben, csak nem nice], es pontosan megadni, hogy a ket logika milyen ertelemben ekvivalens. Egy altalanos ekvivalencia-definicio(k) meg sok egyeb helyen is jol jonne az anyagban. Agi

(1) The logic $\mathcal{L}_{\mathrm{FOL}}$ is slightly more general than the more traditional forms of first-order logic in that here the *logic* does not tell us in advance which relation symbol has what rank (that is why it is called *rank-free*). This information is postponed slightly, because it is not considered to be purely logical. The information about the ranks of the relation symbols will be provided by the models, or equivalently, by the non-logical axioms of some theory. However, we can simulate the more traditional form of first-order logic in $\mathcal{L}_{\mathrm{FOL}}$ as follows.

Any language (or similarity type) of traditional first-order logic is a *theory* of our $\mathcal{L}_{\mathrm{FOL}}$. Namely, such a language includes the rank $\varrho(R)$ of each relation symbol $R \in P$. So, a traditional language is given by a pair $\langle P, \varrho\rangle$. To such a language we associate the following theory T_ϱ (given as a set of formulas):

$$T_\varrho \stackrel{\text{def}}{=} \{\forall v_i\big((\exists v_i R) \leftrightarrow R\big) \ : \ R \in P \text{ and } i \geqslant \varrho(R)\}\,.$$

The theory T_ϱ spells out for each $R \in P$ that the rank of R is $\varrho(R)$. After T_ϱ has been postulated, whenever one sees R as a formula, one can read it as an abbreviation of $R(v_0 \dots v_{\varrho(R)-1})$. To *any* theory T it is usual to associate a "sublogic" of $\mathcal{L}_{\mathrm{FOL}}$ as follows:

$$\mathcal{L}_T \stackrel{\mathrm{def}}{=} \langle F_{\mathrm{FOL}}, Mod(T), mng_{\mathrm{FOL}}, \models_{\mathrm{FOL}}\rangle.$$

For our T_ϱ, the sublogic $\mathcal{L}_{T_\varrho}$ is strongly equivalent with the most traditional first-order logic of language $\langle P, \varrho\rangle$.[6]

(2) The other feature of traditional first-order logic which might seem to be missing from $\mathcal{L}_{\mathrm{FOL}}$ is *substitution* of individual variables, that is, $\mathcal{L}_{\mathrm{FOL}}$ includes atomic formulas with a fixed order of variables only. The reason for this is that Tarski discovered in the 40's that substitution can be expressed with quantification and equality. Namely, if we want to substitute v_1 for v_0 in formula φ then the resulting formula is equivalent to $\exists v_0(v_0 = v_1 \wedge \varphi)$. E.g. $R(v_1, v_1, v_2)$ is equivalent to

$$\exists v_0\big(v_0 = v_1 \wedge R(v_0, v_1, v_2)\big).$$

What happens if we want to *interchange* v_0 and v_1, i.e. we want to express $R(v_1, v_0, v_2)$. Then write

$$\exists v_3 \exists v_4\big[v_3 = v_0 \wedge v_4 = v_1 \wedge \exists v_0 \exists v_1\big(v_0 = v_4 \wedge v_1 = v_3 \wedge R(v_0, v_1, v_2)\big)\big].$$

Someone might object that *before* writing up the theory T_ϱ (cf. item (1) above) one cannot interchange variables. There are two answers: *(i)* This does not really matter if we want to simulate traditional first-order logic. *(ii)* This can be easily done by adding extra unary connectives p_{ij} $(i, j \in \omega)$ to those of $\mathcal{L}_{\mathrm{FOL}}$. The semantics of p_{ij} is given by

$$\mathfrak{M} \models p_{ij}\varphi[q] \quad \stackrel{\mathrm{def}}{\Longleftrightarrow} \quad \mathfrak{M} \models \varphi[\langle q_0, \dots, q_{i-1}, q_j, q_{i+1}, \dots, q_{j-1}, q_i, q_{j+1}, \dots\rangle],$$

if $i \leqslant j$, and similarly otherwise. Adding such connectives does not change the basic properties of the logic. For more on the properties of $\mathcal{L}_{\mathrm{FOL}}$ see e.g. the Appendix of Blok–Pigozzi [22], Andréka–Gergely–Németi [4] and reference Henkin–Tarski [39] of [37, Part I].

Exercises 3.2.25.

(1) Write up a detailed definition of $\mathcal{L}_{\mathrm{FOL}}$ as a multi-modal logic.

Hint: Define the modal models as

$$M_m \stackrel{\mathrm{def}}{=} \{\langle W, v\rangle \in M_S : W \subseteq {}^\omega U \text{ for some set } U, \text{ and for each } R \in P, \\ (\exists n \in \omega)(\exists R_1 \subseteq {}^n M) v(R) = \{s \in {}^\omega U : \langle s_0, \dots, s_n\rangle \in R_1\}\}\,.$$

The rest of the hint is in Exercise 3.2.22 (1).

[6] This equivalence is the strongest possible one. The models are practically the same, and the formulas are alphabetical variants of each other in the following sense. To each "traditional" formula ψ of $\langle P, \varrho\rangle$ there is $\varphi \in F_{\mathrm{FOL}}$ such that their meanings coincide in every model. (Same holds in the other direction: for every $\varphi \in F_{\mathrm{FOL}}$ there is a "traditional" ψ, etc.)

(2) Take the multi-modal form of $\mathcal{L}_{\mathrm{FOL}}$ obtained in (1) above. Consider the "modality" $(\exists v_i)$. Can you write down its meaning definition in the $\Vdash$-style of modal logics, that is, the logics studied before $\mathcal{L}_n$?

Hint: Let $s \in W$. (Recall that $W = {}^{\omega}U$.) Then

$$s \Vdash \exists v_i \varphi \quad \text{iff} \quad (\exists q \in W)\forall j(j \neq i \Rightarrow s_i = q_i \text{ and } q \Vdash \varphi)\,.$$

What is the $\Vdash$-style definition of the zero-ary modality $(v_i = v_j)$?

(3) Consider the modal forms of $\mathcal{L}_n$ and $\mathcal{L}_{\mathrm{FOL}}$. Prove that D is expressible in $\mathcal{L}_n$. Prove that $\Diamond_2$ is expressible in $\mathcal{L}_n$ if $n > 3$. Is D expressible in $\mathcal{L}_{\mathrm{FOL}}$? Is $\Diamond_2$ expressible in $\mathcal{L}_{\mathrm{FOL}}$?

(4) Prove that the following are expressible in $\mathcal{L}_{\mathrm{FOL}}$ about its models

$\mathfrak{M} = \langle M, R^{\mathfrak{M}}\rangle_{R\in P} \in M_{\mathrm{FOL}}$.

(4.1) $|M| > 1$.

(4.2) $|M| = 2$.

(4.3) $|M| > n$ for any fixed number n.

(4.4) $|M| < n$ for any fixed number n.

(5) What part of (4) above carries over to $\mathcal{L}_n$?

(6) Prove that $\mathcal{L}_1$ is decidable. Do you think that $\mathcal{L}_2$ is decidable? Do you think that $\mathcal{L}_3$ is decidable? Do you think that $\mathcal{L}_{\mathrm{FOL}}$ is decidable??

(7) Do you think that the valid formulas of $\mathcal{L}_{\mathrm{FOL}}$ are recursively enumerable?

Exercises 3.2.26. (1) Write up a detailed definition of $\mathcal{L}_{\mathrm{FOL}}$ as a modal logic. (Hint: See Exercise 3.2.25 (1) above.)

(2) Prove that $\mathcal{L}_{\mathrm{FOL}}$ is as expressive as the traditional form of first-order logic. Prove that traditional first-order logic with a language $\langle P, \rho\rangle$ is strongly equivalent with the sublogic $\mathcal{L}_{T_\rho}$ as described in Remark 3.2.24.

(3) Assume $\mathfrak{M} = \langle M, R^{\mathfrak{M}}\rangle \in M_{\mathrm{FOL}}$ with $R^{\mathfrak{M}} \subseteq M \times M$.

Express that R is a transitive relation. (This means that you are asked to write up a formula $\varphi \in F_{\mathrm{FOL}}$ such that for every $\langle M, R\rangle$ with $R \subseteq M \times M$, if $\langle M, R\rangle \models \varphi$ then R is transitive.)

Express that R is a partial ordering (transitive, reflexive and antisymmetric).

Express that R is a dense ordering (density is the property $\forall x, y(xRy \Rightarrow \exists z(xRz \text{ and } zRy))$.)

Express that R is an equivalence relation.

(4) Think of $\mathcal{L}_{\mathrm{FOL}}$ again as a multi-modal logic as in the previous list of exercises. Are there two models $\mathfrak{M}, \mathfrak{N}$ such that they are not distinguishable in $\mathcal{L}_{\mathrm{FOL}}$ but they are distinguishable in any of $\mathcal{L}_D$, $\mathcal{L}_{n\text{-times}}$ for $n \in \omega$? (Hint: no.) What is the answer for $\mathcal{L}_{\kappa\text{-times}}$ with some infinite κ (say $\kappa > 2^\omega$)?

Exercises 3.2.27.

(1) Let $\mathcal{L}_i = \langle F_i, M_i, mng_i, \models_i \rangle$ with $i \leqslant 2$ be two logics. Call $\mathcal{L}_0$ and $\mathcal{L}_1$ *weakly equivalent* iff

$$F_0 = F_1 \text{ and } (\forall \Gamma \subseteq F_0)(\forall \varphi \in F_0)(\Gamma \models_0 \varphi \Leftrightarrow \Gamma \models_1 \varphi)\,.$$

Prove that the following logics are weakly equivalent: $\mathcal{L}_S$ and $\mathcal{L}_S^0$ from Exercise 3.1.1.

(2) Let $\mathcal{L}_i$, $i \leqslant 2$ be as above. Assume that $F_i \subseteq Z_i$ for some set of "symbols" Z_i. That is, we are assuming that the formulas are finite sets of symbols. For a function $f : Z_0 \longrightarrow Z_1$ define its natural extension $\tilde{f} : Z_0^* \longrightarrow Z_1^*$ the usual way: $f\langle a+1, \ldots, a_n \rangle = \langle f(a_1), \ldots, f(a_n) \rangle$. We call $\mathcal{L}_0$ and $\mathcal{L}_1$ *reasonably equivalent* iff there is a function $f : Z_0 \longrightarrow Z_1$ such that $\tilde{f}(F_0) = F_1$ and

(i) $(\forall \Gamma \cup \{\varphi\} \subseteq F_0)\big(\Gamma \models_0 \varphi \text{ iff } \tilde{f}(\Gamma) \models_1 \tilde{f}(\varphi)\big)$,

(ii) $(\forall \Gamma \cup \{\varphi\} \subseteq F_1)\big(\Gamma \models_1 \varphi \text{ iff } \tilde{f}^{-1}(\Gamma) \models_0 \tilde{f}^{-1}(\varphi)\big)$, and

(iii) $(\forall \varphi, \psi \in F_0)\big(\tilde{f}(\varphi) = \tilde{f}(\psi) \Rightarrow (\varphi \models_0 \psi \text{ and } \psi \models_0 \varphi)\big)$.

Prove that any two weakly equivalent logics are reasonably equivalent.

(3) Consider propositional logic with logical connectives $\{\wedge, \vee, \neg\}$ and another version of the same logic with $\{\wedge, \rightarrow, \mathrm{false}\}$. Clearly these two versions of propositional logic are equivalent in some natural sense. Prove that they are not equivalent in the sense of (1), (2) above. Try to broaden the scope of equivalence such that these two versions of $\mathcal{L}_S$ become equivalent.

(4) Let $\mathcal{L}_i$ be as in (1). Consider the existence of two "semantical" functions

$$m_{01} : M_0 \longrightarrow (\text{Subsets of } M_1) \quad \text{and} \quad m_{10} : M_1 \longrightarrow (\text{Subsets of } M_0)\,.$$

We call $\mathcal{L}_0$ and $\mathcal{L}_1$ *semantically equivalent* iff $F_1 = F_2$ and there are m_{01}, m_{10} as above such that for every $\varphi \in F_0$, $\mathfrak{M} \in M_0$ and $\mathfrak{N} \in M_1$,

$$(\mathfrak{M} \models \varphi \Leftrightarrow m_{01}(\mathfrak{M}) \models \varphi) \quad \text{and} \quad (\mathfrak{N} \models \varphi \Leftrightarrow m_{10}(\mathfrak{N}) \models \varphi)\,.$$

Prove that $\mathcal{L}_S$ and $\mathcal{L}_S^0$ (in Exercise 3.1.1) are strongly semantically equivalent.

(5) Combine the equivalences defined in (2) and (4) above. Call this combined concept semantical equivalence. Find logics which are semantically equivalent.

(6) Try to combine (5) and (3) above!

3.2.2 Summary

DISTINGUISHED LOGICS

$\mathcal{L}_S$	propositional logic
$S5$	modal logic, where the accessibility relation is $W \times W$ for some set W
$\mathcal{L}_D$	difference logic (or "some-other-time" logic)
Tw	twice logic
$\mathcal{L}_{\kappa\text{-times}}$	κ-times logic (κ is any cardinal)
$\mathcal{L}_{\text{bin}}$	
$\mathcal{L}_{\text{more}}$	
$\mathcal{L}_{\text{PAIR}}$	set of worlds is arbitrary $W \subseteq U \times U$ for some U, extra-Boolean is $\circ$
$\mathcal{L}_{\text{REL}}$	set of worlds is $U \times U$ for some U, extra-Boolean is $\circ$
$\mathcal{L}_{\text{ARROW}}$	set of worlds is arbitrary $W \subseteq U \times U$ for some U, extra-Booleans are $\circ$, $\smile$, *Id*
$\mathcal{L}_{\text{RA}}$	(logic of relation algebras) set of worlds is $U \times U$, extra-Booleans are $\circ$, $\smile$, *Id*
$\mathcal{L}_n$	first-order logic restricted to the first n variables ($n \in \omega$
$\mathcal{L}_{\text{FOL}}$	(rank-free) first-order logic

DISTINGUISHED PROPERTIES to be checked for every logic $\mathcal{L}$:

The reason for looking at these properties is that they distinguish first-order like logics from propositional-like logics.

dec The set of all valid formulas of $\mathcal{L}$ is decidable.
(Briefly: $\mathcal{L}$ is decidable.)

r.e. The set of all valid formulas of $\mathcal{L}$ is recursively enumerable.
(Briefly: $\mathcal{L}$ is r.e. .)

fmp $\mathcal{L}$ has the *finite model property* that is,

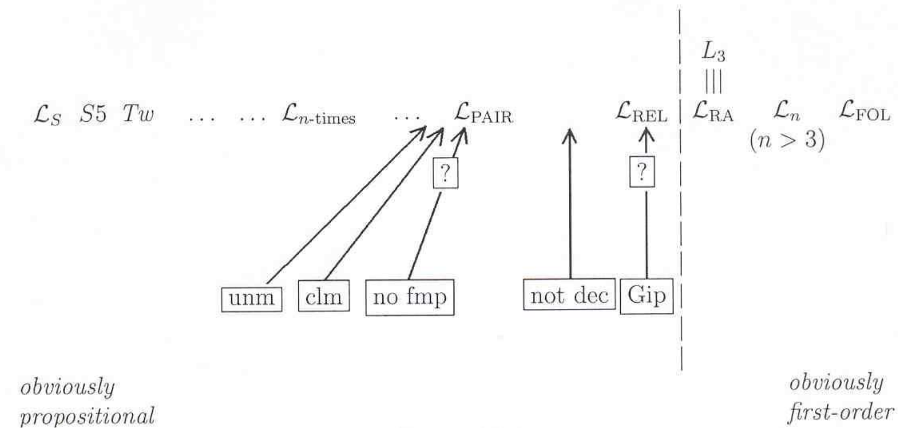


Figure 3.2: Comparison

	$(\forall\varphi \in F_{\mathcal{L}})\big(\models_{\mathcal{L}} \varphi \iff (\forall\mathfrak{M} \in M_{\mathcal{L}})(\mathfrak{M}$ is finite[7] $\Rightarrow \mathfrak{M} \models_{\mathcal{L}} \varphi)\big)$.
Gip	$\mathcal{L}$ has *Gödel's incompleteness property*, that is, $(\exists\varphi \in F_{\mathcal{L}})(\forall T \subseteq F_{\mathcal{L}})\big((\varphi \in T$ and T is consistent$) \implies$ $\implies Csq_{\mathcal{L}}(T)$ is undecidable$)\big)$.
clm	We say that the *distinction between set-models and class-models counts* in $\mathcal{L}$ ($\mathcal{L}$ has clm for short) iff (roughly speaking)[8] even in the case when the set P of atomic formulas of $\mathcal{L}$ is finite, we have ($\exists$ class-model $\mathfrak{M}$) $\big(Th_{\mathcal{L}}(\mathfrak{M})$ is not definable without parameters in our Set Theory$\big)$.
unm	Assuming again that the set P of atomic formulas of $\mathcal{L}$ is finite, there is some $\mathfrak{M} \in M_{\mathcal{L}}$ such that $Th_{\mathcal{L}}(\mathfrak{M})$ is undecidable (unm abbreviates existence of <u>un</u>decidable <u>m</u>odel).

Exercise 3.2.28. Prove that if $\mathcal{L}$ is r.e. and $\mathcal{L}$ has the fmp the $\mathcal{L}$ is decidable.

COMPARISON OF LOGICS w.r.t. the properties above:

(An arrow points to the place where the property in question becomes true "moving from left to right". Hence in principle it should always point to a gap between two logics.)

[7] $\mathfrak{M} = \langle W, v\rangle$ is called *finite* iff W is a finite set.

[8] Recall that for fixed $\mathfrak{M}$, $mng_{\mathfrak{M}}(\varphi)$ was defined by recursion on the complexity of φ in case of each of our distinguished logics discussed so far. (This was so in $\mathcal{L}_S, \ldots,$ in $\mathcal{L}_n$, and also in $\mathcal{L}_{FOL}$ to mention only a few.) Saying that $Th(\mathfrak{M})$ is undefinable implies that our recursive definition of $mng_{\mathfrak{M}}$ *becomes incorrect as a definition* if we permit $\mathfrak{M}$ to be a class model. Roughly, $\mathcal{L}$ has *clm* of Tarski's Undefinability of Truth Theorem is applicable to $\mathcal{L}$. For more on this property of logics see [17, Appendix B].

Exercise 3.2.29. Check which claims represented on Figure 3.2 were asked as an exercise in the text. Try to prove (and claim, if necessary) the missing ones too.

* * *

The following logics are of a different "flavor" than the ones seen so far. They include Lambek Calculus, some fragments of Linear Logic, Pratt's Action Logic, Dynamic Logic, different kinds of semantics than seen so far. The main purpose of giving them is to indicate that the methods of algebraic logic are applicable almost to any unusual logic coming from completely different paradigms of logical or linguistic or computer science research areas, and are not restricted to the kinds of logics discussed so far. If the reader is already convinced, then he may safely skip Definitions 3.2.30–3.2.33.

Some *further logics*, which are even *less similar* to the ones discussed so far, are collected in chapter 7. It is advisable to look into the chapter 7 because our theorems apply to all the logics discussed there. The only reason why those logics are postponed to chapter 7 is that we did not want to postpone the main theorems too much. For example, infinite valued logics, relevant logics and partial logics are in chapter 7.

Definition 3.2.30. (Lambek Calculus [slightly extended]) Recall the logic $\mathcal{L}_{\mathrm{RA}}$ from Def. 3.2.19. The connectives of Lambek calculus $\mathcal{L}_{\mathrm{LC}}$ are $\{\wedge, \circ, \backslash, /, \rightarrow\}$. This defines the formulas F_{LC} of Lambek Calculus. Now,

$$\mathcal{L}_{\mathrm{LC}} \stackrel{\mathrm{def}}{=} \langle F_{\mathrm{LC}}, M_{\mathrm{RA}}, mng_{\mathrm{LC}}, \models_{\mathrm{LC}} \rangle\,,$$

where for all $\varphi, \psi \in F_{\mathrm{LC}}$ and all $\mathfrak{M} \in M_{\mathrm{RA}}$

$$\begin{aligned} mng_{\mathrm{LC}}(\varphi\backslash\psi, \mathfrak{M}) &\stackrel{\mathrm{def}}{=} mng_{\mathrm{RA}}\big(\neg(\varphi^{\smile} \circ \neg\psi), \mathfrak{M}\big), \\ mng_{\mathrm{LC}}(\varphi/\psi, \mathfrak{M}) &\stackrel{\mathrm{def}}{=} mng_{\mathrm{RA}}\big(\neg(\neg\varphi \circ \psi^{\smile}), \mathfrak{M}\big), \\ mng_{\mathrm{LC}}(\varphi \rightarrow \psi, \mathfrak{M}) &\stackrel{\mathrm{def}}{=} mng_{\mathrm{RA}}(\neg\varphi \vee \psi, \mathfrak{M}), \end{aligned}$$

and $\models_{\mathrm{LC}}$ is defined analogously to $\models_{\mathrm{RA}}$.

Remark 3.2.31. Original Lambek Calculus is only a fragment of $\mathcal{L}_{\mathrm{LC}}$ because in the original case the use of "$\rightarrow$" is restricted. (In any formula, "$\rightarrow$" can be used only once, and it is the outer most connective.) The methods of the present work yielded quite a few results for Lambek Calculus and for some further fragments of Linear Logic, cf. Andréka–Mikulás [10].

Definition 3.2.32. (Language model for Lambek Calculus and other logics [e.g. arrow logic])

(1) *Notation:* Recall that U^* denotes the set of all finite sequences over the set U. A set $X \subseteq U^*$ is called a language (in the syntactic sense). Let $X, Y \subseteq U^*$.

Then $X * Y = \{s^\frown q : s \in X \text{ and } q \in Y\}$, where $s^\frown q$ is the concatenation of s and q.

$$M_L \stackrel{\text{def}}{=} \{\langle U, f\rangle \ : \ U \text{ is a set and } f : P \longrightarrow \mathcal{P}(U^*)\}\,.$$

We write $mng(\varphi)$ instead of $mng_L(\varphi, \langle U, f\rangle)$.

$$\begin{aligned}
mng(p_i) &\stackrel{\text{def}}{=} f(p_i) \ \text{ for } p_i \in P,\\
mng(\varphi \wedge \psi) &\stackrel{\text{def}}{=} mng(\varphi) \cap mng(\psi),\\
mng(\varphi \circ \psi) &\stackrel{\text{def}}{=} mng(\varphi) * mng(\psi),\\
mng(\varphi \to \psi) &\stackrel{\text{def}}{=} [U^* \smallsetminus mng(\varphi)] \cup mng(\psi),\\
mng(\varphi \backslash \psi) &\stackrel{\text{def}}{=} \{q : \big(\forall s \in mng(\varphi)\big)\ s^\frown q \in mng(\psi)\},\\
mng(\varphi / \psi) &\stackrel{\text{def}}{=} \{s : \big(\forall q \in mng(\psi)\big)\ s^\frown q \in mng(\varphi)\}\,.
\end{aligned}$$

Now, $\models_L$ is defined as before.

(2) *Lambek calculus with language models* is

$$\mathcal{L}_{\text{LCL}} \stackrel{\text{def}}{=} \langle F_{\text{LC}}, M_L, mng_L, \models_L\rangle\,.$$

This is quite a well investigated logic, and in some respects behaves slightly differently from $\mathcal{L}_{\text{LC}}$.

Now we can extend the definition of mng_L to the connectives $\neg$, $\smile$ and Id as follows:

$$\begin{aligned}
mng(\neg\varphi) &\stackrel{\text{def}}{=} U^* \smallsetminus mng(\varphi),\\
mng(\varphi^\smile) &\stackrel{\text{def}}{=} \{\langle s_n, \ldots, s_1\rangle \ : \ \langle s_1, \ldots, s_n\rangle \in mng(\varphi)\},\\
mng(Id) &\stackrel{\text{def}}{=} \{\langle\rangle\},
\end{aligned}$$

where $\langle\rangle$ denotes the sequence of length 0.

(3) *Extended Lambek calculus with language models*: F^+_{LC} has all the Booleans as connectives in addition to F_{LC}, and the semantics described in (1) above.

$$\mathcal{L}^+_{\text{LCL}} = \langle F^+_{\text{LC}}, M_L, mng_L, \models_L\rangle\,.$$

(4) *Arrow Logic with language models* is

$$\mathcal{L}_{\text{ARROWL}} = \langle F^+_{\text{ARROW}}, M_L, mng_L, \models_L\rangle\,.$$

Definition 3.2.33. (Dynamic Arrow Logic) Recall the definition of $\mathcal{L}_{\text{RA}}$. Add the unary connective * sending φ to φ^*. The set of formulas (denoted as F_{DL}) of Dynamic Arrow Logic is defined as that of $\mathcal{L}_{\text{RA}}$ together with the following clause:

$$\varphi \in F_{\text{DL}} \implies \varphi^* \in F_{\text{DL}}\,.$$

The semantics of this connective is defined by

$$mng_{\mathrm{DL}}(\varphi^*, \mathfrak{M}) \stackrel{\text{def}}{=} \text{"reflexive and transitive closure of the relation } mng_{\mathrm{DL}}(\varphi, \mathfrak{M})\text{"}.$$

This defines $\models^*$ from $\models_{\mathrm{RA}}$. Now, Dynamic Arrow Logic is

$$\mathcal{L}_{\mathrm{DL}} = \langle F_{\mathrm{DL}}, M_{\mathrm{RA}}, mng_{\mathrm{DL}}, \models^* \rangle .$$

Pratt's original dynamic logic can easily and naturally be interpreted into $\mathcal{L}_{\mathrm{DL}}$. For more on Dynamic Arrow Logic cf. e.g. van Benthem [79], Marx [53].

3.2.3 Solutions for some exercises of subsections 3.2.1 and 3.2.2

Exercises 3.2.8

(2) $\mathcal{L}_{\mathrm{PAIR}}$ is decidable.

There is a model $\mathfrak{M} \in M_{\mathrm{PAIR}}$ such that $Th_{\mathcal{L}_{\mathrm{PAIR}}}(\mathfrak{M})$ is not even recursively enumerable. See the hint for Exercises 3.2.16 (3).

(5) A. Simon proved that for finite Ax, $Csq_{\mathcal{L}_{\mathrm{PAIR}}}(Ax)$ is decidable. He proved that the logic $\mathcal{L}_{\mathrm{PAIR}}$+"$\Diamond$ of $S5$" is still decidable; then, using $\Diamond$, $Ax \models \varphi$ is equivalent to validity of a single formula (see Simon [76]).

Exercises 3.2.18

(3) $\mathcal{L}_{\mathrm{REL}}$ is undecidable. This hint is for the case you know that the word problem of semigroups (or equivalently, the quasi-equational theory of semigroups) is undecidable. Define a computable function f which to every quasi-equation q in the language of semigroups associates $f(q) \in F_{\mathrm{REL}}$ such that

$$\models_{\mathrm{REL}} f(q) \quad \Longleftrightarrow \quad \text{Semigroups} \models q.$$

Conclude that $\mathcal{L}_{\mathrm{REL}}$ cannot be decidable because that would provide a decision algorithm for the quasi-equations of semigroups. There are other ways of handling this problem besides the "semigroup" one, cf. e.g. the important book Tarski–Givant [77].

There is a formula $\varphi \in F_{\mathrm{REL}}$ such that $Csq_{\mathcal{L}_{\mathrm{REL}}}(\{\varphi\})$ is undecidable. Moreover, $\mathcal{L}_{\mathrm{REL}}$ has the *Gödel's incompleteness property* that is,

$$(\exists\varphi \in F_{\mathrm{REL}})(\forall T \subseteq F_{\mathrm{REL}})\big((\varphi \in T \text{ and } T \text{ is consistent}) \implies \\ \implies Csq_{\mathcal{L}_{\mathrm{REL}}}(T) \text{ is undecidable}\big).$$

Observe the contrast between $\mathcal{L}_{\mathrm{PAIR}}$ and $\mathcal{L}_{\mathrm{REL}}$!

(4) The others (Exercises 3.2.3 (3)–(5) for $\mathcal{L}_{\mathrm{REL}}$) follow from the corresponding answers for Exercises 3.2.16 above.

Exercises 3.2.14(4) Here we give a very detailed hint for solving this exercise, i.e. for proving that $\mathcal{L}_{\mathrm{more}}$ is decidable.

Let $\mathfrak{A} = \langle A, \leqslant, +, O, I\rangle$ be a structure where $\leqslant$ is a binary relation on A, $+$ is a partial binary operation on A (i.e. $Dom(+) \subseteq A \times A$), $I \subseteq A$ and $O \in I$. The *diagram of* $\mathfrak{A}$, in symbols $\Delta(\mathfrak{A})$, is defined as follows. Let $a_0, \dots, a_n$ be a repetition-free enumeration of $A \smallsetminus I$. Let $x_0, \dots, x_n$ be variables. For any $i, j \leqslant n$ let

$$\pi(x_i, x_j) \stackrel{\text{def}}{=} \begin{cases} x_i + x_j = x_k & \text{if } \ a_i + a_j = a_k \text{ in } \mathfrak{A} \\ x_i = x_j & \text{if } \ a_i + a_j \text{ is not defined in } \mathfrak{A}, \end{cases}$$

$$\varrho(x_i, x_j) \stackrel{\text{def}}{=} \begin{cases} x_i \leqslant x_j & \text{if } a_i \leqslant a_j \text{ in } \mathfrak{A} \\ x_i \not\leqslant x_j & \text{if } a_i \not\leqslant a_j \text{ in } \mathfrak{A}, \end{cases}$$

$$\delta(x_i, x_j) \stackrel{\text{def}}{=} \pi(x_i, x_j) \wedge \varrho(x_i, x_j).$$

$$\Delta(\mathfrak{A}) \stackrel{\text{def}}{=} \exists x_0 \dots x_n \big(\bigwedge \{\delta(x_i, x_j) \ : \ i, j \leqslant n\}\big).$$

We note that $\Delta(\mathfrak{A})$ is a (first-order) formula containing only $+$ and $\leqslant$, therefore it is decidable whether this formula is valid in standard arithmetic or not.

We say that $\mathfrak{A}$ is a *cardinality structure* iff the following hold for all $a, b \in A$:

$\leqslant$ is a linear ordering on A;

O is the smallest element, i.e. $O \leqslant a$ for every $a \in A$;

I is an end segment, i.e. $a \in I$ and $a \leqslant b$ imply $b \in I$;

$O + a = a + O = a, \quad a + b = b$ if $a \leqslant b$ and $b \in I$;

$a + b \in I$ implies ($a \in I$ or $b \in I$);

$+$ is commutative and associative in the sense that

if $a + b$ exists then $b + a$ exists and $a + b = b + a$;

$a+b,\ (a+b)+c$ exist iff $b+c,\ a+(b+c)$ exist and $(a+b)+c = a+(b+c)$;

$\langle \mathbf{N}, \leqslant, +\rangle \models \Delta(\mathfrak{A})$.

We say that $(\mathfrak{A}, \kappa)$ is an *abstract cardinality model*, in symbols $(\mathfrak{A}, \kappa) \in ACMod$, iff

$\mathfrak{A}$ is a cardinality structure;

$\kappa : \mathcal{P}(P) \to A$ (where P is the set of atomic formulas of $\mathcal{L}_{\mathrm{more}}$);

$\sum\langle \kappa(H) : H \in \mathcal{H}\rangle$ exists for all $\mathcal{H} \subseteq \mathcal{P}(P)$, where $\sum$ refers to addition in $\mathfrak{A}$.

Now let $(\mathfrak{A}, \kappa) \in ACMod$ and $\chi \in F_{\text{more}}$. We define $\sigma(\chi) \subseteq \mathcal{P}(P)$ by induction on the complexity of the formula χ as follows.

$\sigma(p) \stackrel{\text{def}}{=} \{H \in \mathcal{P}(P) : p \in H\}$ if $p \in P$;

$\sigma(\varphi \wedge \psi) \stackrel{\text{def}}{=} \sigma(\varphi) \cap \sigma(\psi)$;

$\sigma(\neg\varphi) \stackrel{\text{def}}{=} \mathcal{P}(P) \smallsetminus \sigma(\varphi)$;

$$\sigma\big(\blacklozenge(\varphi, \psi)\big) \stackrel{\text{def}}{=} \begin{cases} \mathcal{P}(P) & \text{if } \sum\langle \kappa(H) : H \in \sigma(\varphi)\rangle \geq \sum\langle \kappa(H) : H \in \sigma(\psi)\rangle \\ \emptyset & \text{otherwise.} \end{cases}$$

$$(\mathfrak{A}, \kappa) \models \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \sigma(\varphi) = \mathcal{P}(P).$$

Show that the following gives an algorithm for deciding validity of φ:

$$\begin{gathered} \varphi \text{ is valid in } \mathcal{L}_{\text{more}} \\ \text{iff} \\ (\mathfrak{A}, \kappa) \models \varphi \text{ for all } (\mathfrak{A}, \kappa) \in ACMod \text{ such that } |A| \leqslant 2^{2^{|P|}}. \end{gathered}$$

Chapter 4

Bridge between the world of logics and the world of algebras

Many of our readers might enjoy looking up section 5 of [62] where we discuss the question of what logic is.

The algebraic counterpart of classical sentential logic $\mathcal{L}_S$ is the variety BA of Boolean algebras. Why is this so important? The answer lies in the general experience that it is usually much easier to solve a problem concerning $\mathcal{L}_S$ by translating it to BA, solving the algebraic problem, and then translating the result back to $\mathcal{L}_S$ (than solving it directly in $\mathcal{L}_S$).

In this section we extend applicability of BA to $\mathcal{L}_S$ to applicability of algebra in general to logics in general. We will introduce a standard translation method from logic to algebra, which to each logic $\mathcal{L}$ associates a class of algebras $\mathsf{Alg}_{\models}(\mathcal{L})$. (Of course, $\mathsf{Alg}_{\models}(\mathcal{L}_S)$ will be BA.) Further, this translation method will tell us how to find the algebraic question corresponding to a logical question. If the logical question is about $\mathcal{L}$ then its algebraic equivalent will be about $\mathsf{Alg}_{\models}(\mathcal{L})$. For example, if we want to decide whether $\mathcal{L}$ has the proof theoretic property called Craig's interpolation property, then it is sufficient to decide whether $\mathsf{Alg}_{\models}(\mathcal{L})$ has the so called amalgamation property (for which there are powerful methods in the literature of algebra). If the logical question concerns connections between several logics, say between $\mathcal{L}_1$ and $\mathcal{L}_2$, then the algebraic question will be about connections between $\mathsf{Alg}_{\models}(\mathcal{L}_1)$ and $\mathsf{Alg}_{\models}(\mathcal{L}_2)$. (The latter are quite often simpler, hence easier to investigate.)

A feature of the present approach seems to be striving for flexibility and building bridges between appearantly distant areas. This gives us hope for bringing Tarski's main research directions together explicitly into a single coherent theory. It seems no coincidence that Tarski's main research fields included (i) algebraic logic, (ii) definability theory, (iii) logical foundation of geometry. Today there seems to be a convergence between areas (i)-(iii) together with the theory of spacetime, hence with logical foundation of relativity theory. This is not too surprising if we recall that the first originator of definability was Einstein's friend Hans Reichenbach who, around 1924, argued for the importance of creating a logical theory of definability and the motivation he gave for this was that relativity theory needed such a tool. Further, area (iii) of Tarski naturally generalizes to logical foundation of spacetime, since relativity theory is often identified with a geometrization of certain parts of physics. Recent works on the combination of (i)-(iii) with logic of spacetime and logical analysis of relativity are [9], [50], [7], [8]. [63] is a point where relativity gives a potential feedback to logic (or to the foundation of mathematics).

4.1 Fine-tuning the framework

The definition of a logic in section 3.1 is very wide. Actually, it is too wide for proving interesting theorems about logics. Now we will define a subclass of logics which we will call *nice logics*. Our notion of a nice logic is wide enough to cover the logics mentioned in the previous section, moreover, it is broad enough to cover almost all logics investigated in the literature. (Certain quantifier logics might need a little reformulation for this, but that reformulation does not effect the essential aspects of the logic in question as we will see.) On the other hand, the class of nice logics is narrow enough for proving interesting theorems about them, that is, we will be able to establish typical logical facts that hold for most logics studied in the literature. For more on this "bridge" and its generalizations, recent applications we refer to [49] under the keyword "duality theories", in particular in Appendix A and pp.280-282, 293-296, 325, A1-A18 therein.

4.1.1 Nice and strongly nice logics

In this subsection, $\mathcal{L} = \langle F, M, mng, \models \rangle$ denotes an arbitrary but fixed logic in the sense of Definition 3.1.3 (i.e. F is a set, M is a class, mng is a function with domain $F \times M$, and $\models \subseteq M \times F$).

We will define the concept of a *nice logic* (and its variants: *strongly nice logic*, *structural logic*) via conditions each of which is interesting on its own right. When reading these conditions, it might be useful to contemplate the common features of the logics studied so far, e.g. $\mathcal{L}_S$, $S5$, $\mathcal{L}_{\mathrm{ARW0}}$, $\mathcal{L}_n$ (cf. section 3.2).

The formulas of each of these logics were built up by means of some *logical connectives*. This property is phrased, in general, as follows.

Definition 4.1.1. ($\mathcal{L}$ has logical connectives) We say that $\mathcal{L}$ *has logical connectives* iff (i) and (ii) below hold.

(i) A set $Cn(\mathcal{L})$, called the set of *logical connectives* of $\mathcal{L}$, is fixed. Every $c \in Cn(\mathcal{L})$ has some rank $rank(c) \in \omega$. The set of all logical connectives of rank k is denoted by $Cn_k(\mathcal{L})$.

(ii) There is a set P, called the set of *atomic formulas* (or *proposition letters* or *parameters* or *propositional variables*), such that F is the smallest set satisfying conditions (a–b) below.

(a) $P \subseteq F$,

(b) if $c \in Cn_k(\mathcal{L})$ and $\varphi_1, \ldots, \varphi_k \in F$ then $c(\varphi_1, \ldots, \varphi_k) \in F$.

The word-algebra generated by P using the logical connectives from $Cn(\mathcal{L})$ as algebraic operations is denoted by $\mathfrak{F}$, that is, $\mathfrak{F} = \langle F, c \rangle_{c \in Cn(\mathcal{L})}$. $\mathfrak{F}$ is called the *formula algebra* of $\mathcal{L}$.

Definition 4.1.2. (compositionality) We say that $\mathcal{L}$ is *compositional* iff it has logical connectives and the function

$$mng_{\mathfrak{M}} \stackrel{\text{def}}{=} \langle mng(\varphi, \mathfrak{M}) \ : \ \varphi \in F \rangle$$

is a homomorphism from $\mathfrak{F}$, for every $\mathfrak{M} \in M$.

In words, compositionality means that the meanings of formulas are built up from the meanings of their subformulas in a "regular" and "uniform" way. This is just *Frége's principle of compositionality* (a well-known purely logical criterion).

An equivalent formulation of compositionality is that the kernel $ker(\varphi) = \{\langle a, b \rangle \ : \ mng_{\mathfrak{M}}(a) = mng_{\mathfrak{M}}(b)\}$ of the function $mng_{\mathfrak{M}}$ is a congruence relation of the formula lagebra $\mathfrak{F}$ (cf. subsection 2.2.4, Exercises 2.2.15). Thus $\mathcal{L}$ is compositional iff condition (4.1) below is satisfied for all k-ary connective $c \in Cn_k(\mathcal{L})$ and $\varphi_i, \psi_i \in F_{\mathcal{L}}$, $1 \leqslant i \leqslant k$:

$$\bigwedge_{i=1}^{k} mng_{\mathfrak{M}}(\varphi_i) = mng_{\mathfrak{M}}(\psi_i) \Longrightarrow$$
$$mng_{\mathfrak{M}}(c(\varphi_1, \ldots, \varphi_k)) = mng_{\mathfrak{M}}(c(\psi_1, \ldots, \psi_k)) \,. \quad (4.1)$$

By a *derived connective* we mean a term (more precisely, a scheme of terms, cf. Def.4.1.19) in the language of the word-algebra $\mathfrak{F}$. For example, in all the logics studied so far the biconditional $\leftrightarrow$ is available as a derived connective, cf. Exercises 3.2.2(2). Other examples for derived connectives are $\vee$ and *True*: $\varphi \vee \psi$ is defined as $\neg(\neg\varphi \wedge \neg\psi)$ while *True* is defined by the *scheme* $\varphi \vee \neg\varphi$. It is easy to see that in all our logics defined so far, we have

(a) $mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(\psi) \Longleftrightarrow \mathfrak{M} \models \varphi \leftrightarrow \psi$

and

(b) $\mathfrak{M} \models \varphi \Longleftrightarrow \mathfrak{M} \models \mathit{True} \leftrightarrow \varphi$

for every model $\mathfrak{M}$ and formulas φ and ψ. The following property, called the filter property, is a generalization of the above (a) and (b) (choosing $m = n = 1$, $\Delta_0 = \leftrightarrow$, $\varepsilon_0(\varphi) = \mathit{True}$, $\delta_0(\varphi) = \varphi$ in Def.4.1.3). As you will see, the filter property is a weaker condition than the above (a) and (b) together.

Definition 4.1.3. (filter property) We say that $\mathcal{L}$ has the *filter property* iff there are derived connectives $\varepsilon_0, \ldots, \varepsilon_{m-1}$ and $\delta_0, \ldots, \delta_{m-1}$ (unary) and $\Delta_0, \ldots, \Delta_{n-1}$ (binary) ($m, n \in \omega$) of $\mathcal{L}$ with the following properties:

(i) For every $\mathfrak{M} \in M$ and $\varphi, \psi \in F$,

$$mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(\psi) \Longleftrightarrow (\forall i < n)\mathfrak{M} \models \varphi \Delta_i \psi \, .$$

(ii) For every $\mathfrak{M} \in M$ and $\varphi \in F$,

$$\mathfrak{M} \models \varphi \Longleftrightarrow (\forall j < m)(\forall i < n)\mathfrak{M} \models \varepsilon_j(\varphi)\Delta_i \delta_j(\varphi) \, .$$

Exercises 4.1.4. 1. Assume that $\mathcal{L}$ is compositional and has the filter property. Prove that condition (3.1) of Def.3.1.3(v) holds for $\mathcal{L}$.

2. We have seen that conditions (a) and (b) together provide a special case of the filter property. Prove that (a) plus (b) imply the following connectio between $\models$ and *mng*.

$$(\forall \varphi, \psi \in F)\big((\models \varphi \text{ and } \models \psi) \Rightarrow (\forall \mathfrak{M} \in M) mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(\psi)\big) \, . \quad (4.2)$$

This does not follow from the filter property in general.

3. Prove that if $\mathcal{L}$ is compositional and conditions (a) and (b) concerning *True* and $\leftrightarrow$ are satisfied by $\mathcal{L}$ then condition (4.3) below holds.

$$(\forall \varphi \in F)(\forall \mathfrak{M} \in M)\mathfrak{M} \models \varphi \Longleftrightarrow mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(\mathit{True}) \, . \qquad (4.3)$$

The last two properties we list here are two different *substitution properties*.

Definition 4.1.5. ((syntactical) substitution property) We say that $\mathcal{L}$ has the *(syntactical) substitution property* iff

$$(\forall \psi, \varphi_0, \dots, \varphi_k \in F)(\forall p_0, \dots, p_k \in P)\big(\models \psi(\overline{p}) \implies \models \psi(\overline{p}/\overline{\varphi})\big),$$

where $\overline{p} = \langle p_0, \dots, p_k \rangle$, $\overline{\varphi} = \langle \varphi_0, \dots, \varphi_k \rangle$, and $\psi(\overline{p}/\overline{\varphi})$ denotes the formula that we get from ψ after simultaneously substituting φ_i for every occurrence of p_i $(i \leqslant k)$ in ψ.

Definition 4.1.6. (semantactical substitution property) We say that $\mathcal{L}$ has the *semantical substitution property* iff condition (4.4) below holds.

$$\begin{aligned}(\forall s \in {}^P F)(\forall \mathfrak{M} \in M)(\exists \mathfrak{N} \in M)(\forall \varphi(p_{i_0}, \dots, p_{i_k}) \in F)& \\ mng_{\mathfrak{N}}(\varphi) = mng_{\mathfrak{M}}\big(\varphi(p_{i_0}/s(p_{i_0}), \dots, p_{i_k}/s(p_{i_k}))\big) \, .& \quad (4.4)\end{aligned}$$

Let $\hat{s} \in {}^F F$ be the natural extension of s to $\mathfrak{F}$. Then (4.4) says that

$$mng_{\mathfrak{N}}(\varphi) = mng_{\mathfrak{M}}(\hat{s}(\varphi)).$$

The model $\mathfrak{N}$ is called the *substituted version* of $\mathfrak{M}$ along substitution s.

An equivalent form of (4.4) above is the very natural condition

$$(\forall h \in Hom(\mathfrak{F}, \mathfrak{F}))\,(\forall \mathfrak{M} \in M)(\exists \mathfrak{N} \in M)\ mng_{\mathfrak{N}} = mng_{\mathfrak{M}} \circ h\,.$$

Since h is just a substitution, this form makes it explicit that $\mathfrak{N}$ is the h-substituted version of $\mathfrak{M}$. Another equivalent form of (4.4) is the following.

$$(\forall \mathfrak{M} \in M)\big(\forall h \in Hom(\mathfrak{F}, mng_{\mathfrak{M}}(\mathfrak{F}))\big)(\exists \mathfrak{N} \in M)\ mng_{\mathfrak{N}} = h.$$

Exercise 4.1.7. Prove that if $\mathcal{L}$ is compositional, has the filter property and the semantical substitution property then $\mathcal{L}$ has the syntactical substitution property as well. Is this true in general?

Definition 4.1.8. (nice logic, strongly nice logic, structural logic) We say that $\mathcal{L}$ is a *nice logic* iff it is compositional, has the filter property and the syntactical substitution property. $\mathcal{L}$ is called a *strongly nice logic* iff it is a nice logic and it has the semantical substitution property as well. Following the terminology of Blok and Pigozzi (cf. e.g. [22]), logics that are compositional and have both substitution properties are called *structural logics*.

Remark 4.1.9. (Connections with the Blok–Pigozzi approach) Here we mention only a small part of these connections.

The $\langle F_{\mathcal{L}}, \models_{\mathcal{L}}\rangle$ part[1] of a strongly nice, consequence compact (see Def. 4.2.14 below) logic $\mathcal{L}$ is always an *algebraizable deductive system* in the sense of Blok–Pigozzi [22] (which is an algebraizable 1-deductive system in [21]). Conversely, if $\langle F, \vdash\rangle$ is an algebraizable deductive system then $\mathcal{L}_{\vdash}$, as defined in Remark 3.1.2 above, is always a strongly nice consequence compact logic in our sense. Sructural logics and the connections between the two approaches are discussed in more detail in Font–Jansana [30].

A small sample of references of the Blok–Pigozzi approach is [22], [21], [23], [64], Czelakowski [28], Font–Jansana [30].

Exercises 4.1.10.

(1) (*Important!*) Show that all the logics introduced in Defs. 3.2.1–3.2.21 above are strongly nice logics. It is especially important to do it for $\mathcal{L}_n$!

(2) Show that $\mathcal{L}_{\mathrm{FOL}}$ (cf. Def. 3.2.23) is a nice logic.

Exercises 4.1.11. Show logics where $n = 1$ but Δ_0 is not our old biconditional $\leftrightarrow$. (E.g., in $S5$ we can also take $\Box(\Phi_1 \leftrightarrow \Phi_2)$ as $\Phi_1 \Delta_0 \Phi_2$.) Show logics where $n > 1$.

4.1.2 The algebraic counterpart of $\langle F, M, mng, \models\rangle$

Recall from Def.2.2.10 that for any class K of similar algebras,

$$\mathbf{I}K \stackrel{\text{def}}{=} \{\mathfrak{M} \;:\; (\exists \mathfrak{N} \in K)\mathfrak{M} \text{ is isomorphic to } \mathfrak{N}\}\,.$$

[1] Here $\models_{\mathcal{L}}$ denotes the semantical consequence relation induced by the validity relation of $\mathcal{L}$.

Definition 4.1.12. (algebraic counterpart of a logic) Let $\mathcal{L} = \langle F, M, mng, \models\rangle$ be a compositional logic.

(i) Let $K \subseteq M$. Then for every $\varphi, \psi \in F$

$$\varphi \sim_K \psi \overset{\text{def}}{\iff} (\forall \mathfrak{M} \in K)\ mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(\psi).$$

Then $\sim_K$ is an equivalence relation, which is a congruence on $\mathfrak{F}$ by compositionality. $\mathfrak{F}/{\sim_K}$ denotes the factor-algebra of $\mathfrak{F}$, factorized by $\sim_K$. Now,

$$\mathsf{Alg}_{\models}(\mathcal{L}) \overset{\text{def}}{=} \mathbf{I}\{\mathfrak{F}/{\sim_K} :\ K \subseteq M\}\,.$$

(ii) Further

$$\mathsf{Alg}_m(\mathcal{L}) \overset{\text{def}}{=} \{mng_{\mathfrak{M}}(\mathfrak{F})\ :\ \mathfrak{M} \in M\}\,,$$

where $mng_{\mathfrak{M}}$ was defined in Definition 4.1.2, and for any homomorphism $h : \mathfrak{A} \longrightarrow \mathfrak{B}$, $h(\mathfrak{A})$ is the homomorphic image of $\mathfrak{A}$ along h i.e., $h(\mathfrak{A})$ is the smallest subalgebra of $\mathfrak{B}$ such that $h : \mathfrak{A} \longrightarrow h(\mathfrak{A})$ (cf. p.??).

Remark 4.1.13. In the definition of $\mathsf{Alg}_m(\mathcal{L})$ above, it is important that $\mathsf{Alg}_m(\mathcal{L})$ is not an abstract class in the sense that it is not closed under isomorphisms. The reason for defining $\mathsf{Alg}_m(\mathcal{L})$ in such a way is that since $\mathsf{Alg}_m(\mathcal{L})$ is the class of algebraic counterparts of the *models* of $\mathcal{L}$, we need these algebras as concrete algebras and replacing them with their isomophic copies would lead to loss of information (about semantic-model theoretic matters). See e.g. items 6.0.44–6.0.50 in section 6 about the algebraic characterization of the *weak Beth definability property*.

Fact 4.1.14. *Let $\mathcal{L}$ be a compositional logic satisfying condition (i) of the filter property (see Definition 4.1.3) Then*

$$\mathsf{Alg}_{\models}(\mathcal{L}) = \mathbf{I}\langle\mathfrak{F}/{\sim_{Mod_{\mathcal{L}}(\Gamma)}} :\ \Gamma \subseteq F\rangle.$$

Proof. For every $K \subseteq M$, $\quad \mathfrak{F}/{\sim_K} = \mathfrak{F}/{\sim_{Mod_{\mathcal{L}}(Th_{\mathcal{L}}(K))}}$ holds (cf. Definitions 3.1.5 and 3.1.6). □

Theorem 4.1.15. *For any compositional logic $\mathcal{L} = \langle F, M, mng, \models\rangle$, (i)–(iii) below hold.*

(i) $\mathsf{Alg}_m(\mathcal{L}) \subseteq \mathsf{Alg}_{\models}(\mathcal{L})$.

(ii) $\mathsf{Alg}_{\models} \subseteq \mathbf{SP}\mathsf{Alg}_m(\mathcal{L})$.

(iii) $\mathbf{SP}\mathsf{Alg}_{\models}(\mathcal{L}) = \mathbf{SP}\mathsf{Alg}_m(\mathcal{L})$.

Proof. **(i):** Let $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$, that is, assume that $(\exists \mathfrak{M} \in M)\mathfrak{A} = mng_{\mathfrak{M}}(\mathfrak{F})$. Let $K \overset{\text{def}}{=} \{\mathfrak{M}\}$. Then $\sim_K = ker(mng_{\mathfrak{M}})$, and $\mathfrak{A} \cong \mathfrak{F}/{\sim_K} \in \mathsf{Alg}_{\models(\mathcal{L})}$. Thus $\mathfrak{A} \in \mathsf{Alg}_{\models}(\mathcal{L})$ since $\mathsf{Alg}_{\models}(\mathcal{L})$ is closed under isomorphisms.

(ii): Let $\mathfrak{A} \in \mathsf{Alg}_{\models}(\mathcal{L})$, that is, assume that $(\exists K \subseteq M)\mathfrak{A} \cong \mathfrak{F}/{\sim_K}$. Since F is a set, $(\exists K' \subseteq K)(K'$ is a set and $\mathfrak{F}/{\sim_K} = \mathfrak{F}/{\sim_{K'}})$. From now on, let K denote such a K'. Consider the following function

$$h : \mathfrak{F}/{\sim_K} \longrightarrow \Pi\langle mng_{\mathfrak{M}}(\mathfrak{F}) \ : \ \mathfrak{M} \in K\rangle\,.$$

For any $\varphi \in F$, $h(\varphi/{\sim_K}) \stackrel{\text{def}}{=} \langle mng_{\mathfrak{M}}(\varphi) \ : \ \mathfrak{M} \in K\rangle$. It is easy to see that h is a homomorphism, moreover, it is an embedding. Thus $\mathfrak{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathcal{L})$.

(iii): $\mathbf{SP}\mathsf{Alg}_m(\mathcal{L}) \subseteq \mathbf{SP}\mathsf{Alg}_{\models}(\mathcal{L})$ holds by (i) and by the fact that $\mathbf{SP}$ is isotone (by being a closure operator). The other direction $\mathbf{SP}\mathsf{Alg}_m(\mathcal{L}) \supseteq \mathbf{SP}\mathsf{Alg}_{\models}(\mathcal{L})$ holds by (ii) and by the fact that $\mathbf{SP}$ is idempotent. □

Corollary 4.1.16. $\mathsf{Alg}_{\models}(\mathcal{L})$ *and* $\mathsf{Alg}_m(\mathcal{L})$ *have the same quasi-equational and equational theories.*

Proof. This follows from Theorems 4.1.15, 2.5.10, and 2.5.11. □

Exercise 4.1.17. Why did we need the fact that K in the proof of (ii) of Thm.4.1.15 is a *set* and not a proper class?

Exercises 4.1.18. Prove that

(i) $\mathsf{Alg}_m(\mathcal{L}_S) \subseteq$"class of all Boolean set algebras"

(ii) $\mathsf{Alg}_m(S5) \subseteq$"class of all one-dimensional cylindric set algebras".

4.1.3 Hilbert-type inference systems

By Def.3.1.3, by a logic we meant only a 4-tuple $\langle F, M, mng, \models\rangle$, even though, at the beginning of section 3.2, we said that, often, a *syntactical consequence relation* $\vdash \subseteq \mathcal{P}(F) \times F$ also belongs to the picture. $\vdash$ is also called an *inference system*, referring to its "computational" nature. Namely, inference systems are syntactical devices serving to recapture (or at least approximate) the semantical consequence relation $\models \subseteq \mathcal{P}(F) \times F$ of the logic. The idea is the following. Suppose $\Sigma \models_{\mathcal{L}} \varphi$. This means that, in the logic $\mathcal{L}$, the assumptions collected in Σ semantically imply the conclusion φ. (In any model $\mathfrak{M}$ of $\mathcal{L}$, that is, in any $\mathfrak{M} \in M_{\mathcal{L}}$, whenever Σ is valid in $\mathfrak{M}$, then also φ is valid in $\mathfrak{M}$.) Then we would like to be able to reproduce this relationship between Σ and φ by purely syntactical, "finitistic" means. That is, by applying some formal rules of inference (and some axioms of the logic $\mathcal{L}$) we would like to be able to derive φ from Σ by using "paper and pencil" only. In particular, such a derivation will always be a finite string of symbols. If we can do this, that will be denoted by $\Sigma \vdash \varphi$.

Definition 4.1.19. (formula scheme) Let $\mathcal{L}$ be a logic having the set $Cn(\mathcal{L})$ of logical connectives. Fix a countable set $A = \{A_i : i < \omega\}$, called the set of *formula variables*. The set $Fms_{\mathcal{L}}$ of *formula schemes* of $\mathcal{L}$ is the smallest set satisfying conditions (a–b) below.

(a) $A \subseteq Fms_{\mathcal{L}}$,
(b) if $c \in Cn_k(\mathcal{L})$ and $\Phi_1, \dots, \Phi_k \in Fms_{\mathcal{L}}$ then $c(\Phi_1, \dots, \Phi_k) \in Fms_{\mathcal{L}}$.

An *instance of a formula scheme* is given by substituting formulas for the formula variables in it.

Definition 4.1.20. (Hilbert-style inference system) Assume that a logic $\mathcal{L}$ has logical connectives (see Definition 4.1.1). An *inference rule* of $\mathcal{L}$ is a pair $\big\langle \langle B_1, \dots, B_n \rangle, B_0 \big\rangle$, where every B_i $(i \leqslant n)$ is a formula scheme. This inference rule will be denoted by

$$\frac{B_1, \dots, B_n}{B_0}\,.$$

An *instance of an inference rule* is given by substituting formulas for the formula variables in the formula schemes occurring in the rule.

A *Hilbert-style inference system* (or *calculus*) for $\mathcal{L}$ is a *finite* set of formula schemes (called *axiom schemes* or *axioms*) together with a *finite* set of inference rules.

Definition 4.1.21. (derivability) Let $\mathcal{L}$ be a logic and assume that $\mathcal{L}$ has logical connectives. Let $\vdash$ be a Hilbert-style inference system for $\mathcal{L}$. Assume $\Sigma \cup \{\varphi\} \subseteq F_{\mathcal{L}}$. We say that φ is $\vdash$*-derivable* (or $\vdash$*-provable*) from Σ iff there is a finite sequence $\langle \varphi_1, \dots, \varphi_n \rangle$ of formulas (an $\vdash$*-proof of* φ *from* Σ) such that φ_n is φ and for every $1 \leqslant i \leqslant n$

- $\varphi_i \in \Sigma$ or
- φ_i is an instance of an axiom scheme (an *axiom* for short) of $\vdash$ or
- there are $j_1, \dots, j_k < i$, and there is an inference rule of $\vdash$ such that $\frac{\varphi_{j_1}, \dots, \varphi_{j_k}}{\varphi_i}$ is an instance of this rule.

We write $\Sigma \vdash \varphi$ if φ is $\vdash$-provable from Σ. (We will often identify an inference system $\vdash$ with the corresponding derivability relation.)

Definition 4.1.22. (complete and sound Hilbert-type inference system) Let $\mathcal{L}$ be a logic and assume that $\mathcal{L}$ has logical connectives. Let $\vdash$ be a Hilbert-type inference system for $\mathcal{L}$. Then

- $\vdash$ is *weakly complete* for $\mathcal{L}$ iff $(\forall \varphi \in F_{\mathcal{L}})\,(\models_{\mathcal{L}} \varphi \Rightarrow\, \vdash \varphi)\,;$
- $\vdash$ is *strongly complete* for $\mathcal{L}$ iff $(\forall \Sigma \subseteq F_{\mathcal{L}})(\forall \varphi \in F_{\mathcal{L}})\,(\Sigma \models_{\mathcal{L}} \varphi \Rightarrow \Sigma \vdash \varphi)\,;$
- $\vdash$ is *finitely complete* for $\mathcal{L}$ iff $(\forall \Sigma \subseteq_{\omega} F_{\mathcal{L}})(\forall \varphi \in F_{\mathcal{L}})\,(\Sigma \models_{\mathcal{L}} \varphi \Rightarrow \Sigma \vdash \varphi)$ (we consider only finite Σ's);
- $\vdash$ is *weakly sound* for $\mathcal{L}$ iff $(\forall \varphi \in F_{\mathcal{L}})\,(\vdash \varphi \Rightarrow\, \models_{\mathcal{L}} \varphi)\,;$
- $\vdash$ is *strongly sound* for $\mathcal{L}$ iff $(\forall \Sigma \subseteq F_{\mathcal{L}})(\forall \varphi \in F_{\mathcal{L}})\,(\Sigma \vdash \varphi \Rightarrow \Sigma \models_{\mathcal{L}} \varphi)\,.$

4.2 Algebraic characterizations of completeness and compactness properties

In the proofs of the main theorems we make a careful distinction between $\models_{\mathcal{L}}$ and $\models$, using the former symbol for the validity (and semantical consequence) relation of logic $\mathcal{L}$ and $\models$ for the usual first-order validity relation.

Theorem 4.2.1. *(i) Let $\mathcal{L} = \langle F, M, mng, \models_{\mathcal{L}} \rangle$ be a strongly nice logic. Let m be as in Def. 4.1.3. Then for any formulas $\varphi_0, \varphi_1, \dots, \varphi_k$,*

$$\{\varphi_1, \dots, \varphi_k\} \models_{\mathcal{L}} \varphi_0 \qquad \textit{iff} \qquad \textit{for each } j < m,$$

$$\mathsf{Alg}_m(\mathcal{L}) \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \leqslant s \leqslant k,\ i < m\} \to \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big).$$

(ii) Let $\mathcal{L}$ be a strongly nice logic in the sense of Def. 4.1.8. Let n be as in Def. 4.1.3. Then for any quasi-equation q of form

$$(\tau_1 = \tau_1' \wedge \dots \wedge \tau_k = \tau_k') \to \tau_0 = \tau_0',$$
$$\mathsf{Alg}_m(\mathcal{L}) \models q \iff \{\tau_s \Delta_j \tau_s' : 1 \leqslant s \leqslant k,\ j < n\} \models_{\mathcal{L}} \tau_0 \Delta_i \tau_0'$$

for each $i < n$.

Proof. Proof of (i):

Direction "$\Longrightarrow$": Assume $p_0, \dots, p_\ell$ are the only atomic formulas occurring in $\varphi_0, \dots, \varphi_k$ and assume that

$$\{\varphi_1(p_0, \dots, p_\ell), \dots, \varphi_k(p_0, \dots, p_\ell)\} \models_{\mathcal{L}} \varphi_0(p_0, \dots, p_\ell).$$

Let $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$. Then $\mathfrak{A} = mng_{\mathfrak{M}}(\mathfrak{F})$ for some $\mathfrak{M} \in M$. Let $a \in {}^P A$ be arbitrary. For every $i \leqslant \ell$ we denote $a_i \stackrel{\text{def}}{=} a(p_i)$. Clearly, for every $i \leqslant \ell, a_i = mng_{\mathfrak{M}}(\gamma_i)$ for some $\gamma_i \in F$. For every $s \leqslant k$,

$$\varphi_s[a_0, \dots, a_\ell]^{\mathfrak{A}} = \varphi_s[mng_{\mathfrak{M}}(\gamma_0), \dots, mng_{\mathfrak{M}}(\gamma_\ell)]^{\mathfrak{A}} = mng_{\mathfrak{M}}\big(\varphi_s(\gamma_0, \dots, \gamma_\ell)\big),$$

since $mng_{\mathfrak{M}}$ is a homomorphism by compositionality.

Assume that for every $1 \leqslant s \leqslant k$ and $j < m$, $\mathfrak{A} \models \big(\varepsilon_j(\varphi_s) = \delta_j(\varphi_s)\big)[a]$. This holds iff

$$mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi_s(\gamma_0, \dots, \gamma_\ell))\big) = mng_{\mathfrak{M}}\big(\delta_j(\varphi_s(\gamma_0, \dots, \gamma_\ell))\big) \quad (1 \leqslant s \leqslant k,\ j < m).$$

Then, by the semantical substitution property, there exists an $\mathfrak{N}$ as described in

Def. 4.1.6 for s sending p_0 to γ_0, ..., p_k to γ_k an for $\mathfrak{M}$. Let this $\mathfrak{N}$ be fixed.

$$\Longrightarrow mng_{\mathfrak{N}}\big(\varepsilon_j(\varphi_s)\big) = mng_{\mathfrak{N}}\big(\delta_j(\varphi_s)\big) \quad (1 \leqslant s \leqslant k,\ j < m)$$

$$\overset{\text{(by Def. 4.1.3)}}{\Longleftrightarrow} \mathfrak{N} \models_{\mathcal{L}} \varphi_s \quad (1 \leqslant s \leqslant k)$$

$$\overset{\text{(by our assumption)}}{\Longrightarrow} \mathfrak{N} \models_{\mathcal{L}} \varphi_0$$

$$\overset{\text{(by Def. 4.1.3)}}{\Longleftrightarrow} mng_{\mathfrak{N}}\big(\varepsilon_j(\varphi_0)\big) = mng_{\mathfrak{N}}\big(\delta_j(\varphi_0)\big) \quad (j < m)$$

$$\overset{\text{(by Def. 4.1.6)}}{\Longrightarrow} mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi_0(\gamma_0, \dots, \gamma_\ell))\big) = mng_{\mathfrak{M}}\big(\delta_j(\varphi_0(\gamma_0, \dots, \gamma_\ell))\big) \quad (j < m)$$

$$\Longleftrightarrow \mathfrak{A} \models \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big)[a], \quad (j < m)$$

proving Thm. 4.2.1 (i) direction "$\Longrightarrow$", since a was chosen arbitrarily.

Direction "$\Longleftarrow$": Assume that

$$\mathsf{Alg}_m(\mathcal{L}) \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \leqslant s \leqslant k,\ i < m\} \to \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big).$$

Let $\mathfrak{M} \in M$. Assume that for every $1 \leqslant s \leqslant k$ $\mathfrak{M} \models_{\mathcal{L}} \varphi_s$.

$$\overset{\text{(by Def. 4.1.3)}}{\Longrightarrow} mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi_s)\big) = mng_{\mathfrak{M}}\big(\delta_j(\varphi_s)\big) \quad (1 \leqslant s \leqslant k,\ j < m)$$

$$\overset{\text{(by our assumption)}}{\Longleftrightarrow} mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi_0)\big) = mng_{\mathfrak{M}}\big(\delta_j(\varphi_0)\big) \quad (j < m)$$

$$\overset{\text{(by Def. 4.1.3)}}{\Longrightarrow} \mathfrak{M} \models_{\mathcal{L}} \varphi_0$$

proving Thm. 4.2.1 (i) direction "$\Longleftarrow$".

Proof of (ii):
Direction "$\Longrightarrow$": Assume that for every $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$ and for every valuation $a \in {}^P A$ $\mathfrak{A} \models q[a]$. Let $\mathfrak{M} \in M$ such that

$$\mathfrak{M} \models_{\mathcal{L}} \{\tau_s \Delta_i \tau'_s : 1 \leqslant s \leqslant k, i < n\}.$$

Then by Def. 4.1.3 (i), $mng_{\mathfrak{M}}(\tau_s) = mng_{\mathfrak{M}}(\tau'_s)$ for each $1 \leqslant s \leqslant k$. Now we let $\mathfrak{A} \overset{\text{def}}{=} mng_{\mathfrak{M}}(\mathfrak{F})$ and let $a \in {}^P A$ be such that for each $p \in P$, $a(p) \overset{\text{def}}{=} mng_{\mathfrak{M}}(p)$. Then

$$\mathfrak{A} \models (\tau_1 = \tau'_1 \wedge \cdots \wedge \tau_k = \tau'_k)[a],$$

which implies by our assumption that $\mathfrak{A} \models (\tau_0 = \tau'_0)[a]$. This is the same as $mng_{\mathfrak{M}}(\tau_0) = mng_{\mathfrak{M}}(\tau'_0)$, thus again by Definition 4.1.3 (i), $\mathfrak{M} \models_{\mathcal{L}} \tau_0 \Delta_i \tau'_0$ for each $i < n$, which proves direction "$\Longrightarrow$" of Thm. 4.2.1(ii).

Direction "$\Longleftarrow$": Assume $\{\tau_s \Delta_j \tau'_s : 1 \leqslant s \leqslant k,\ j < n\} \models_{\mathcal{L}} \tau_0 \Delta_i \tau'_0$ for each $i < n$. Assume $p_0, \dots, p_\ell$ are the only atomic formulas occurring in $\tau_0, \tau'_0, \dots, \tau_k, \tau'_k$.

Let $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$. Then $\mathfrak{A} = mng_{\mathfrak{M}}(\mathfrak{F})$ for some $\mathfrak{M} \in M$. Let $a \in {}^P A$ be arbitrary. For every $i \leqslant \ell$ we denote $a_i \stackrel{\text{def}}{=} a(p_i)$. Clearly for every $i \leqslant \ell \quad a_i = mng_{\mathfrak{M}}(\gamma_i)$ for some $\gamma_i \in F$. For every $\varphi \in F$

$$\varphi[a_0, \ldots, a_\ell]^{\mathfrak{A}} = \varphi[mng_{\mathfrak{M}}(\gamma_0), \ldots, mng_{\mathfrak{M}}(\gamma_\ell)]^{\mathfrak{A}} = mng_{\mathfrak{M}}\big(\varphi(\gamma_0, \ldots, \gamma_\ell)\big)$$

since $mng_{\mathfrak{M}}$ is a homomorphism by compositionality. Assume that for every $1 \leqslant s \leqslant k$, $\mathfrak{A} \models \tau_s = \tau'_s[a]$.

$\iff mng_{\mathfrak{M}}\big(\tau_s(\gamma_0, \ldots, \gamma_\ell)\big) = mng_{\mathfrak{M}}\big(\tau'_s(\gamma_0, \ldots, \gamma_\ell)\big)$

$\overset{\text{(by Def. 4.1.6)}}{\Longrightarrow}$ There is $\mathfrak{N}$ as described in Def. 4.1.6 for s sending p_0 to γ_0, …, p_k to γ_k and for $\mathfrak{M}$. Let this $\mathfrak{N}$ be fixed.

$\Longrightarrow mng_{\mathfrak{N}}(\tau_s) = mng_{\mathfrak{N}}(\tau'_s) \quad (1 \leqslant s \leqslant k)$

$\overset{\text{(by Def. 4.1.3 (i))}}{\iff} \mathfrak{N} \models_{\mathcal{L}} \tau_s \Delta_i \tau'_s \quad (1 \leqslant s \leqslant k,\ i < n)$

$\overset{\text{(by our assumption)}}{\Longrightarrow} \mathfrak{N} \models \tau_0 \Delta_i \tau'_0 \quad (i < n)$

$\overset{\text{(by Def. 4.1.6)}}{\Longrightarrow} mng_{\mathfrak{M}}\big(\tau_0(\gamma_0, \ldots, \gamma_\ell)\big) = mng_{\mathfrak{M}}\big(\tau'_0(\gamma_0, \ldots, \gamma_\ell)\big)$

$\iff \mathfrak{A} \models \big(\tau_0 = \tau'_0\big)[a]$,

proving Thm. 4.2.1 (ii) direction "$\Longleftarrow$", since a was chosen arbitrarily.

$\square$

Discussion: of Theorem 4.2.1: For proving the "$\Longleftarrow$" direction of (i), it is enough to assume that $\mathcal{L}$ is compositional and has the filter property. For proving the "$\Longrightarrow$" direction of (ii), it is enough to assume that $\mathcal{L}$ is compositional and satisfies condition (i) of the filter property. However, there exist compositional logics satisfying (i) of the filter property for which direction "$\Longleftarrow$" of (ii) does not hold. For proving this direction we do not have to assume condition (ii) of the filter property.

Corollary 4.2.2. *Let $\mathcal{L}$ be a nice logic. Let $\varepsilon, \delta, \Delta, m, n$ be as in Def. 4.1.3. Then (i) and (ii) below hold.*

(i) For any formula φ,

$$\models_{\mathcal{L}} \varphi \quad \iff \quad \mathsf{Alg}_m(\mathcal{L}) \models \varepsilon_j(\varphi) = \delta_j(\varphi) \quad \textit{for each } j < m.$$

(ii) For any equation $\tau = \tau'$,

$$\mathsf{Alg}_m(\mathcal{L}) \models \tau = \tau' \quad \iff \quad \models_{\mathcal{L}} \tau \Delta_i \tau' \quad \textit{for each } i < n.$$

Proof. Item (ii) is a special case of item (ii) of Thm. 4.2.1, but now we have to prove (i) for *nice* logics, cf. Def. 4.1.8.

Assume $\models_{\mathcal{L}} \varphi(p_0, \ldots, p_\ell)$. Let $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$. Then $\mathfrak{A} = mng_{\mathfrak{M}}(\mathfrak{F})$ for some $\mathfrak{M} \in M$. Let $a \in {}^P A$ be arbitrary. We denote $a_0 \stackrel{\text{def}}{=} a(p_0), \ldots, a_\ell \stackrel{\text{def}}{=} a(p_\ell)$. Clearly $(\forall s \leqslant \ell)$ $(a_s = mng_{\mathfrak{M}}(\gamma_s)$ for some $\gamma_s \in F)$.

$$\varphi[a_0, \ldots, a_\ell]^{\mathfrak{A}} = \varphi[mng_{\mathfrak{M}}(\gamma_0), \ldots, mng_{\mathfrak{M}}(\gamma_\ell)]^{\mathfrak{A}} = mng_{\mathfrak{M}}\left(\varphi(\gamma_0, \ldots, \gamma_\ell)\right),$$

since $mng_{\mathfrak{M}}$ is a homomorphism.

$\models_{\mathcal{L}} \varphi(p_0, \ldots, p_\ell)$ implies, by the substitution property, that $\models_{\mathcal{L}} \varphi(\gamma_0, \ldots, \gamma_\ell)$. Thus by the filter property, for each $j < m$

$$mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi(\gamma_0, \ldots, \gamma_\ell))\big) = mng_{\mathfrak{M}}\big(\delta_j(\varphi(\gamma_0, \ldots, \gamma_\ell))\big).$$

But

$$\begin{aligned} mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi(\gamma_0, \ldots, \gamma_\ell))\big) &= \varepsilon_j(\varphi)[a]^{\mathfrak{A}} \text{ and} \\ mng_{\mathfrak{M}}\big(\delta_j(\varphi(\gamma_0, \ldots, \gamma_\ell))\big) &= \delta_j(\varphi)[a]^{\mathfrak{A}} \qquad (j < m). \end{aligned}$$

Thus we have $\mathfrak{A} \models \big(\varepsilon_j(\varphi) = \delta_j(\varphi)\big)[a]$ for each $j < m$, completing the proof since a was chosen arbitrarily. ☐

In Theorem 4.2.3 below, we will give a sufficent and necessary condition for a strongly nice logic to have a finitely complete Hilbert-style inference system.

Theorem 4.2.3. *Assume $\mathcal{L}$ is a strongly nice logic and $Cn(\mathcal{L})$ is finite*[2]. *Then*

$\mathsf{Alg}_m(\mathcal{L})$ *generates a finitely axiomatizable quasi-variety*

$\Longleftrightarrow$

$(\exists$ *Hilbert-style* $\vdash)(\vdash$ *is finitely complete and strongly sound for* $\mathcal{L})$.

Proof. **(Proof of ($\Longrightarrow$))**

Notation Let $\Phi_0, \Phi_1, \ldots$ denote formula variables, $\tau_0, \tau_1, \ldots$ denote formula schemes, $\overline{\Phi}$ denote sequence of formula variables and $\overline{x}$ denote sequence of variables. Let m and n $(m, n \in \omega)$ denote the number of ε_j's and Δ_i's, respectively. For any formula schemes τ, τ', let $\tau \boldsymbol{\Delta} \tau'$ abbreviate the system $\tau\Delta_0\tau', \ldots, \tau\Delta_{n-1}\tau'$ of formula schemes.

Now assume that Ax is a finite set of quasi-equations axiomatizing the quasi-variety generated by $\mathsf{Alg}_m(\mathcal{L})$ and define a Hilbert-style inference system $\vdash_{Ax}$ as follows:

[2] One can eliminate the assumption of $Cn(\mathcal{L})$ being finite. Then the finitary character of a Hilbert-style inference system has to be ensured in a more subtle way. Also, "finitely axiomatizable quasi-variety" must be replaced by "finite schema axiomatizable quasi-variety" in the second clause, cf. e.g. Monk [55], Németi [60].

AXIOM SCHEMES: $\Phi_0\Delta_i\Phi_0 \qquad (i<n)$.
INFERENCE RULES: If $\left[\left(\tau_1(\overline{x})=\tau_1'(\overline{x})\wedge\cdots\wedge\tau_k(\overline{x})=\tau_k'(\overline{x})\right)\Rightarrow\tau_0(\overline{x})=\tau_0'(\overline{x})\right]\in Ax$, then

$$\frac{\tau_1(\overline{\Phi})\boldsymbol{\Delta}\tau_1'(\overline{\Phi}),\ldots,\tau_k(\overline{\Phi})\boldsymbol{\Delta}\tau_k'(\overline{\Phi})}{\tau_0(\overline{\Phi})\Delta_i\tau_0'(\overline{\Phi})}$$

is a rule for each $i<n$. Other rules are:

$$(\forall i<n)\quad\frac{\Phi_0\boldsymbol{\Delta}\Phi_1,\ \Phi_1\boldsymbol{\Delta}\Phi_2}{\Phi_0\Delta_i\Phi_2},$$

$$(\forall i<n)\quad\frac{\Phi_0\boldsymbol{\Delta}\Phi_1}{\Phi_1\Delta_i\Phi_0},$$

$$(\forall c\in Cn_\ell(\mathcal{L}))(\forall i<n)\quad\frac{\Phi_1\boldsymbol{\Delta}\Phi_1',\ldots,\Phi_\ell\boldsymbol{\Delta}\Phi_\ell'}{c(\Phi_1,\ldots,\Phi_\ell)\Delta_i c(\Phi_1',\ldots,\Phi_\ell')},$$

$$\frac{\varepsilon_0(\Phi_0)\boldsymbol{\Delta}\delta_0(\Phi_0),\ldots,\varepsilon_{m-1}(\Phi_0)\boldsymbol{\Delta}\delta_{m-1}(\Phi_0)}{\Phi_0},$$

$$(\forall i<n)(\forall j<m)\quad\frac{\Phi_0}{\varepsilon_j(\Phi_0)\Delta_i\delta_j(\Phi_0)}\,.$$

We will show that the inference system $\vdash_{Ax}$ is finitely complete and strongly sound for $\mathcal{L}$.

For any set Σ of formulas we define

$$\psi\sim_\Sigma\psi'\quad\stackrel{\text{def}}{\Longleftrightarrow}\quad\Sigma\vdash_{Ax}\{\psi\Delta_i\psi':i<n\}.$$

Note that, by the definition of $\vdash_{Ax}$ and by the definition of derivability (Def. 4.1.21), $\sim_\Sigma$ is a congruence relation on $\mathfrak{F}$ for any Σ. □

Claim 4.2.4. *For any* $\Sigma\subseteq F$, $\quad(\mathfrak{F}/\sim_\Sigma)\models Ax$.

Proof. **(Proof of Claim 4.2.4)** Let $q\in Ax$ and assume that q is of form

$$\left(\tau_1(\overline{x})=\tau_1'(\overline{x})\wedge\cdots\wedge\tau_k(\overline{x})=\tau_k'(\overline{x})\right)\Rightarrow\tau_0(\overline{x})=\tau_0'(\overline{x})\,.$$

Let $\mathfrak{A}\stackrel{\text{def}}{=}(\mathfrak{F}/\sim_\Sigma)$. We want to prove that, for every valuation a of the variables into $\mathfrak{A}$, $\mathfrak{A}\models q[a]$.

So let a be an arbitrary valuation into $\mathfrak{A}$. Then $(\forall i\in\omega)\ a(x_i)=\varphi_i/\sim_\Sigma$ for some $\varphi_i\in F$. Assume that

$$\mathfrak{A}\models\tau_1\left[\overline{\varphi/\sim_\Sigma}\right]=\tau_1'\left[\overline{\varphi/\sim_\Sigma}\right]\wedge\cdots\wedge\tau_k\left[\overline{\varphi/\sim_\Sigma}\right]=\tau_k'\left[\overline{\varphi/\sim_\Sigma}\right].$$

Then

$$\left(\tau_1(\overline{\varphi})\right)/\sim_\Sigma=\left(\tau_1'(\overline{\varphi})\right)/\sim_\Sigma,\ldots,\left(\tau_k(\overline{\varphi})\right)/\sim_\Sigma=\left(\tau_k'(\overline{\varphi})\right)/\sim_\Sigma,$$

since $\sim_\Sigma$ is a congruence on $\mathfrak{F}$. Then

$$\tau_1\left(\overline{\varphi}\right)\sim_\Sigma\tau_1'\left(\overline{\varphi}\right),\ldots,\tau_k\left(\overline{\varphi}\right)\sim_\Sigma\tau_k'\left(\overline{\varphi}\right)$$

that is,

$$\Sigma \vdash_{Ax} \{\tau_j(\overline{\varphi})\,\Delta_i \tau'_j(\overline{\varphi}) : 1 \leqslant j \leqslant k,\ i < n\}$$

by the definition of $\sim_\Sigma$. In $\vdash_{Ax}$, we have the following rule for each $i < n$ (corresponding to quasiequation q):

$$\frac{\tau_1(\overline{\Phi})\boldsymbol{\Delta}\tau'_1(\overline{\Phi}), \ldots, \tau_k(\overline{\Phi})\boldsymbol{\Delta}\tau'_k(\overline{\Phi})}{\tau_0(\overline{\Phi})\Delta_i\tau'_0(\overline{\Phi})}\,.$$

By these rules, we get that $\Sigma \vdash_{Ax} \tau_0(\overline{\varphi})\,\Delta_i\tau'_0(\overline{\varphi})$ for each $i < n$. Then $\tau_0(\overline{\varphi}) \sim_\Sigma \tau'_0(\overline{\varphi})$, whence $\big(\tau_0(\overline{\varphi})\big)/{\sim_\Sigma} = \big(\tau'_0(\overline{\varphi})\big)/{\sim_\Sigma}$ that is, $\mathfrak{A} \models \tau_0\big[\overline{\varphi/{\sim_\Sigma}}\big] = \tau'_0\big[\overline{\varphi/{\sim_\Sigma}}\big]$ which implies $\mathfrak{A} \models \big(\tau_0(\overline{x}) = \tau'_0(\overline{x})\big)[a]$. By this we proved Claim 4.2.4. $\square$

Now let $\Sigma \overset{\text{def}}{=} \{\varphi_1, \ldots, \varphi_k\}$ and assume $\Sigma \models_{\mathcal{L}} \varphi_0$. Then, by Thm. 4.2.1 (i),

$$\mathsf{Alg}_m(\mathcal{L}) \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \leqslant s \leqslant k,\ i < m\} \to \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big) \quad (j < m)$$

$$\Longrightarrow Ax \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \leqslant s \leqslant k,\ i < m\} \to \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big) \quad (j < m)$$

$$\overset{\text{(Claim 4.2.4)}}{\Longrightarrow} (\mathfrak{F}/{\sim_\Sigma}) \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \leqslant s \leqslant k,\ i < m\} \Rightarrow$$

$$\Rightarrow \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big) \quad (j < m)$$

$$\Longrightarrow \big[\text{if } \big((\forall i < m)(\forall 1 \leqslant s \leqslant k)\ \varepsilon_i(\varphi_s) \sim_\Sigma \delta_i(\varphi_s)\big) \text{ then } (\forall j < m)\ \varepsilon_j(\varphi_0) \sim_\Sigma \delta_j(\varphi_0)\big]$$

$$\Longleftrightarrow \big[\text{if } \big((\forall \ell < n)(\forall i < m)(\forall 1 \leqslant s \leqslant k)\ \Sigma \vdash_{Ax} \varepsilon_i(\varphi_s)\Delta_\ell\delta_i(\varphi_s)\big)$$

$$\text{then } (\forall \ell < n)(\forall j < m)\ \Sigma \vdash_{Ax} \varepsilon_j(\varphi_0)\Delta_\ell\delta_j(\varphi_0)\big]. \qquad (\bullet)$$

By the rules $\frac{\Phi_0}{\varepsilon_i(\Phi_0)\Delta_\ell\delta_i(\Phi_0)}$ we have $\Sigma \vdash_{Ax} \varepsilon_i(\varphi_s)\Delta_\ell\delta_i(\varphi_s)$ for every $i < m$, $\ell < n$, $1 \leqslant s \leqslant k$. Thus, by $(\bullet)$, $\Sigma \vdash_{Ax} \varepsilon_j(\varphi_0)\Delta_\ell\delta_j(\varphi_0)$ holds for each $\ell < n$, $j < m$. Now using the rule $\frac{\varepsilon_0(\Phi_0)\boldsymbol{\Delta}\delta_0(\Phi_0),\ldots,\varepsilon_{m-1}(\Phi_0)\boldsymbol{\Delta}\delta_{m-1}(\Phi_0)}{\Phi_0}$ we get $\Sigma \vdash_{Ax} \varphi_0$, proving the finite completeness of $\vdash_{Ax}$.

The strong soundness of $\vdash_{Ax}$ can be proved by induction on the length of the $\vdash_{Ax}$-proof of φ_0 from $\{\varphi_1, \ldots, \varphi_k\}$. We only show one part of the induction step, namely the case when φ_0 is 'obtained' by one of the inference rules corresponding to a quasi-equation $q \in Ax$. Say q has the form

$$\big(\tau_1(\overline{x}) = \tau'_1(\overline{x}) \wedge \cdots \wedge \tau_r(\overline{x}) = \tau'_r(\overline{x})\big) \Rightarrow \tau_0(\overline{x}) = \tau'_0(\overline{x})\,,$$

where $\overline{x} = \langle x_1, \ldots, x_z\rangle$. Then a corresponding inference rule is

$$\frac{\tau_1(\overline{\Phi})\boldsymbol{\Delta}\tau'_1(\overline{\Phi}), \ldots, \tau_r(\overline{\Phi})\boldsymbol{\Delta}\tau'_r(\overline{\Phi})}{\tau_0(\overline{\Phi})\Delta_i\tau'_0(\overline{\Phi})}\,,$$

for some $i < n$. Assume that φ_0 is obtained with the help of this rule by substituting the members of the sequence $\overline{\gamma} = \langle\gamma_1, \ldots, \gamma_z\rangle$ of formulas for the members of the sequence $\overline{\Phi} = \langle\Phi_1, \ldots, \Phi_z\rangle$ of formula variables, i.e. φ_0 has the form $\tau_0(\overline{\gamma})\Delta_i\tau'_0(\overline{\gamma})$.

Now fix a model $\mathfrak{M}$ and assume that

$$\mathfrak{M} \models_{\mathcal{L}} \tau_1(\overline{\gamma})\boldsymbol{\Delta}\tau_1'(\overline{\gamma}), \ldots, \mathfrak{M} \models_{\mathcal{L}} \tau_r(\overline{\gamma})\boldsymbol{\Delta}\tau_r'(\overline{\gamma}).$$

We have to show that $\mathfrak{M} \models_{\mathcal{L}} \tau_0(\overline{\gamma})\Delta_i\tau_0'(\overline{\gamma})$.

Let $\mathfrak{A} \stackrel{\text{def}}{=} mng_{\mathfrak{M}}(\mathfrak{F}) \in \mathsf{Alg}_m(\mathcal{L})$. and let a be a valuation of $\mathfrak{A}$ such that for every $1 \leqslant v \leqslant z \quad a(x_v) \stackrel{\text{def}}{=} mng_{\mathfrak{M}}(\gamma_v)$. Then by condition (i) of the filter property,

$$\begin{array}{rcl} (\forall 1 \leqslant j \leqslant r) \;\; mng_{\mathfrak{M}}\big(\tau_j(\overline{\gamma})\big) = mng_{\mathfrak{M}}\big(\tau_j'(\overline{\gamma})\big) & & \\ & \iff & \mathfrak{A} \models \big(\tau_1(\overline{x}) = \tau_1'(\overline{x}) \wedge \cdots \wedge \tau_r(\overline{x}) = \tau_r'(\overline{x})\big)[a] \\ (\text{by } \mathsf{Alg}_m(\mathcal{L}) \models Ax) & \Longrightarrow & \mathfrak{A} \models \big(\tau_0(\overline{x}) = \tau_0'(\overline{x})\big)[a] \\ (\text{by (i) of filter prop.}) & \Longrightarrow & \mathfrak{M} \models_{\mathcal{L}} \tau_0(\overline{\gamma})\Delta_i\tau_0'(\overline{\gamma}). \end{array}$$

This completes the proof of direction "$\Longrightarrow$" of Theorem 4.2.3. □

Proof. **(Proof of ($\Longleftarrow$))** Let $\Phi_1, \ldots, \Phi_z$ denote formula variables, $\tau_0, \tau_1, \ldots, \tau_k$ denote formula schemes, let $\overline{\Phi} \stackrel{\text{def}}{=} \langle \Phi_1, \ldots, \Phi_z \rangle$, and let $\overline{x} \stackrel{\text{def}}{=} \langle x_1, \ldots, x_z \rangle$ be a sequence of variables. Assume that $\vdash$ is a finitely complete and strongly sound Hilbert-type inference system for the logic $\mathcal{L}$, and define the finite set Ax of quasi-equations as follows:

(1) If $\tau_0(\overline{\Phi})$ is an axiom scheme of $\vdash$ then let "$\varepsilon_j\big(\tau_0(\overline{x})\big) = \delta_j\big(\tau_0(\overline{x})\big)$" belong to Ax for each $j < m$.

(2) If $\frac{\tau_1(\overline{\Phi}), \ldots, \tau_k(\overline{\Phi})}{\tau_0(\overline{\Phi})}$ is an inference rule of $\vdash$ then let

$$\text{“} \bigwedge\{\varepsilon_i\big(\tau_s(\overline{x})\big) = \delta_i\big(\tau_s(\overline{x})\big) : 1 \leqslant s \leqslant k,\ i < m\} \Rightarrow \varepsilon_j\big(\tau_0(\overline{x})\big) = \delta_j\big(\tau_0(\overline{x})\big)\text{”}$$

belong to Ax for each $j < m$.

(3) Let "$\varepsilon_j(x_0\Delta_i x_0) = \delta_j(x_0\Delta_i x_0)$" belong to Ax for each $i < n,\ j < m$.

(4) Let "$\bigwedge\{\varepsilon_j(x_0\Delta_i x_1) = \delta_j(x_0\Delta_i x_1) : j < m,\ i < n\} \Rightarrow (x_0 = x_1)$" belong to Ax.

We will show that Ax axiomatizes the quasi-variety generated by $\mathsf{Alg}_m(\mathcal{L})$.

Claim 4.2.5. $\mathsf{Alg}_m(\mathcal{L}) \models Ax$.

Proof. **(Proof of Claim 4.2.5)** Quasi-equations of type (3) and (4) above obviously hold in $\mathsf{Alg}_m(\mathcal{L})$ by the filter property.

Now consider a quasi-equation of type (2). Let $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$ and let a be an arbitrary valuation of the variables into $\mathfrak{A}$. Let $\mathfrak{M}$ be such that $\mathfrak{A} = mng_{\mathfrak{M}}(\mathfrak{F})$. Then for every $i \in \omega \quad a(x_i) = mng_{\mathfrak{M}}(\varphi_i)$ for some $\varphi_i \in F$. Assume that

$$\mathfrak{A} \models \bigwedge\{\varepsilon_i\big(\tau_s(\overline{x})\big) = \delta_i\big(\tau_s(\overline{x})\big) : 1 \leqslant s \leqslant k,\ i < m\}[a].$$

Then the filter property

$$\mathfrak{M} \models_{\mathcal{L}} \tau_s(x_1/\varphi_1, \ldots, x_z/\varphi_z) \quad \text{(for each } 1 \leqslant s \leqslant k). \tag{$\bullet\bullet$}$$

But $\frac{\tau_1(\overline{\Phi}), \ldots, \tau_k(\overline{\Phi})}{\tau_0(\overline{\Phi})}$ is an inference rule of $\vdash$, therefore $\{\tau_1(\overline{\varphi}), \ldots, \tau_k(\overline{\varphi})\} \vdash \tau_0(\overline{\varphi})$. This implies by the strong soundness of $\vdash$ that $\{\tau_1(\overline{\varphi}), \ldots, \tau_k(\overline{\varphi})\} \models_{\mathcal{L}} \tau_0(\overline{\varphi})$. Now, by ($\bullet\bullet$) above, $\mathfrak{M} \models \tau_0(\overline{\varphi})$, hence again by the filter property, $\mathfrak{A} \models \big(\varepsilon_j(\tau_0(\overline{x})) = \delta_j(\tau_0(\overline{x}))\big)[a]$ for each $j < m$, which was desired. □

The case of equations of type (1) can be proved similarly. □

Claim 4.2.6. *For any formulas $\varphi_0, \varphi_1, \ldots, \varphi_k$,*

$$\{\varphi_1, \ldots, \varphi_k\} \vdash \varphi_0 \quad \Longrightarrow$$
$$\Longrightarrow \quad Ax \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \leqslant s \leqslant k,\ i < m\} \to \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big)$$

for each $j < m$.

Proof. **(Proof of Claim 4.2.6)** It can be proved by induction on the length of the $\vdash$-proof of φ_0 from $\{\varphi_1, \ldots, \varphi_k\}$. We only show one part of the induction step, namely the case when φ_0 is 'obtained' by an inference rule $\frac{\tau_1(\overline{\Phi}), \ldots, \tau_r(\overline{\Phi})}{\tau_0(\overline{\Phi})}$, where $\overline{\Phi} = \langle \Phi_1, \ldots, \Phi_z \rangle$. Then there are formulas $\gamma_1, \ldots, \gamma_z$ such that $\varphi_0 = \tau_0(\gamma_1, \ldots, \gamma_z)$ and for every $1 \leqslant \ell \leqslant r$ $\{\varphi_1, \ldots, \varphi_k\} \vdash \tau_\ell(\overline{\gamma})$. Then by the induction hypothesis

$$Ax \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \leqslant s \leqslant k,\ i < m\} \Rightarrow$$
$$\Rightarrow\ \varepsilon_j\big(\tau_\ell(\overline{\gamma})\big) = \delta_j\big(\tau_\ell(\overline{\gamma})\big) \quad \text{(for each } j < m,\ 1 \leqslant \ell \leqslant r). \tag{4.5}$$

By the definition of Ax

$$Ax \models \bigwedge\{\varepsilon_i\big(\tau_\ell(\overline{x})\big) = \delta_i\big(\tau_\ell(\overline{x})\big) : 1 \leqslant \ell \leqslant r,\ i < m\} \Rightarrow$$
$$\Rightarrow\ \varepsilon_j\big(\tau_0(\overline{x})\big) = \delta_j\big(\tau_0(\overline{x})\big) \quad \text{(for each } j < m). \tag{4.6}$$

Let $\mathfrak{B}$ be an algebra with $\mathfrak{B} \models Ax$ and let b be any valuation of the variables into B. Now we can define a valuation b' with $b'(x_v) \stackrel{\text{def}}{=} \gamma_v[b]^{\mathfrak{B}}$ $\quad (1 \leqslant v \leqslant z)$. Then for every $0 \leqslant \ell \leqslant r$ $\quad \tau_\ell(\overline{x})[b']^{\mathfrak{B}} = \tau_\ell(\overline{\gamma})[b]^{\mathfrak{B}}$. Thus, by (4.5) and (4.6),

$$\mathfrak{B} \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \leqslant s \leqslant k,\ i < m\} \Rightarrow \big(\varepsilon_j(\tau_0(\overline{\gamma})) = \delta_j(\tau_0(\overline{\gamma}))\big)[b]$$

for each $j < m$, which was desired. □

Now we can prove that each quasi-equation which holds in $\mathsf{Alg}_m(\mathcal{L})$ is a consequence of Ax. Assume that

$$\mathsf{Alg}_m(\mathcal{L}) \models \big(\tau_1 = \tau_1' \wedge \cdots \wedge \tau_k = \tau_k'\big) \Rightarrow \tau_0 = \tau_0'.$$

$$\overset{\text{(Thm. 4.2.1 (ii))}}{\Longrightarrow} \quad \{\tau_s \Delta_j \tau_s' : 1 \leqslant s \leqslant k,\ j < n\} \models_{\mathcal{L}} \tau_0 \Delta_i \tau_0' \quad \text{for each } i < n$$

$$\overset{\text{(finite completeness)}}{\Longrightarrow} \quad \{\tau_s \Delta_j \tau_s' : 1 \leqslant s \leqslant k,\ j < n\} \vdash \tau_0 \Delta_i \tau_0' \quad \text{for each } i < n$$

$$\overset{\text{(Claim 4.2.6)}}{\Longrightarrow} \quad Ax \models \bigwedge \{\varepsilon_\ell(\tau_s \Delta_j \tau_s') = \delta_\ell(\tau_s \Delta_j \tau_s') : \ell < m,\ 1 \leqslant s \leqslant k,\ j < n\} \Rightarrow$$
$$\Rightarrow \ \big(\varepsilon_p(\tau_0 \Delta_i \tau_0') = \delta_p(\tau_0 \Delta_i \tau_0')\big) \quad \text{for all } p < m,\ i < n.$$

But, since quasi-equations of type (3) and (4) belong to Ax, this implies to

$$Ax \models \big(\tau_1 = \tau_1' \wedge \cdots \wedge \tau_k = \tau_k'\big) \Rightarrow \tau_0 = \tau_0',$$

completing the proof of direction "$\Longleftarrow$" of Theorem 4.2.3. □

Having found the algebraic counterpart of "finitely complete", let us try to characterize "weakly complete". Since weak completeness is slightly weaker than finite completeness, we have to weaken the algebraic counterpart of finite completeness for characterizing weak completeness. This way we obtain condition (4.7) below, where $Eq_{\mathcal{L}}$ and $Qeq_{\mathcal{L}}$ denote the set of all equations and the set of all quasi-equations, respectively, of the language of $\mathsf{Alg}_m(\mathcal{L})$ (cf. [71]).

$$(\exists Ax \subseteq_\omega Qeq_{\mathcal{L}}) \quad \big((\forall e \in Eq_{\mathcal{L}})\,(\mathsf{Alg}_m(\mathcal{L}) \models e \Longrightarrow Ax \models e) \text{ and } \mathsf{Alg}_m(\mathcal{L}) \models Ax\big)\,. \tag{4.7}$$

That is, the equational theory of $\mathsf{Alg}_m(\mathcal{L})$ is finitely axiomatizable by quasi-equations valid in $\mathsf{Alg}_m(\mathcal{L})$. In other words, (4.7) says that there is a finitely axiomatizable quasi-variety K such that $\mathbf{HSP}\mathsf{Alg}_m(\mathcal{L}) = \mathbf{H}\mathsf{K}$.

Theorem 4.2.7. *Assume that $\mathcal{L}$ is nice and $Cn(\mathcal{L})$ is finite*[3]*. Then*

$$(4.7) \iff (\exists \text{ Hilbert-style} \vdash) \quad (\vdash \ \text{is weakly complete and strongly sound for } \mathcal{L})\,. \tag{4.8}$$

In particular, if the equational theory of $\mathsf{Alg}_m(\mathcal{L})$ is finitely axiomatizable, then $\mathcal{L}$ admits a weakly complete Hilbert-style inference system.

Proof. It is similar to the proof of Theorem 4.2.3. The only important difference is that Theorem 4.2.7 already holds for *nice* logics. However, the only part of the proof of Theorem 4.2.3 which used the additional criterion for strong niceness (Definition 4.1.6) was Thm. 4.2.1 (i). Here one has to use Cor. 4.2.2 (i) instead. □

[3]Cf. the footnote of Theorem 4.2.3.

Exercise 4.2.8. Give weakly complete and sound calculi for the logics $\mathcal{L}_S$ and $S5$. (Hint: Use that the **SP**-closure of the Alg_m-image of these logics are finitely axiomatizable varieties, so (4.7) is satisfied.)

Theorem 4.2.7 motivates the following question. Recall from, e.g. [15], that RCA_n denotes the variety of n-dimensional representable cylindric algebras.

Problem 4.2.9. *Is there a finitely axiomatizable quasi-variety* $\mathsf{K} \subseteq \mathsf{RCA}_n$ *such that* $\mathbf{H}\mathsf{K} = \mathsf{RCA}_n$*? In other words,* K *should be such that* RCA_n *is the variety generated by* K.

The above Problem is the corrected version of Open Problem 3.24 in [62] where the original one contains a fatal typo.

Definition 4.2.10. (deduction theorem, deduction term)
Let $\mathcal{L} = \langle F_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}}\rangle$ be a logic having logical connectives. We say that $\mathcal{L}$ *has a deduction theorem*, iff

$$(\exists(\Phi_1\nabla\Phi_2) \in Fms_{\mathcal{L}}))\,(\forall\Sigma \subseteq F_{\mathcal{L}})(\forall\varphi,\psi \in F_{\mathcal{L}})$$
$$(\Sigma \cup \{\varphi\} \models_{\mathcal{L}} \psi \Longleftrightarrow \Sigma \models_{\mathcal{L}} \varphi\nabla\psi)\ , \quad (4.9)$$

where "$\varphi\nabla\psi$" denotes an instance of scheme "$\Phi_1\nabla\Phi_2$". Such a "$\Phi_1\nabla\Phi_2$" is called a *deduction term* for $\mathcal{L}$.

Proposition 4.2.11. $\mathcal{L}_S$ *and* $S5$ *have deduction terms.*

Proof. It is not hard to show that "$\Phi_1 \to \Phi_2$" and "$\Box\Phi_1 \to \Box\Phi_2$" (where $\Box$ is the abbreviation of $\neg\Diamond\neg$) are suitable deduction terms for $\mathcal{L}_S$ and $S5$, respectively. $\Box$

The following theorem states that for any nice logic the existence of a deduction term and that of a weakly complete Hilbert-style calculus provides a finitely complete inference system.

Theorem 4.2.12. *Assume $\mathcal{L}$ is a logic having logical connectives. Assume $\mathcal{L}$ has a deduction theorem, and there is some Hilbert-style inference system which is weakly complete and strongly sound for $\mathcal{L}$. Then*

$$(\exists \text{ Hilbert-style } \vdash)(\vdash \text{ is finitely complete and strongly sound for } \mathcal{L})\,.$$

First we note the following fact (its proof is straightforward by the assumptions on ∇).

Fact 4.2.13. *The inference rule modus ponens w.r.t.* ∇ *(*MP_∇*) that is,*

$$\frac{\Phi_1, \quad \Phi_1\nabla\Phi_2}{\Phi_2} \qquad (\mathrm{MP}_\nabla)$$

is strongly sound for $\mathcal{L}$. $\Box$

Proof. **(Proof of Theorem 4.2.12)** Assume that there is some Hilbert-style inference system which is weakly complete and strongly sound for $\mathcal{L}$. Let such an inference system be fixed and let us add (MP_∇) to it. We denote this (extended) inference system by $\vdash$.

To prove finite completeness, assume $\{\varphi_0, \ldots, \varphi_n\} \models \psi$. Then, applying the deduction theorem $n+1$ times, we get:

$$\begin{aligned}
\{\varphi_0, \ldots, \varphi_{n-1}\} &\models (\varphi_n \nabla \psi) \\
\{\varphi_0, \ldots, \varphi_{n-2}\} &\models (\varphi_{n-1} \nabla (\varphi_n \nabla \psi)) \\
&\vdots \\
&\models \underbrace{(\varphi_0 \nabla (\varphi_1 \nabla \ldots (\varphi_n \nabla \psi) \ldots)}_{\gamma_0}.
\end{aligned}$$

Then $\vdash \gamma_0$ by weak completeness of $\vdash$. Then, using (MP_∇) $n+1$ times, we get:

$$\begin{aligned}
\{\varphi_0\} \vdash \{\varphi_0, \gamma_0\} &\vdash \underbrace{\varphi_1 \nabla (\varphi_2 \nabla \ldots (\varphi_n \nabla \psi) \ldots)}_{\gamma_1} \\
\{\varphi_0, \varphi_1\} \vdash \{\varphi_1, \gamma_1\} &\vdash \underbrace{\varphi_2 \nabla (\varphi_2 \nabla \ldots (\varphi_n \nabla \psi) \ldots)}_{\gamma_2} \\
\vdots \qquad\qquad & \vdots \\
\{\varphi_0, \varphi_1, \ldots, \varphi_n\} \vdash \{\varphi_n, \gamma_n\} &\vdash \psi \quad , \text{ where } \gamma_n = (\varphi_n \nabla \psi).
\end{aligned}$$

Thus we received the following $\vdash$-proof of ψ from $\{\varphi_0, \ldots, \varphi_n\}$:

$$\langle \gamma_0, \varphi_0, \gamma_1, \varphi_1, \gamma_2, \varphi_2, \ldots, \gamma_n \varphi_n, \psi \rangle \,,$$

which proves Theorem 4.2.12. $\square$

We will study strong completeness in item 4.2.28. As a preparation, first we study compactness.

Definition 4.2.14. (compactness of a logic) Let $\mathcal{L} = \langle F_\mathcal{L}, M_\mathcal{L}, mng_\mathcal{L}, \models_\mathcal{L} \rangle$ be a logic. We say that

(i) $\mathcal{L}$ is *satisfiability compact* (sat. compact for short), if

$$(\forall \Gamma \subseteq F_\mathcal{L}) \big[(\forall \Sigma \subseteq_\omega \Gamma)\, (\Sigma \text{ has a model}) \implies (\Gamma \text{ has a model}) \big] \,, \text{ and}$$

(ii) $\mathcal{L}$ is *consequence compact* (cons. compact), if for every $\Gamma \cup \{\varphi\} \subseteq F_\mathcal{L}$

$$\Gamma \models_\mathcal{L} \varphi \quad \implies \quad (\exists \Sigma \subseteq_\omega \Gamma)\ \Sigma \models_\mathcal{L} \varphi.$$

Exercise 4.2.15. Prove that even for nice logics we have

- sat. compact $\not\Longrightarrow$ cons. compact;
- sat. compact $\not\Longleftarrow$ cons. compact.

(Hint for (1): Let the logical connectives be Δ (binary), and *True*, $k_0, \ldots, k_n, \ldots$ all zero-ary. A model $\mathfrak{M}$ is a function $\mathfrak{M} : \{True, p_i, k_i : i \in \omega\} \to \{0, 1\}$. $mng_{\mathfrak{M}}(True) = 1$ for every $\mathfrak{M}$ and meaning of Δ is the standard meaning of the biconditional $\leftrightarrow$. Exclude those models from M in which $(\forall\, i > 0)\ \mathfrak{M}(k_i) = 1$ but $\mathfrak{M}(k_0) = 0$. [This logic is not strongly nice!] Observe that for $\mathfrak{M} = \{True, p_i, k_i : i \in \omega\} \times \{1\}$ we have $\mathfrak{M} \models_{\mathcal{L}} F_{\mathcal{L}}$. Hence sat. completeness trivially holds.)

(Hint for (2): Let $\mathcal{L}$ have *True* and Δ as the only logical connectives. Exclude the models $\mathfrak{M}$ with $\mathfrak{M} \models_{\mathcal{L}} F_{\mathcal{L}}$. Then sat. completeness fails (we have infinitely many propositional variables). Show that cons. completeness remains true.)

Exercise 4.2.16. Find natural conditions under which "$\Longrightarrow$" and/or "$\Longleftarrow$" of Exercise 4.2.15 above hold.

- We say that $\mathcal{L}$ *has weak false* if $(\exists\varphi \in F_{\mathcal{L}})$ such that $(\forall\mathfrak{M} \in M_{\mathcal{L}})\ \mathfrak{M} \not\models_{\mathcal{L}} \varphi$. Show that under this assumption

$$\text{cons. compact} \Longrightarrow \text{sat. compact}.$$

- We say that $\mathcal{L}$ *has negation* if

$$(\forall\varphi \in F_{\mathcal{L}})(\exists\psi \in F_{\mathcal{L}})(\forall\mathfrak{M} \in M_{\mathcal{L}})[\mathfrak{M} \models_{\mathcal{L}} \psi \Longleftrightarrow \mathfrak{M} \not\models_{\mathcal{L}} \varphi]\,.$$

  Show that under this assumption

$$\text{sat. compact} \Longrightarrow \text{cons. compact}.$$

- Try to find weaker sufficient conditions.
- Show that for nice logics

$$\mathcal{L} \text{ has weak false} \Longleftrightarrow \mathcal{L} \text{ has negation}.$$

For more information about the two notions of compactness, see [17].

* * *

Recall that in Definition 4.1.8 (and also in the logics studied so far), there was a parameter P, which was the set of atomic formulas. The choice of P influenced what the set F of formulas would be. Thus in fact, our old definition of a logic yields a family

$$\big\langle \langle F^P, M^P, mng^P, \models^P \rangle : \ P \text{ is a set} \big\rangle$$

of logics.

Definition 4.2.17. (general logic) A *general logic* is a class

$$\mathbf{L} \stackrel{\text{def}}{=} \langle \mathcal{L}^P \ : \ P \text{ is a set} \rangle,$$

where for each set P, $\quad \mathcal{L}^P = \langle F^P, M^P, mng^P, \models^P \rangle$ is a logic in the sense of Def. 3.1.3.

$\mathbf{L}$ is called a *nice [strongly nice, structural] general logic* iff conditions (1–4) below hold for $\mathbf{L}$.

(1) $\mathcal{L}^P$ is a nice [strongly nice, structural] logic (cf. Def. 4.1.8) for each set P, and P is the set of atomic formulas of logic $\mathcal{L}^P$.

(2) For any sets P and Q, $Cn(\mathcal{L}^P) = Cn(\mathcal{L}^Q) \stackrel{\text{def}}{=} Cn(\mathbf{L})$. The "special" connectives ε_j, δ_j $(j < m)$ and Δ_i $(i < n)$ are the same for any logic $\mathcal{L}^P$ (cf. Def. 4.1.3).

(3) For any sets P, Q, if there is a bijection $f : P \to Q$ then logic $\mathcal{L}^Q$ is an "isomorphic copy" of logic $\mathcal{L}^P$, i.e. there are bijections $f^F : F^P \to F^Q$ and $f^M : M^P \to M^Q$ such that

   (a) f^F is an isomorphism from $\mathfrak{F}^P$ onto $\mathfrak{F}^Q$ extending f;

   (b) for all $\varphi \in F^P$, $\mathfrak{M} \in M^P$

$$\begin{gathered} mng^P(\varphi, \mathfrak{M}) = mng^Q\big(f^F(\varphi), f^M(\mathfrak{M})\big) \\ \mathfrak{M} \models^P \varphi \quad \iff \quad f^M(\mathfrak{M}) \models^Q f^F(\varphi). \end{gathered}$$

(4) For all sets $P \subseteq Q$,

$$\left\{ mng^P_{\mathfrak{M}} : \mathfrak{M} \in M^P \right\} = \left\{ (mng^Q_{\mathfrak{M}}) \lceil F^P : \mathfrak{M} \in M^Q \right\}.$$

(Intuitively, condition (4) says that $\mathcal{L}^P$ is the "natural" restriction of $\mathcal{L}^Q$.)

Remark 4.2.18. We note that if $\mathbf{L}$ is a nice general logic then $\mathbf{L}$ has the following property. For all sets $P \subseteq Q$,

$$\{\{\varphi \in F^P : \mathfrak{M} \models^P \varphi\} : \mathfrak{M} \in M^P\} = \{\{\varphi \in F^P : \mathfrak{N} \models^Q \varphi\} : \mathfrak{N} \in M^Q\}.$$

Moreover, for all $\Gamma \cup \{\varphi\} \subseteq F^P$,

$$\Gamma \models^P \varphi \iff \Gamma \models^Q \varphi.$$

However, (5) below does not automatically hold for all strongly nice logics.

(5) For each $P \subseteq Q$ there is a "reduct-function" $r : M^Q \longrightarrow M^P$ with $Rng(r) = M^P$ such that $(\forall \mathfrak{M} \in M^Q)(\forall \varphi \in F^P)$

$$[(\mathfrak{M} \models^Q \varphi \iff r(\mathfrak{M}) \models^P \varphi) \text{ and } mng^Q_{\mathfrak{M}}(\varphi) = mng^P_{r(\mathfrak{M})}(\varphi)].$$

We will not assume and use condition (5), but it can be useful for investigations of the Beth definability properties (and related issues).

Definition 4.2.19. (algebraic counterpart of a general logic) Let $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set}\rangle$ be a nice or structural general logic. Then

$$\mathsf{Alg}_{\models}(\mathbf{L}) \stackrel{\text{def}}{=} \bigcup \left\{ \mathsf{Alg}_{\models}(\mathcal{L}^P) \ : \ P \text{ is a set} \right\},$$
$$\mathsf{Alg}_m(\mathbf{L}) \stackrel{\text{def}}{=} \bigcup \left\{ \mathsf{Alg}_m(\mathcal{L}^P) \ : \ P \text{ is a set} \right\}$$

(cf. Def. 4.1.12).

Exercise 4.2.20. Prove that

(i) $\mathsf{Alg}_m(\mathbf{L}_S) =$"class of all Boolean set algebras"

(ii) $\mathsf{Alg}_m(\mathbf{L}_{\mathrm{S5}}) =$"class of all one-dimensional cylindric set algebras"

(cf. Defs. 3.2.1 and 3.2.4). (Hint: The part "$\subseteq$" will be easy. If you would encounter cardinality difficulties in the other direction, e.g. a Boolean algebra $\mathfrak{A}$ with $|A|$ too big, then choose the set P of atomic formulas to be bigger than $|A|$.)

Theorem 4.2.21. *For structural general logics*

$$\mathsf{Alg}_{\models}(\mathbf{L}) = \mathbf{SP}\mathsf{Alg}_m(\mathbf{L}).$$

Proof. Proof First we not that, by Thm. 4.1.15, $\mathsf{Alg}_{\models}(\mathcal{L}^P) \subseteq \mathbf{SP}\mathsf{Alg}_m(\mathcal{L}^P)$ for any set P, thus $\mathsf{Alg}_{\models}(\mathbf{L}) \subseteq \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$ holds. □

To prove $\mathbf{SP}\mathsf{Alg}_m(\mathbf{L}) \subseteq \mathsf{Alg}_{\models}(\mathbf{L})$ we need Claims 4.2.22 and 4.2.23 below.

Claim 4.2.22. *For any sets P, Q, algebra $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L}^Q)$ and homomorphism $h : \mathfrak{F}^P \to \mathfrak{A}$,*

$$(\exists \mathfrak{N} \in M^P)(\forall \varphi \in F^P)\ h(\varphi) = mng^P_{\mathfrak{N}}(\varphi).$$

Proof. **(Proof of Claim 4.2.22)** Let $\mathfrak{M} \in M^Q$ be such that $\mathfrak{A} = mng^Q_{\mathfrak{M}}$. Then

$$(\forall p \in P)(\exists \varphi_p \in F^Q)\ h(p) = mng^Q_{\mathfrak{M}}(\varphi_p). \tag{$*$}$$

Because of condition (3) of Def. 4.2.17 without loss of generality we can assume that either $P \subseteq Q$ or $Q \subseteq P$ hold.

1^{st} *case* : $Q \subseteq P$

Then, by (4) of Def. 4.2.17,

$$(\exists \mathfrak{M}' \in M^P)(\forall p \in P)\ mng^P_{\mathfrak{M}'}(\varphi_p) = mng^Q_{\mathfrak{M}}(\varphi_p) = h(p). \tag{$**$}$$

Let $s : P \to F^P$ be defined by $s(p) \stackrel{\text{def}}{=} \varphi_p$, for any $p \in P$. Then, by the semantical substitution property,

$$(\exists \mathfrak{N} \in M^P)(\forall p \in P)\ mng^P_{\mathfrak{N}}(p) = mng^P_{\mathfrak{M}'}(\varphi_p). \tag{$***$}$$

Now $(*)$, $(**)$ and $(***)$ together imply that $(\forall p \in P)\ h(p) = mng^P_{\mathfrak{N}}(p)$.

2^{nd} *case* : $P \subseteq Q$

Let $s : Q \to F^Q$ be defined by

$$s(p) \stackrel{\text{def}}{=} \begin{cases} \varphi_p, & \text{if } p \in P \\ \text{any element of } F^Q, & \text{else.} \end{cases}$$

Then, by the semantical substitution property,

$$(\exists \mathfrak{M}' \in M^Q)(\forall p \in P)\ mng^Q_{\mathfrak{M}'}(p) = mng^Q_{\mathfrak{M}}(\varphi_p). \qquad (\dagger)$$

By (4) of Def. 4.2.17,

$$(\exists \mathfrak{N} \in M^P)(\forall p \in P)\ mng^P_{\mathfrak{N}}(p) = mng^Q_{\mathfrak{M}'}(p). \qquad (\dagger\dagger)$$

Then, by $(*)$, $(\dagger)$ and $(\dagger\dagger)$, $(\forall p \in P)\ mng^P_{\mathfrak{N}}(p) = h(p)$ holds again. □

Claim 4.2.23. *Let* $\mathfrak{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$ *and let* $h : \mathfrak{F}^P \twoheadrightarrow \mathfrak{A}$ *be a surjective homomorphism for some set* P*. Then*

$$(\exists K \subseteq M^P)\ (ker(h) =\sim_K \ \textit{that is, } \mathfrak{A} \cong \mathfrak{F}^P/\sim_K).$$

Proof. **(Proof of Claim 4.2.23)** Let $\mathfrak{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$. Then there are some sets I and Q_i $(i \in I)$ and $\mathfrak{A}_i \in \mathsf{Alg}_m(\mathcal{L}^{Q_i})$ such that $\mathfrak{A} \subseteq \Pi_{i\in I}\mathfrak{A}_i$. For each $i \in I$ let π_i denote the projection function into $\mathfrak{A}_i$. Then, by Claim 4.2.22, $(\forall i \in I)(\exists \mathfrak{N}_i \in M^P)(\forall p \in P)\ (\pi_i \circ h)(p) = mng^P_{\mathfrak{N}_i}(p)$. Let $K \stackrel{\text{def}}{=} \{\mathfrak{N}_i : i \in I\}$. Then it is easy to check that for any $\varphi, \psi \in F^P$,

$$h(\varphi) = h(\psi) \quad \text{iff} \quad \varphi \sim_K \psi$$

that is, $\mathfrak{A} \cong \mathfrak{F}^P/\sim_K$. □

Now, to prove $\mathbf{SP}\mathsf{Alg}_m(\mathbf{L}) \subseteq \mathsf{Alg}_{\models}(\mathbf{L})$, assume $\mathfrak{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$. Let $h : \mathfrak{F}^A \twoheadrightarrow \mathfrak{A}$ be the usual extension of the identity map of A to a homomorphism. Then, by Claim 4.2.23, $(\exists K \subseteq M^A)\ \mathfrak{A} \cong \mathfrak{F}^A/\sim_K$ that is, $\mathfrak{A} \in \mathsf{Alg}_{\models}(\mathbf{L})$, completing the proof of Theorem 4.2.21. □

Definition 4.2.24. (compactness of a general logic) A general logic $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set}\rangle$ is *satisfiability (consequence) compact* if for each set P the logic $\mathcal{L}^P$ is satisfiability (consequence) compact.

Recall that for an arbitrary class K of algebras,

$$\mathbf{Up}\mathsf{K} \stackrel{\text{def}}{=} \mathbf{I}\,\{\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F} : \mathcal{F} \text{ is an ultrafilter over the set } I, \text{ and } (\forall i \in I)\ \mathfrak{A}_i \in \mathsf{K}\}\ .$$

We say that K is $\mathbf{Up}$-*closed* if $\mathbf{Up}\mathsf{K} \subseteq \mathsf{K}$, in other words, K is $\mathbf{Up}$-closed if it is closed under taking ultraproducts (cf. [71]).

Our next theorem gives a sufficent condition for sat. compactness of a strongly nice general logic.

$$
\begin{tikzcd}
& & \omega \arrow[dll, "mng_0"'] \arrow[d, "mng_k"] \arrow[drr, "v"] & & \\
\mathfrak{A}_0 & \dots & \mathfrak{A}_k & \dots \quad \underset{\lim}{\Longrightarrow} & \mathfrak{A}
\end{tikzcd}
$$

Figure 4.1: Proof of Theorem 4.2.25

Theorem 4.2.25. *Let* **L** *be a strongly nice general logic. Then*

$$(\mathsf{Alg}_{\models}(\mathbf{L}) \textit{ is } \mathbf{Up}\textit{-closed}) \quad \Longrightarrow \quad (\mathbf{L} \textit{ is sat. compact}).$$

Proof. We let $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set}\rangle$ We give a proof for the case of $P = \omega$ that is, for the compactness of $\mathcal{L}^\omega = \langle F^\omega, M^\omega, mng^\omega, \models^\omega\rangle$. For other sets the proof is similar and is left to the reader. Assume $\Gamma \subseteq F^\omega$ and

$$(\forall \Sigma \subseteq_\omega \Gamma)\ \Sigma \text{ has a model}.$$

Then we may assume that Γ is countable, say $\Gamma = \{\varphi_0, \varphi_1, \dots, \varphi_n, \dots\}_{n\in\omega}$ and

$$(\forall k \in \omega)(\exists \mathfrak{M}_k \in M^\omega)\ \mathfrak{M}_k \models^\omega \{\varphi_0, \dots, \varphi_k\}.$$

Let such $\mathfrak{M}_k$'s be fixed. Let $\mathfrak{A}_k \stackrel{\text{def}}{=} mng^\omega_{\mathfrak{M}_k}(\mathfrak{F}^\omega) \in \mathsf{Alg}_m(\mathbf{L})$. Then $\mathfrak{A}_k \in \mathsf{Alg}_{\models}(\mathbf{L})$ also holds (by Thm.4.1.15). Let $mng_k \stackrel{\text{def}}{=} mng^\omega_{\mathfrak{M}_k}\lceil\omega$. Since ω is the set of atomic formulas of $\mathcal{L}^\omega$, the function $mng_k : \omega \longrightarrow A_k$ is a valuation of the (propositional) variables into $\mathfrak{A}_k$. Let $\mathcal{F}$ be a non-principal ultrafilter over ω, and let $\mathfrak{A} \stackrel{\text{def}}{=} \Pi_{k\in\omega}\mathfrak{A}_k/\mathcal{F}$ denote the ultraproduct of algebras $\mathfrak{A}_k$ w.r.t. $\mathcal{F}$. We define the function $v : \omega \longrightarrow A$ as follows:

$$v(i) \stackrel{\text{def}}{=} \langle mng_k(i) \ : \ k \in \omega\rangle/\mathcal{F}.$$

See Figure 4.1 below.

By assumption, $\mathfrak{M}_k \models^\omega \varphi_i$ for every $i \leqslant k$. Thus, for every $i \leqslant k \in \omega$, we have the following:

$$
\begin{aligned}
\mathfrak{M}_k &\models^\omega \varphi_i \\
&\Updownarrow \ \text{by (ii) of filter prop.} \\
\mathfrak{M}_k &\models^\omega \varepsilon_j(\varphi_i)\Delta_\ell\delta_j(\varphi_i) \quad \text{for each } j < m,\ \ell < n \\
&\Updownarrow \ \text{by (i) of the filter prop.} \\
mng^\omega_{\mathfrak{M}_k}\big(\varepsilon_j(\varphi_i)\big) &= mng^\omega_{\mathfrak{M}_k}\big(\delta_j(\varphi_i)\big) \quad \text{for each } j < m \\
&\Updownarrow \\
\mathfrak{A}_k &\models \big(\varepsilon_j(\varphi_i) = \delta_j(\varphi_i)\big)[mng_k] \quad \text{for each } j < m.
\end{aligned}
$$

We derived that $(\forall k \in \omega)(\forall i \leqslant k)\ \mathfrak{A}_k \models \bigwedge_{j<m}\big(\varepsilon_j(\varphi_i) = \delta_j(\varphi_i)\big)[mng_k]$, i.e. for every $i \in \omega$, $\{k \in \omega : \mathfrak{A}_k \models \bigwedge_{j<m}\big(\varepsilon_j(\varphi_i) = \delta_j(\varphi_i)\big)[mng_k]\} \in \mathcal{F}$. Using Łos's theorem (cf. [71]), we have that

$$(\forall i \in \omega) \quad \mathfrak{A} \models \bigwedge_{j<m} \big(\varepsilon_j(\varphi_i) = \delta_j(\varphi_i)\big)[v]\,.$$

Since by our assumption $\mathsf{Alg}_{\models}(\mathbf{L})$ is $\mathbf{Up}$-closed, $\mathfrak{A} \in \mathsf{Alg}_{\models}(\mathbf{L})$. Thus, Def. 4.2.17 (3), ($\exists$ set $P \supseteq \omega$) $(\exists K \subseteq M^P)$ $\mathfrak{A} \cong \mathfrak{F}^P/\!\sim_K$. Let *iso* denote this isomorphism. Let $\mathfrak{B} \overset{\text{def}}{=} \mathfrak{F}^P/\!\sim_K$, and let $w \overset{\text{def}}{=} iso \circ v$. Then

$$(\forall i \in \omega) \quad \mathfrak{B} \models \bigwedge_{j<m} \big(\varepsilon_j(\varphi_i) = \delta_j(\varphi_i)\big)[w]$$

that is,

$$(\forall i \in \omega)(\forall j < m) \quad \varepsilon_j(\varphi_i)[w(p_{i_0}), \dots, w(p_{i_z})]^{\mathfrak{B}} = \delta_j(\varphi_i)[w(p_{i_0}), \dots, w(p_{i_z})]^{\mathfrak{B}},$$

where all the atomic formulas (elements of ω) occurring in φ_i are among $\{p_{i_0}, \dots, p_{i_z}\}$. Let $s : P \longrightarrow F^P$ be such that for all $p \in \omega$ $\;s(p)$ is an element of the congruence class $w(p)$. For every $i \in \omega$, let $\hat{\varphi}_i \in F^P$ be $\varphi_i(p_{i_0}/s(p_{i_0}), \dots, p_{i_z}/s(p_{i_z}))$. Then for every $i \in \omega$, $j < m$ we have,

$$\begin{gathered}
\varepsilon_j(\varphi_i)[s(p_{i_0})/\!\sim_K, \dots, s(p_{i_z})/\!\sim_K]^{\mathfrak{B}} = \delta_j(\varphi_i)[s(p_{i_0})/\!\sim_K, \dots, s(p_{i_z})/\!\sim_K]^{\mathfrak{B}} \\
\Downarrow \quad (\sim_K \text{ is a congruence on } \mathfrak{F}^P) \\
\varepsilon_j\big(\varphi_i(s(p_{i_0}), \dots, s(p_{i_z}))\big)/\!\sim_K = \delta_j\big(\varphi_i(s(p_{i_0}), \dots, s(p_{i_z}))\big)/\!\sim_K \\
\Updownarrow \\
\varepsilon_j(\hat{\varphi}_i) \sim_K \delta_j(\hat{\varphi}_i).
\end{gathered} \tag{\dagger}$$

We have that $(\forall \mathfrak{M} \in K)(\forall i \in \omega)(\forall j < m)$

$$mng^P_{\mathfrak{M}}\big(\varepsilon_j(\hat{\varphi}_i)\big) = mng^P_{\mathfrak{M}}\big(\delta_j(\hat{\varphi}_i)\big).$$

Let $\mathfrak{M}$ be any model belonging to K. Then, by the semantical substitution property, $(\exists \mathfrak{N}' \in M^P)$ $(\forall i \in \omega)$ $(\forall j < m)$

$$mng^P_{\mathfrak{N}'}\big(\varepsilon_j(\varphi_i)\big) = mng^P_{\mathfrak{M}}\big(\varepsilon_j(\hat{\varphi}_i)\big) = mng^P_{\mathfrak{M}}\big(\delta_j(\hat{\varphi}_i)\big) = mng^P_{\mathfrak{N}'}\big(\delta_j(\varphi_i)\big).$$

Since for each $i \in \omega$, φ_i belongs to F^ω, by (3) of Def. 4.2.17,

$$(\exists \mathfrak{N} \in M^\omega)(\forall i \in \omega)(\forall j < m)\ mng^\omega_{\mathfrak{N}}\big(\varepsilon_j(\varphi_i)\big) = mng^\omega_{\mathfrak{N}}\big(\delta_j(\varphi_i)\big).$$

Then, by the filter property,

$$(\forall i \in \omega)\ \mathfrak{N} \models^\omega \varphi_i,$$

which proves Theorem 4.2.25. $\square$

Our next theorem states that the condition of Theorem 4.2.25 above is sufficient and also necessary for cons. compactness, and so for strong completeness (cf. Theorem 4.2.28 below).

Theorem 4.2.26 (cf. [17] Thm. 2.8). *Assume* $\mathbf{L}$ *is a strongly nice general logic. Then*

$$(\mathsf{Alg}_{\models}(\mathbf{L}) \text{ is } \mathbf{Up}\text{-closed}) \quad \iff \quad (\mathbf{L} \text{ is cons. compact}).$$

Proof. **(Proof of** $(\Longrightarrow)$**)** One can push through the proof of Thm. 4.2.25 for this case, as follows. Now we want to prove $\{\varphi_i : i \in \omega\} \not\models^\omega \psi$ from the assumption $\{\varphi_0, \dots, \varphi_k\} \not\models^\omega \psi$ for each $k \in \omega$. Change $\mathfrak{M}_k$ in the above proof such that $\mathfrak{M}_k \models^\omega \{\varphi_0, \dots, \varphi_k\}$ *and* $\mathfrak{M}_k \not\models^\omega \psi$. Drag this "$\not\models^\omega \psi$" part through the whole argument in exactly the same style as "$\models^\omega \varphi_k$" was treated in the original proof. Then in line (†) of the proof above we have

$$\begin{aligned} (\forall i \in \omega)(\forall j < m) \qquad & \varepsilon_j(\hat{\varphi}_i) \sim_K \delta_j(\hat{\varphi}_i) \quad \text{and} \\ (\exists j < m) \qquad & \varepsilon_j(\hat{\psi}) \not\sim_K \delta_j(\hat{\psi}). \end{aligned} \tag{‡}$$

Now we cannot choose an arbitrary $\mathfrak{M} \in K$ but we can infer that there exists some $\mathfrak{M} \in K$ such that $(\forall i \in \omega)$ $(\forall j < m)$

$$mng^P_{\mathfrak{M}}\big(\varepsilon_j(\hat{\varphi}_i)\big) = mng^P_{\mathfrak{M}}\big(\delta_j(\hat{\varphi}_i)\big)$$

and $(\exists j < m)$

$$mng^P_{\mathfrak{M}}\big(\varepsilon_j(\hat{\psi})\big) \neq mng^P_{\mathfrak{M}}\big(\delta_j(\hat{\psi})\big).$$

Thus, again by the semantical substitution property and by (3) of Def. 4.2.17, there is an $\mathfrak{N} \in M^\omega$ with $\mathfrak{N} \models^\omega \{\varphi_i : i \in \omega\}$ and $\mathfrak{N} \not\models^\omega \psi$, as was desired. □

Proof. **(Proof of** $(\Longleftarrow)$**)** Fix any set I and assume that for each $i \in I$, $\mathfrak{A}_i \in \mathsf{Alg}_{\models}(\mathbf{L})$.

We let $\mathfrak{P} \stackrel{\text{def}}{=} \Pi_{i \in I}\mathfrak{A}_i$. For each $X \subseteq I$ define the congruence R_X of $\mathfrak{P}$ as follows.

$$R_X \stackrel{\text{def}}{=} \{(a, b) \in P \times P : a\lceil X = b\lceil X\}.$$

Then for each $X \subseteq I$, $\mathfrak{P}/R_X \cong \Pi_{i \in X}\mathfrak{A}_i$ obviously holds. Therefore

$$\mathfrak{P}/R_X \in \mathbf{P}\mathsf{Alg}_{\models}(\mathbf{L}) \overset{\text{Thm. 4.1.15}}{\subseteq} \mathbf{PSP}\mathsf{Alg}_m(\mathbf{L}) \subseteq \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$$

(cf. e.g. [71] for $\mathbf{PSP} \subseteq \mathbf{SP}$). Let $h : \mathfrak{F}^P \twoheadrightarrow \mathfrak{P}$ be the natural extension of the identity map on P to a homomorphism and let $g_X : \mathfrak{P} \twoheadrightarrow \mathfrak{P}/R_X$ be the quotient map corresponding to R_X. Then, by Claim 4.2.23, for each $X \subseteq I$ there is some class $K_X \subseteq M^P$ such that $ker(g_X \circ h) = \sim_{K_X}$ that is,

$$(\forall \varphi, \psi \in F^P) \ \big[\big(h(\varphi), h(\psi)\big) \in R_X \quad \iff \quad \varphi \sim_{K_X} \psi\big]. \tag{$*$}$$

Moreover, an inspection of the proof of Claim 4.2.23 shows, that

$$X \subseteq Y \subseteq I \quad \Longrightarrow \quad K_X \subseteq K_Y. \qquad (**)$$

For each $X \subseteq I$, let $\Gamma_X \stackrel{\text{def}}{=} Th(K_X)$. Recall (cf. Fact 4.1.14) that $\sim_{K_X} = \sim_{Mod(\Gamma_X)}$ holds.

Claim 4.2.27. *Let $\mathcal{F}$ be any filter on I and let $\Gamma \stackrel{\text{def}}{=} \bigcup\{\Gamma_X : X \in \mathcal{F}\}$. Then for every $\varphi, \psi \in F^P$*

$$\varphi \sim_{Mod(\Gamma)} \psi \quad \Longleftrightarrow \quad (\exists X \in \mathcal{F})\ \varphi \sim_{Mod(\Gamma_X)} \psi.$$

Proof. **(Proof of Claim 4.2.27)** First, assume that $(\exists X \in \mathcal{F})\ \varphi \sim_{Mod(\Gamma_X)} \psi$. Then, since $\Gamma_X \subseteq \Gamma$, $\varphi \sim_{Mod(\Gamma)} \psi$ obviously holds.

On the other hand, assume $\varphi \sim_{Mod(\Gamma)} \psi$. Then, by the filter property, $(\forall i < n)\ \Gamma \models^P \varphi\Delta_i\psi$. Then, by the cons. compactness of $\mathcal{L}^P$, for each $i < n$ there is some $\Sigma_i \subseteq_\omega \Gamma$ with $\Sigma_i \models^P \varphi\Delta_i\psi$. Then there is some $\Sigma \subseteq_\omega \Gamma$ such that for each $i < n$ $\Sigma \models^P \varphi\Delta_i\psi$. Say, $\Sigma = \{\chi_0, \ldots, \chi_{z-1}\}$. Since $\Sigma \subseteq \Gamma$, $(\forall j < z)(\exists X_j \in \mathcal{F})\ \chi_j \in \Gamma_{X_j}$. Let $X \stackrel{\text{def}}{=} \bigcap\{X_j : j < z\}$. Then $X \in \mathcal{F}$, since $\mathcal{F}$ is a filter. Now $\Sigma \subseteq \Gamma_{X_0} \cup \cdots \cup \Gamma_{X_{z-1}} \subseteq \Gamma_X$ holds by $(**)$ above, thus for each $i < n$, $\Gamma_X \models^P \varphi\Delta_i\psi$, which implies $\varphi \sim_{Mod(\Gamma_X)} \psi$. ☐

Now we want to prove that $\mathfrak{P}/\mathcal{F} \in \mathsf{Alg}_\models(\mathbf{L})$. We show that $\mathfrak{P}/\mathcal{F} \cong \mathfrak{F}^P/{\sim_{Mod(\Gamma)}}$ (cf. Claim 4.2.27 above for the definition of Γ). That is,

$$(\forall \varphi, \psi \in F^P)\ \big[h(\varphi) \sim_\mathcal{F} h(\psi) \quad \Longleftrightarrow \quad \varphi \sim_{Mod(\Gamma)} \psi\big]$$

holds. Indeed,

$$\begin{aligned} & h(\varphi) \sim_\mathcal{F} h(\psi) \\ \Longleftrightarrow \quad & (\exists X \in \mathcal{F})\ \big(h(\varphi), h(\psi)\big) \in R_X \\ \stackrel{(*)}{\Longleftrightarrow} \quad & (\exists X \in \mathcal{F})\ \varphi \sim_{Mod(\Gamma_X)} \psi \\ \stackrel{\text{Cl. 4.2.27}}{\Longleftrightarrow} \quad & \varphi \sim_{Mod(\Gamma)} \psi, \end{aligned}$$

which completes the proof of Theorem 4.2.26. We note that we proved that $\mathsf{Alg}_\models(\mathbf{L})$ is closed under taking arbitrary *reduced* products (not only ultraproducts). ☐

Theorem 4.2.28. *Assume $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set}\rangle$ is a strongly nice general logic. Then*

$$\mathsf{Alg}_\models(\mathbf{L}) \textit{ is a finitely axiomatizable quasi-variety}$$
$$\Longleftrightarrow$$
$$(\exists \textit{ Hilbert-style } \vdash)(\forall \textit{ set } P)(\vdash \textit{ is strongly complete and strongly sound for } \mathcal{L}^P).$$

Proof. To prove Theorem 4.2.28 we need the following lemma.

Lemma 4.2.29. *For every infinite set P and for every quasi-equation q*

$$\mathsf{Alg}_m(\mathcal{L}^P) \models q \quad\Longrightarrow\quad \mathsf{Alg}_m(\mathbf{L}) \models q.$$

Proof. **(Proof of Lemma 4.2.29)** Fix an infinite set P and a quasi-equation q such that $\mathsf{Alg}_m(\mathcal{L}^P) \models q$. Let $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L}^Q)$ for some set Q. Then there is some $\mathfrak{M} \in M^Q$ with $\mathfrak{A} = mng^Q_{\mathfrak{M}}(\mathfrak{F}^Q)$. By (3) of Def. 4.2.17, without loss of generality we can assume that either $P \subseteq Q$ or $Q \subseteq P$ hold.

First assume that $Q \subseteq P$. Then, by (4) of Def. 4.2.17, $(\exists \mathfrak{N} \in M^P)\ mng^P_{\mathfrak{N}} \lceil \mathfrak{F}^Q = mng^Q_{\mathfrak{M}}$. Then $\mathfrak{A} \subseteq mng^P_{\mathfrak{N}}(\mathfrak{F}^P) \in \mathsf{Alg}_m(\mathcal{L}^P)$, thus $\mathfrak{A} \models q$, since quasi-equations are preserved under taking subalgebras.

Now let $Q \supseteq P$ and assume that $\mathfrak{A} \not\models q[k]$ for some evaluation k of the variables. Say, let $k(x_i) \overset{\text{def}}{=} mng^Q_{\mathfrak{M}}(\gamma_i))$ $(1 \leqslant i \leqslant n)$, assuming that $x_1, \ldots, x_n$ are the only variables occurring free in q. Assume that the atomic formulas occurring in the formulas $\gamma_1, \ldots, \gamma_n$ are among $p_{i_1}, \ldots, p_{i_m}$ and let s be the following substitution:

$$(\forall 1 \leqslant j \leqslant m)\ s(p_j) \overset{\text{def}}{=} p_{i_j}.$$

Then, by the semantical substitution property,

$$(\exists \mathfrak{N} \in M^Q)(\forall 1 \leqslant i \leqslant n)\ mng^Q_{\mathfrak{M}}(\gamma_i) = mng^Q_{\mathfrak{N}}\big(\gamma_i(p_{i_1}/p_1, \ldots, p_{i_m}/p_m)\big).$$

By (4) of Definition 4.2.17, $(\exists \mathfrak{N}' \in M^P) mng^Q_{\mathfrak{N}} \lceil \mathfrak{F}^P = mng^P_{\mathfrak{N}'}$. Now, let $\mathfrak{B} \overset{\text{def}}{=} mng^P_{\mathfrak{N}'}$ and let $k'(x_i) \overset{\text{def}}{=} mng^P_{\mathfrak{N}'}\big(\gamma_i(p_1, \ldots, p_m)\big)$. Then $\mathfrak{A} \not\models q[k]$ implies $\mathfrak{B} \not\models q[k']$, which contradicts to $\mathfrak{B} \in \mathsf{Alg}_m(\mathcal{L}^P)$. □

□

Proof. **(Proof of $(\Longrightarrow)$ of Theorem 4.2.28)** Assume that Ax is a finite set of quasi-equations axiomatizing $\mathsf{Alg}_{\models}(\mathbf{L})$. Since $\mathsf{Alg}_{\models}(\mathbf{L}) = \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$ (cf. Theorem 4.2.21), by Lemma 4.2.29 above, Ax also axiomatizes the quasi-variety generated by $\mathsf{Alg}_m(\mathcal{L}^P)$ for each infinite set P. Thus, by Theorem 4.2.3, for each infinite P there is a finitely complete and strongly sound Hilbert-style inference system $\vdash$ for $\mathcal{L}^P$. Moreover, checking the proof of Theorem 4.2.3 one can observe that the same inference system $\vdash$ works for all infinite sets P.

We show that for any set Q, $\vdash$ is strongly complete for $\mathcal{L}^Q$. Assume that for some $\Gamma \cup \{\varphi\} \subseteq F^Q$ $\quad \Gamma \models^Q \varphi$. Then there is some infinite set P such that $\Gamma \cup \{\varphi\} \subseteq F^P$ and $\Gamma \models^P \varphi$ (cf. Remark 4.2.18 above). Since quasi-varieties are **Up**-closed, $\mathcal{L}^P$ is cons. compact by Theorem 4.2.26. Therefore there is a finite subset Σ of Γ such that $\Sigma \models^P \varphi$. Thus, by finite completeness $\Sigma \vdash \varphi$, which implies $\Gamma \vdash \varphi$ by the definition of derivability (Def. 4.1.21). □

Proof. **(Proof of $(\Longleftarrow)$ of Theorem 4.2.28)** If $\vdash$ is strongly complete then it is also finitely complete. Thus, by Theorem 4.2.3, the quasi-variety generated by $\mathsf{Alg}_m(\mathcal{L}^P)$ is finitely axiomatizable for each set P.

On the other hand, strong completeness implies cons. compactness, as follows. Assume that for some P, $\Gamma \cup \{\varphi\} \subseteq F^P \quad \Gamma \models^P \varphi$. Then $\Gamma \vdash \varphi$, which implies by Definition 4.1.21 that there is a finite subset Σ of Γ such that $\Sigma \vdash \varphi$. Then, by strong soundness, $\Sigma \models^P \varphi$. Now, by Theorem 4.2.26, $\mathsf{Alg}_{\models}(\mathbf{L})$ is **Up**-closed. But by Theorem 4.2.21, it is also closed under **S** and **P**, thus it is a quasi-variety (cf. "quasi-variety characterization" in [71]). This and the fact that the quasi-varieties generated by $\mathsf{Alg}_m(\mathcal{L}^P)$ are finitely axiomatizable (with the same set Ax of quasi-equations, as the proof of Theorem 4.2.3 shows) imply that $\mathsf{Alg}_{\models}(\mathbf{L})$ is a finitely axiomatizable quasi-variety. $\square$

Exercise 4.2.30. Show that $\mathcal{L}_S$ and $S5$ have strongly complete and sound Hilbert-style inference systems. Give such calculi. (Hint: Use that the corresponding classes of algebras ($\mathsf{Alg}_m(\mathbf{L}_S) = \mathsf{BA}$ and $\mathsf{Alg}_m(\mathbf{L}_{S5}) = \mathsf{Cs}_1$) generate finitely axiomatizable varieties.)

In all the above we investigated only some logical properties, e.g. completeness and compactness. However, the literature contains similar theorems for a very large number of *further logical properties*. Such are e.g. Craig's interpolation property, the various definability properties (e.g. Beth's), the property of having a deduction theorem, the property of admitting Gabbay-style inference systems, to mention only a few. Some of these are discussed in chapter 6.

Chapter 5

Generalizations

First we relax the assumption on our logic having derived connectives "ε_j", "δ_j" ($j < m$) (cf. Def. 4.1.3). We will omit condition (ii) of Def. 4.1.3, obtaining the notion of a *semi-nice logic*.

Definition 5.0.31. ((strongly) semi-nice (general) logic)
Let $\mathcal{L} = \langle F, M, mng, \models\rangle$ be a logic in the sense of Def. 3.1.3. Then

(i) $\mathcal{L}$ is said to be *semi-nice* iff it is compositional, satifies (i) of the filter property, and has the syntactical substitution property.

(ii) $\mathcal{L}$ is said to be *strongly semi-nice* if $\mathcal{L}$ is semi-nice and it also has the semantical substitution property (Def. 4.1.6).

(iii) A *(strongly) semi-nice general logic* is obtained by replacing "nice logic" with "semi-nice logic" in condition (1) of Def. 4.2.17 (i.e. by doing the natural change in the definition of a (strongly) nice general logic).

Semi-nice logics, even without condition the syntactical substitution property, were investigated in [17] but investigation of the $\models$ relation was restricted to the case of one Δ_i and to formulas of the form $(\varphi\Delta_0\psi)$. Below we indicate how to extend investigation to all formulas, i.e. how to extend the theory described in the present work to semi-nice logics.

To algebraize (in a reversible way) these more general logics, we add a new unary operation symbol "c" to (the language of) our algebras. So the new version $\mathsf{Alg}_i^+(\mathcal{L})$ of $\mathsf{Alg}_i(\mathcal{L})$ ($i \in \{\models, m\}$) will consist of algebras which have an extra operation "C" not available in $\mathsf{Alg}_i(\mathcal{L})$. However, in order to make our approach work, we have to permit "c" to be a *partial operation*. This means that for certain elements of our algebras "c" may not be defined. (A classical example of a partial operation is inversion $x \to x^{-1}$ in the field of real numbers. Zero has no inverse, so $^{-1}$ is undefined at argument 0.) Universal algebra for partial algebras (i.e. algebras with partial operations) is well defined, cf. e.g. Burmeister [24], Andréka–Németi [12]. Therefore generalizing our previous theorems to the new algebras causes no real difficulty. Those readers who would prefer avoiding partial algebras are asked to consult Remark 5.0.33 below. It is shown there how to eliminate the partial operation symbol "c".

Definition 5.0.32. ($\mathsf{Alg}^+_{\models}(\mathcal{L})$, $\mathsf{Alg}^+_m(\mathcal{L})$) Let $\mathcal{L} = \langle F, M, mng, \models\rangle$ be a logic. Assume $\mathcal{L}$ is compositional.

Let $K \subseteq M$. Then we define the partial function $c_K : F \to F$ in the following way. For any $\varphi \in F$,

$$\begin{array}{llll} \text{if} & K \models \varphi & \text{then} & c_K(\varphi) \text{ is defined and } c_K(\varphi) = \varphi; \quad \text{while} \\ \text{if} & K \not\models \varphi & \text{then} & c_K(\varphi) \text{ is undefined.} \end{array}$$

Clearly, $\langle \mathfrak{F}, c_K\rangle$ is a partial algebra for every $K \subseteq M$. The equivalence relation $\sim_K$ (defined in Def. 4.1.12) is a congruence not only on $\mathfrak{F}$ but also on $\langle \mathfrak{F}, c_K\rangle$ (c_K was defined in a way to ensure this). Now,

$$\mathsf{Alg}^+_{\models}(\mathcal{L}) \stackrel{\text{def}}{=} \mathbf{I}\,\{\langle \mathfrak{F}, c_K\rangle/{\sim_K} \;:\; K \subseteq M\}\,.$$

Let us turn to defining $\mathsf{Alg}^+_m(\mathcal{L})$. First we define a new partial function c on the algebra $\mathfrak{A}(\mathfrak{M}) \stackrel{\text{def}}{=} mng_{\mathfrak{M}}(\mathfrak{F})$ as follows. For every $\varphi \in F$,

$$\begin{array}{l} c\big(mng_{\mathfrak{M}}(\varphi)\big) \stackrel{\text{def}}{=} mng_{\mathfrak{M}}(\varphi) \quad \text{if} \quad \mathfrak{M} \models \varphi; \quad \text{else} \\ c\big(mng_{\mathfrak{M}}(\varphi)\big) \quad \text{is undefined.} \end{array}$$

The new partial algebra $\mathfrak{A}^+(\mathfrak{M})$ associated to $\mathfrak{M}$ is

$$\mathfrak{A}^+(\mathfrak{M}) \stackrel{\text{def}}{=} \langle \mathfrak{A}(\mathfrak{M}), c\rangle.$$

Now

$$\mathsf{Alg}^+_m(\mathcal{L}) \stackrel{\text{def}}{=} \big\{\mathfrak{A}^+(\mathfrak{M}) \;:\; \mathfrak{M} \in M\big\}\,.$$

As we mentioned, universal algebra for partial algebras is well developed (cf. op cit). For completeness we recall those notions which are most needed. Since any partial algebra $\langle \mathfrak{A}, c\rangle$ is a model in the model theoretic sense (consider "c" as a binary relation), the model theoretic operations like direct products ($\mathbf{P}$), ultraproducts ($\mathbf{Up}$), reduced products ($\mathbf{P^r}$) need not de defined. Subalgebras are submodels closed under "c", i.e. to each element x of our subalgebra, if c is defined on x then $c(x)$ is also in our subalgebra.

Let τ be a term, $\mathfrak{A}$ a partial algebra and $k \in {}^{\omega}A$ an evaluation of the variables. Then, τ is said to be *defined* at evaluation k (in $\mathfrak{A}$) iff every subterm of τ is defined at k.[1]

Now, $\mathfrak{A} \models (\tau = \sigma)[k]$ (i.e. evaluation k satisfies the equation $\tau = \sigma$) iff both τ and σ are defined at evaluation k and their values coincide.

[1]Variables are always defined and $c(\tau)$ is defined if $\big[\tau$ is defined and c is defined at the value of $\tau\big]$.

With this we defined the satisfaction for atomic formulas (i.e. equations) of the language of partial algebras. The logical connectives are interpreted the usual way, hence satisfaction (and thus validity) is defined for all formulas of partial algebras. In particular, quasi-equations $(\tau_1 = \sigma_1 \wedge \cdots \wedge \tau_n = \sigma_n) \to \tau_0 = \sigma_0$ are defined and interpreted in the usual way. A class K is said to be a *quasi-variety* iff it is definable by a set of quasi-equations. It is a *variety* iff it is definable by equations. The usual theorems carry over, e.g.

$$\mathsf{K} \text{ is a quasi-variety} \quad \text{iff} \quad \mathsf{K} = \mathbf{SPUp}\ \mathsf{K} = \mathbf{SP^r}\ \mathsf{K}.$$

For more cf. [24], [12].

With the above in mind, it seems reasonable to repeat for semi-nice logics and $\mathsf{Alg}_i^+(\mathcal{L})$ $(i \in \{\models, m\})$ what we did in section 4.1 for nice logics and $\mathsf{Alg}_i(\mathcal{L})$ $(i \in \{\models, m\})$.

We note that Blok and Pigozzi (cf. [22], [21] and the references therein) have strong results in this direction (in perhaps a slightly different formulation). Before turning to generalizing section 4.1 to the present more general setting, we should mention an equivalent form of what we are doing.

Remark 5.0.33. If the reader would like to avoid using partial algebras, then the following equivalent more natural approach works. Instead of "c" we add a new unary predicate "$T(x)$" (T for truth). Imitating the definition of "c_K", we let $T_K \stackrel{\text{def}}{=} \{\varphi \in F : K \models \varphi\}$ for any $K \subseteq M$. Similarly, the algebraic counterpart of a model $\mathfrak{M}$ looks like $\langle \mathfrak{A}, T\rangle$, where $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$ and $T \subseteq A$ such that

$$(\forall \varphi \in F)\big(\mathfrak{M} \models \varphi \quad \Longleftrightarrow \quad mng_{\mathfrak{M}}(\varphi) \in T\big);$$

holds for T.

This approach is practically equivalent to the one using "c" instead of "T". Further, this is very-very closely related to what is called "matrix semantics" in Blok–Pigozzi [22], [21], Czelakowski [27] and in the papers quoted in these works. In these papers there are several strong results about the presently outlined approach.

Now, many of the results proved for nice logics so far, can be pushed through for semi-nice logics (with $\mathsf{Alg}_{\models}^+$, Alg_m^+ in place of $\mathsf{Alg}_{\models}$, Alg_m).

For example, the proof of

$$(\mathsf{Alg}_{\models}^+(\mathbf{L}) \text{ is } \mathbf{Up}\text{-closed}) \quad \Longrightarrow \quad (\mathbf{L} \text{ is sat. compact})$$

(cf. Thm. 4.2.25) should go through with the natural modifications for semi-nice logics.

For some of the results the formulation of the result needs a minor modification. E.g. the algebraic equation corresponding to logical formula φ is now $c(\varphi) = \varphi$ (instead of $\varepsilon_j(\varphi) = \delta_j(\varphi)$ for all $j < m$). But again we have

$$\models_{\mathcal{L}} \varphi \quad \Longleftrightarrow \quad \mathsf{Alg}_m^+(\mathcal{L}) \models c(\varphi) = \varphi$$

(cf. Thm. 4.2.1).

Exercises 5.0.34. (1) Replace the definition of the validity relation $f \models_\infty \varphi$ of logic $\mathcal{L}_\infty$ (cf. Def. 3.2.30) by

$$f \models'_\infty \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad mn_f(\varphi) > 0.9$$

and show that the resulting logic is not nice but semi-nice.

(2) Push through the proof of Thm. 4.2.25 for strongly semi-nice general logics.

(3) Check what is needed for the other direction, i.e. for Thm. 4.2.26 to go through.

(4) Repeat the proof of Fact 4.1.14 in the new (semi-nice) setting.

(5) Look at the major theorems in subsection 4.2 one by one and check if their proofs can be pushed through in the new setting. Where it does not seem to go through, check whether some change in the formulation of the result permits you to push the proof through.

(6) Try to find out whether we could use a total operation instead of our partial one "c". E.g. try to define $c_K(\varphi) = \varphi$ if $K \models \varphi$ else $(\varphi\Delta_0\varphi)$ (assume that only Δ_0 is available as "special" connective). Now our algebra is not partial! Can this approach work? Show that the validity relation $\models$ can be recovered from the new total "c", so the coding is faithful. But do the results go through? Check them! Show that Thm.4.1.15 fails. Show that Thm. 4.2.3 does not want to go through even with modifications.

* * *

If we want to drop condition the filter property altogether, then a possibility is to restrict the validity relation $\models$ to *sequents* $(\varphi \Rightarrow \psi)$ of formulas (instead of having it for all formulas). Here "$\Rightarrow$" is not a logical connective, but rather a metalevel symbol. If $\varphi, \psi \in F$ then $(\varphi \Rightarrow \psi)$ is a sequent (sequents are *not* formulas). Further,

$$\mathfrak{M} \models (\varphi \Rightarrow \psi) \quad \text{iff} \quad mng_{\mathfrak{M}}(\varphi) \subseteq mng_{\mathfrak{M}}(\psi).$$

This approach is applicable to more logics, hence more kinds of algebras show up in $\mathsf{Alg}_i(\mathcal{L})$ $(i \in \{\models, m\})$. However, similarly to the way we had to introduce "c" above to code validity in a model, now we have to introduce a pre-ordering "$\leqslant$" on our algebras to code "$\Rightarrow$". However, this is not needed if we restrict the validity relation $\models$ a little bit more, namely to pairs $\{(\varphi \Rightarrow \psi), (\psi \Rightarrow \varphi)\}$ of sequents. Then we do not need new symbols like "$\leqslant$" in our algebras. This approach is investigated e.g. in [17] to quite some extent. See also investigations on k-deductive systems in Blok–Pigozzi [21]. For a general method using sequents see Guzman–Verdu [34], Font–Verdu [31].

We could also try to drop the two substitution properties i.e. permutability of atomic formulas. This would enable us to treat traditional first-order logic more comfortably (with less preparation to do). This can be done, the only thing needed is the universal algebraic concept of a *free algebra over some defining relations*. The details are available in [17].

Chapter 6

Further equivalence results

In this section we give algebraic characterizations for further logical properties, such as *decidability of the validity problem*, various *Beth's definability properties* and *Craig's interpolation properties*.

First recall that a logic is called *decidable* iff the set of its validities is a decidable subset of the set of all formulas (cf. Definition 3.1.7).

Theorem 6.0.35. *Assume that $\mathcal{L}$ is a nice logic. Then*

(i) *$\mathcal{L}$ is decidable $\iff$ the equational theory of $\mathsf{Alg}_{\models}(\mathcal{L})$ is decidable.*

(ii) *The validities of $\mathcal{L}$ are recursively enumerable $\iff$ the equational theory of $\mathsf{Alg}_{\models}(\mathcal{L})$ is recursively enumerable.*

Proof. It is a straightforward corollary of Cor. 4.2.2 way above. $\square$

Let $\mathcal{L}$ be a nice logic. An inference rule $B_1, \dots, B_n \vdash B_0$ is called *admissible* for $\mathcal{L}$ iff it is strongly sound for $\mathcal{L}$. We note that, in the style of Theorem 6.0.35, the set of admissible rules of $\mathcal{L}$ is decidable iff the quasi–equational theory of $\mathsf{Alg}_{\mathcal{L}}$ is decidable.

Next we turn to the algebraic characterization of some definability properties. Beth definability properties of logics were introduced, e.g., in Barwise–Feferman [20] and in Sain [70]. Here we give the definitions in the framework of the present paper. The proofs of Theorems 6.0.40 and 6.0.46 below can be found in Németi [57] and in Hoogland [41].[1]

Definition 6.0.36. (implicit definition, explicit definition, local explicit definition) Let $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set} \rangle$ be a general logic. Let $P \subsetneqq Q$ be sets with $F^P \neq \emptyset$, and let $R \stackrel{\text{def}}{=} Q \setminus P$.

[1] Actually, Theorem 6.0.46 is not in [57], an early version of Theorem 6.0.46 is in [70] and the full version is in [41].

A set $\Sigma \subseteq F^Q$ of formulas *defines* R *implicitly in* Q iff

$$\big(\forall \mathfrak{M}, \mathfrak{N} \in Mod^Q(\Sigma)\big)\big(mng^Q_{\mathfrak{M}} \lceil F^P = mng^Q_{\mathfrak{N}} \lceil F^P \implies mng^Q_{\mathfrak{M}} = mng^Q_{\mathfrak{N}}\big)\,.$$

Σ *defines* R *explicitly in* Q iff

$$(\forall r \in R)(\exists \varphi_r \in F^P)(\forall \mathfrak{M} \in Mod^Q(\Sigma))\ mng^Q_{\mathfrak{M}}(r) = mng^Q_{\mathfrak{M}}(\varphi_r).$$

Σ *defines* R *local-explicitly in* Q iff

$$(\forall \mathfrak{M} \in Mod^Q(\Sigma))(\forall r \in R)(\exists \varphi_r \in F^P)\ mng^Q_{\mathfrak{M}}(r) = mng^Q_{\mathfrak{M}}(\varphi_r).$$

Definition 6.0.37. ((strong) Beth definability property) Let $\mathbf{L}$ be a general logic. $\mathbf{L}$ has the *(strong) Beth definability property* iff for all P, Q, R and Σ as in Def. 6.0.36 above

$$(\Sigma \text{ defines } R \text{ implicitly in } Q \quad \Longrightarrow \quad \Sigma \text{ defines } R \text{ explicitly in } Q).$$

Definition 6.0.38. (patchwork property of models) Let $\mathbf{L}$ be a general logic. $\mathbf{L}$ has the *patchwork property of models* iff

$$\begin{aligned} &(\forall \text{sets } P, Q)(\forall \mathfrak{M} \in M^P)(\forall \mathfrak{N} \in M^Q)\\ &\big(F^{P\cap Q} \neq \emptyset \text{ and } mng^P_{\mathfrak{M}} \lceil F^{P\cap Q} = mng^Q_{\mathfrak{N}} \lceil F^{P\cap Q}\big) \quad \Longrightarrow\\ &\quad \Longrightarrow\ (\exists \mathfrak{P} \in M^{P\cup Q})\big(mng^{P\cup Q}_{\mathfrak{P}} \lceil F^P = mng^P_{\mathfrak{M}} \text{ and } mng^{P\cup Q}_{\mathfrak{P}} \lceil F^Q = mng^Q_{\mathfrak{N}}\big)\,. \end{aligned}$$

Definition 6.0.39. (morphism, epimorphism) Let K be a class of algebras. By a *morphism of* K we understand a triple $\langle \mathfrak{A}, h, \mathfrak{B}\rangle$, where $\mathfrak{A}, \mathfrak{B} \in K$ and $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism.

A morphism $\langle \mathfrak{A}, h, \mathfrak{B}\rangle$ is an *epimorphism of* K iff for every $\mathfrak{C} \in K$ and every pair $f : \mathfrak{B} \to \mathfrak{C}$, $k : \mathfrak{B} \to \mathfrak{C}$ of homomorphisms we have $(f \circ h = k \circ h \Longrightarrow f = k)$.

Typical examples of epimorphisms are the surjections. But for certain choices of K there are epimorphisms of K which are not surjective. This is the case, e.g.,
[17] when K is the class of distributive lattices.

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Theorem 6.0.40 ([57], [17, sec. II.2], [41]). *Let* $\mathbf{L}$ *be a strongly nice general logic which has the patchwork property of models. Then*

$$\mathbf{L} \text{ has the (strong) Beth definability property} \\ \Longleftrightarrow \\ \text{all the epimorphisms of } \mathsf{Alg}_{\models}(\mathbf{L}) \text{ are surjective.}$$

The **proof** is in [57] and Hoogland [41]. A less general version of this theorem is proved in [37, Thm.5.6.10]. □

Definition 6.0.41. ((strong) local Beth definability property) Let $\mathbf{L}$ be a general logic. $\mathbf{L}$ has the *(strong) local Beth definability property* iff for all P, Q, R and Σ as in Definition 6.0.36 above

$$(\Sigma \text{ defines } R \text{ implicitly in } Q \quad \Longrightarrow \quad \Sigma \text{ defines } R \text{ local-explicitly in } Q).$$

Theorem 6.0.42. *(J. X. Madarász) Let* $\mathbf{L}$ *be a strongly nice general logic which has the patchwork property of models. Then*

$$\mathbf{L} \text{ has the (strong) local Beth definability property}$$
$$\Longleftrightarrow$$

all the epimorphisms of $\mathsf{Alg}_m(\mathbf{L})$ *are surjective.* □

Definition 6.0.43. (strong implicit definition) Let $\mathbf{L}$ be a general logic. Let P, Q, R and Σ be as in Def. 6.0.36 above. Σ *defines* R *implicitly in* Q *in the strong sense* iff

$$\Sigma \text{ defines } R \text{ implicitly in } Q \quad \text{and}$$
$$\big(\forall\mathfrak{M} \in Mod^P(Th^Q\mathsf{Mod}^Q(\Sigma) \cap F^P)\big)(\exists\mathfrak{N} \in Mod^Q(\Sigma))\ mng^Q_{\mathfrak{N}}\lceil F^P = mng^P_{\mathfrak{M}}.$$

Definition 6.0.44. (weak Beth definability property) [2] Let $\mathbf{L}$ be a general logic. $\mathbf{L}$ has the *weak Beth definability property* iff for all P, Q, R and Σ as in Def. 6.0.36 above

$$(\Sigma \text{ defines } R \text{ implicitly in } Q \text{ in a strong sense } \Longrightarrow \Sigma \text{ defines } R \text{ explicitly in } Q).$$

Definition 6.0.45. (K-extensible) Let $K_0 \subseteq K$ be two classes of algebras. Let $\langle\mathfrak{A}, h, \mathfrak{B}\rangle$ be a morphism of K. h is said to be K_0-*extensible* iff for every algebra $\mathfrak{C} \in K_0$ and every homomorphism $f : \mathfrak{A} \to \mathfrak{C}$ there exists some $\mathfrak{N} \in K_0$ and $g : \mathfrak{B} \to \mathfrak{N}$ such that $\mathfrak{C} \subseteq \mathfrak{N}$ and $g \circ h = f$.

It is important to emphasize that $\mathfrak{C}$ is a concrete subalgebra of $\mathfrak{N}$ and *not* only is embeddable into $\mathfrak{N}$.

Theorem 6.0.46 (Hoogland [41], Sain [70]). *Let* $\mathbf{L}$ *be a strongly nice general logic which has the patchwork property of models. Then*

$$\mathbf{L} \text{ has the weak Beth definability property}$$
$$\Longleftrightarrow$$

every $\mathsf{Alg}_m(\mathbf{L})$*-extensible epimorphism of* $\mathsf{Alg}_{\models}(\mathbf{L})$ *is surjective.* □

In the formulation of Theorem 6.0.46 above, it was important that $\mathsf{Alg}_m(\mathbf{L})$ is not an abstract class in the sense that it is not closed under isomorphisms, since the definition of K-extensibility strongly differentiates isomorphic algebras.

Theorem 6.0.46 and Theorem 6.0.48 below are solutions for Problem 14 in [70]. On the other hand, Theorem 6.0.49 together with Definition 6.0.47 aims for being a possible solution for Problem 15 of [70].

Definition 6.0.47. (full algebras of $\mathsf{Alg}_m(\mathbf{L})$) Let $\mathbf{L}$ be a nice general logic. The class $\mathsf{FullAlg}_m(\mathbf{L})$ of algebras is defined as follows.

$$\mathsf{FullAlg}_m(\mathbf{L}) \stackrel{\text{def}}{=} \{\mathfrak{A} \in \mathsf{Alg}_m(\mathbf{L}) : (\forall\mathfrak{B} \in \mathsf{Alg}_m(\mathbf{L}))(\mathfrak{A} \subseteq \mathfrak{B} \Longrightarrow \mathfrak{A} = \mathfrak{B})\}.$$

[2] The weak Beth definability property was introduced in Friedman [32] and has been investigated since then, cf. e.g. [20, pp. 73–76, 689–716].

We will use the notions of "reflective subcategory" and "limits of diagrams of algebras" as in Mac Lane [45]. We will not recall these.

Throughout, by a reflective subcategory we will understand a full and isomorphism closed one.

Theorem 6.0.48. *(Sain–Madarász–Németi (cf. [70, item(9) on p. 223])) Assume the conditions of Theorem 6.0.46. Assume* $\mathsf{Alg}_m(\mathbf{L}) \subseteq \mathbf{S}\mathsf{FullAlg}_m(\mathbf{L})$. *Then*

$$\mathbf{L} \textit{ has the weak Beth definability property}$$
$$\Longleftrightarrow$$
$$\mathsf{Alg}_{\models}(\mathbf{L}) \textit{ is the smallest full reflective subcategory } K \textit{ of } \mathsf{Alg}_{\models}(\mathbf{L}) \textit{ with } \mathsf{FullAlg}_m(\mathbf{L}) \subseteq K.$$ □

Theorem 6.0.49. *Assume the conditions of Theorem 6.0.48. Then*

$$\mathbf{L} \textit{ has the weak Beth definability property}$$
$$\Longleftrightarrow$$
$$\mathsf{FullAlg}_m(\mathbf{L}) \textit{ generates } \mathsf{Alg}_{\models}(\mathbf{L}) \textit{ by taking limits of diagrams of algebras.}$$[3]

On the proof

The proof is based on Theorem 6.0.48 and on the simple lemma denoted as (†) below.

(†) Assume $K_0 = \mathbf{SP}K_0$ and $K_1 \subseteq K_0$ is a set of algebras in K_0. Then the smallest full reflective subcategory K of K_0 containing K_1 exists and coincides with the smallest limit-closed class containing K_1.[4]

Next one uses the fact that

$$\begin{aligned} &(\exists\kappa \in Card)\big(\forall\mathfrak{A} \in \mathsf{FullAlg}_m(\mathbf{L})\big)(\forall H \subseteq A) \\ &\big(|H| < \kappa \Longrightarrow (\exists\mathfrak{B} \subseteq \mathfrak{A})(H \subseteq B \ \&\ \mathfrak{B} \in \mathsf{FullAlg}_m(\mathbf{L}) \ \&\ |B| < \kappa)\big)\,. \end{aligned} \tag{††}$$

(††) follows from the assumption that $\mathbf{L}$ is a structural nice general logic; cf. in particular item (4) in the definition of "general logic". □

$\mathbf{L}_n$ denotes the general logic which we get from $\mathcal{L}_n$ (cf. Def. 3.2.21). Recall from [15] that Cs_n denotes the class of cylindric set algebras of dimension n.

Remark 6.0.50. Note that $\mathsf{FullAlg}_m(\mathbf{L}_n) = \mathsf{FullCs}_n$.

Conjecture 6.0.1. *We conjecture that item (4) in the definition of general logic is essential for Theorem 6.0.49. Indeed, we conjecture that without this condition Theorem 6.0.49 might become independent of ZFC set theory.*

[3] I.e., there is no limit-closed class separating these two classes of algebras.

[4] We conjecture that (†) might become independent of set theory if the restriction that K_1 is a set is omitted. Clearly, (†) becomes false if K_0 is permitted to be an arbitrary complete and co-complete category.

Terminology

Let $\mathbf{L}$ be a nice general logic. Then $\mathrm{Mod}(\mathbf{L}) \stackrel{\text{def}}{=} \bigcup\{M^P : P \text{ is a set}\}$ is the class of all models of $\mathbf{L}$.

Let $\mathfrak{M}, \mathfrak{M}_1 \in \mathrm{Mod}(\mathbf{L})$. Then: $\mathfrak{M}_1$ is an *expansion* of $\mathfrak{M}$ iff $\mathfrak{M}$ is a *reduct* of $\mathfrak{M}_1$ iff $\exists P(\mathfrak{M} = \mathfrak{M}_1 \lceil P)$. Further:

$\mathrm{Mng}(\mathfrak{M}) \stackrel{\text{def}}{=}$ set of meanings of $\mathfrak{M}$ = universe of the meaning-algebra $mng_{\mathfrak{M}}(\mathcal{F})$ of $\mathfrak{M}$.

$\mathrm{Alg}(\mathfrak{M}) \stackrel{\text{def}}{=} mng_{\mathfrak{M}}(\mathcal{F})$ = the meaning-algebra of $\mathfrak{M}$.

Conjecture 6.0.2. *We conjecture that the characterizations of weak Beth in items 6.0.48–6.0.49 can be made more "logic oriented" (or more intuitive) the following way: Let $\mathbf{L}$ be a nice general logic and $\mathfrak{M} \in Mod(\mathbf{L})$. Then $\mathfrak{M}$ is called* full *iff ($\forall$ expansion $\mathfrak{M}_1$ of $\mathfrak{M}$) $Mng(\mathfrak{M}_1) \subseteq Mng(\mathfrak{M})$. Now we define*

$$\mathsf{FuAlg}_m(\mathbf{L}) \stackrel{\text{def}}{=} \{Alg(\mathfrak{M}) : \mathfrak{M} \textit{ is a full model of } \mathbf{L}\}$$

Now, the assumption that

$$\mathsf{Alg}_m(\mathbf{L}) \subseteq \mathbf{S}\mathsf{FullAlg}_m(\mathbf{L}) \tag{$*$}$$

in items 6.0.48–6.0.49 can be replaced with the more intuitive assumption that

$$\textit{every } \mathfrak{M} \in Mod(\mathbf{L}) \textit{ has a full expansion.} \tag{$**$}$$

*We conjecture that the characterizations of weak Beth property in items 6.0.48–6.0.49 remain true if we replace $(*)$ with $(**)$ and "*Full*" with "*Fu*" in them. In particular, $(**) \Longrightarrow \mathsf{Alg}_m(\mathbf{L}) \subseteq \mathbf{S}\mathsf{FuAlg}_m(\mathbf{L})$ holds for structural general logics with the patchwork property. For such a logics we also have $\mathsf{FuAlg}_m(\mathbf{L}) = \mathsf{FullAlg}_m(\mathbf{L})$, hence we conclude that full meaning algebras are exactly the meaning algebras of full models.*

The purpose of the present conjecture is to find a natural (or logic-oriented) characterization of $\mathsf{FullAlg}_m(\mathbf{L})$, which in turn, might be a kind of solution of Problem 15 from [70].

For the origins of our characterizations of weak Beth property (in items 6.0.46, 6.0.48, 6.0.49) see items (8), (9) below Problem 14 in [70]. (In this connection it is useful to read [70] beginning with Problem 12 to the end.)

Next we turn to characterizing Craig's interpolation property.

Definition 6.0.51. (($\models$ interpolation) property) Let $\mathcal{L} = \langle F, M, mng, \models\rangle$ be a nice logic. For each formula $\varphi \in F$ let $atf(\varphi)$ denote the set of atomic formulas occurring in φ. Then $\mathcal{L}$ has the *($\models$ interpolation) property* iff

$$(\forall\varphi,\psi \in F)\big(\{\varphi\} \models \psi \;\Rightarrow\; (\exists\chi \in F)(atf(\chi) \subseteq atf(\varphi) \cap atf(\psi) \text{ and } \{\varphi\} \models \chi \text{ and } \{\chi\} \models \psi)\big)\,.$$

Recall that for any class K of algebras $\mathbf{I}K$ denotes the class of all isomorphic copies of members of K.

Definition 6.0.52. (amalgamation property) Let K be a class of algebras. We say that K *has the amalgamation property* iff for any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathbf{I}K$ with $\mathfrak{B} \supseteq \mathfrak{A} \subseteq \mathfrak{C}$, there are $\mathfrak{N} \in K$ and injective homomorphisms (embeddings) $f : \mathfrak{B} \rightarrowtail \mathfrak{N}$ $h : \mathfrak{C} \rightarrowtail \mathfrak{N}$ such that $f{\restriction}A = h{\restriction}A$.

Theorem 6.0.53 (J. Czelakowski). *Let $\mathcal{L}$ be a strongly nice and consequence compact logic. Assume that usual conjunction "$\wedge$" is in $Cn(\mathcal{L})$. Assume that $\mathcal{L}$ has a deduction theorem. Then*

$\mathcal{L}$ has the ($\models$ interpolation) property $\iff$ $\mathsf{Alg}_{\models}(\mathcal{L})$ has the amalgamation property.

Proof. It can be found in Czelakowski [27], cf. Thm.3 therein. □

Definition 6.0.54. (($\rightarrow$ interpolation) property) Let $\mathbf{L}$ be a general logic having logical connectives, and let $\rightarrow$ be a binary connective of $\mathbf{L}$. We say that $\mathbf{L}$ *has the ($\rightarrow$ interpolation) property* if

$$\begin{aligned}(\forall \varphi, \psi \in F)\big(\models \varphi \rightarrow \psi \ \Rightarrow\ (\exists \chi \in F)(&atf(\chi) \subseteq atf(\varphi) \cap atf(\psi)\\ &\text{and } \models \varphi \rightarrow \chi \text{ and } \models \chi \rightarrow \psi)\big)\,.\end{aligned}$$

By a *partially ordered algebra* we mean a structure $(\mathfrak{A}, \leqslant)$ where $\mathfrak{A}$ is an algebra and $\leqslant$ is a partial ordering on the universe A of $\mathfrak{A}$.

Definition 6.0.55. (super–amalgamation property (cf. Maksimova [52])) A class K of partially ordered algebras has the *super–amalgamation property* if for any $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathsf{K}$ and for any embeddings
$i_1 : \mathfrak{A}_0 \longrightarrow \mathfrak{A}_1$ and $i_2 : \mathfrak{A}_0 \longrightarrow \mathfrak{A}_2$ there exist an $\mathfrak{A} \in \mathsf{K}$ and embeddings
$m_1 : \mathfrak{A}_1 \longrightarrow \mathfrak{A}$ a nd $m_2 : \mathfrak{A}_2 \longrightarrow \mathfrak{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$ and

$$(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leqslant m_k(y) \Rightarrow (\exists z \in A_0)(x \leqslant i_j(z) \quad \text{and} \quad i_k(z) \leqslant y)),$$

where $\{j, k\} = \{1, 2\}$.

Theorem 6.0.56. *(Madarász [48]) Let $\mathbf{L}$ be a strongly nice general logic such that $\mathbf{L}$ contains the classical propositional logic as a fragment (i.e. $\mathsf{Alg}_{\models}(\mathbf{L})$ has a Boolean reduct). Assume that $\mathsf{Alg}_{\models}(\mathbf{L})$ forms a variety. Let $\rightarrow$ be the usual Boolean implication. Assume that $\mathbf{L}$ has the local deduction property in the following sense:*[5]

For all $\varphi, \psi \in F$ there is a unary derived connective, say, $\Box$ of $\mathbf{L}$, such that

$$(\varphi \models \psi \implies \models \Box(\varphi) \rightarrow \psi) \quad \textit{and} \quad \varphi \models \Box(\varphi).$$

Then

[5] The usual deduction property is also sufficient for the conclusion of this theorem.

$$\mathbf{L} \textit{ has the } (\rightarrow \textit{ interpolation}) \textit{ property}$$
$$\Longleftrightarrow$$
$$\mathsf{Alg}_{\models}(\mathbf{L}) \textit{ has the super–amalgamation property,}$$

where super–amalgamation is understood via the following partial ordering: $a \leqslant b \Leftrightarrow a \rightarrow b = True$. □

Further investigations concerning the ($\rightarrow$ interpolation) property, its algebraic characterizability and related algebraic results are in [47], [48] and [46].

$\mathbf{L}_2$ and $\mathcal{L}_{\text{ARROW}}$ denotes the general logic which we get from $\mathcal{L}_2$ (cf. Definition 3.2.21) and $\mathcal{L}_{\text{ARROW}}$ (cf. Definition 3.2.19), respectively.

Definition 6.0.57. $\mathbf{L}_2^+$ is $\mathbf{L}_2$ expanded with atomic formulas of the form $R(v_1, v_0)$. Equivalently we could add the connective $\smile$ of $\mathbf{L}_{\text{ARROW}}$ to $\mathbf{L}_2$ and have the atomic formulas unchanged.

Open problems:

(1) Are all the conditions of Theorem 6.0.53 needed? Try to characterize ($\models$ interpolation) property with fewer assumptions on the logic.

(2) What is the logical counterpart of the algebraic property that "$\mathsf{Alg}_{\models}(\mathcal{L})$ has the *strong amalgamation property*" (i.e., we also require $f(B \smallsetminus A) \cap h(C \smallsetminus A) = \emptyset$ in Definition 6.0.52 above)?

(3) Does $\mathbf{L}_2^+$ have the weak Beth property? Does $\mathbf{L}_2$ have it? Does $\mathbf{L}_2^+$ without equality have weak Beth property (or even (strong) Beth property)? We note that the $\mathsf{Alg}_{\models}\big(\mathbf{L}_2^+ \text{ without equality}\big) = \mathsf{RPA}_2$ where RPA_n is the class of representable polyadic algebras of dimension n.

We note that $\mathbf{L}_2^+$ restricted to models of cardinality $\leqslant 10$ has the weak Beth property but not the Beth property. Hence this logic "$(\mathbf{L}_2^+ \lceil \leqslant 10)$" separates the Beth property from the weak Beth property, showing that Theorems 6.0.46, 6.0.48, 6.0.49 above are not superfluous.

In connection with Figure 4.1 and Exercises 3.2.18 we note the following. The proof theoretical version of $\mathbf{L}_3$ is weaker than $\mathbf{L}_3$. Further, it is proved in [58], [59] that already the proof theoretic version of $\mathbf{L}_3$ enjoys a very strong version of Gödel's incompleteness property, namely Set Theory can be built up in proof theoretic $\mathbf{L}_3$ using a finite number of axioms. In this $\mathbf{L}_3$ version of Set Theory, the usual formula *Con(Set Theory)* expressing its consistence is expressible, but (of course) is not derivable. Then the usual consequences of Gödel's second (stronger) incompleteness results do apply.

The following is open.

Problem 6.0.58. *Does the above extend to* $\mathbf{L}_3$ *without equality? Here substitutions are not allowed, i.e. the logical connectives are the Booleans and* $\exists v_0$, $\exists v_1$, $\exists v_2$. *The question is open for both the proof theoretical and the semantical versions of* $\mathbf{L}_3$ *without equality.*

The algebraic counterpart of proof theoretical $\mathbf{L}_3$ is the class CA_3 of cylindric algebras of dimension 3. The equality-free reduct of CA_3 is the class Df_3 of diagonal-free cylindric algebras, see e.g. [68]. So the following are in [58] and [59].

Theorem 6.0.59. *([58, Thm.1.6, Thm.1], [59, pp.107-108 and Thm.12 (p.65)]) Set Theory can be built up in the equational theory (or language) of* CA_3 *by a single equation* $\tau(x) = 1$ *in one variable* x*. Hence the usual consequences of Gödel's stronger (or second) incompleteness theorem apply to the equational theory of* CA_3*.*

Problem 6.0.60. *Is the same (or part of it) true for* Df_3*?*

As a partial hint for an answer it is proved in [58], namely, that Set Theory can be also built up in the reduct SCA_3 of CA_3 where SCA_3 corresponds to $\mathbf{L}_3$ without equality but with substitutions instead. The operations of SCA_3 are the Booleans, c_0, c_1, c_2 and the s^i_j's $(i, j < 3)$ where $s^i_j(x) = c_i(x \wedge d_{ij})$, where d_{ij} is the algebraic counterpart of $v_i = v_j$; and c_i is the algebraic counterpart of "$\exists v_i$".

The above problem is regarded as quite important. It goes back to something that we might call "Tarski's quest for the weakest logic with Gödel's incompleteness property". It has been extensively studied and discussed in [77]."

Chapter 7

New kinds of logics

In this chapter we collect a few logics which are of a different "flavor" than the ones listed in subsection 3.2.1. The main purpose of these examples is showing that the present Algebraic Logic framework is suitable for handling all sorts of unusual logics coming from completely different paradigms of logical or linguistic or computer science research areas.

Definition 7.0.61. (infinite valued logic $\mathcal{L}_\infty$) Let P be any set, the set of atomic formulas of $\mathcal{L}_\infty$. The logical connectives of $\mathcal{L}_\infty$ are $\wedge$, $\neg$, $\vee$ and $\to$. The set F_∞ of formulas is defined the usual way. Recall that $P \subseteq F_\infty$ is the set of atomic formulas.

$$M_\infty \stackrel{\text{def}}{=} \{f \ : \ (f : P \to [0,1])\} \ ,$$

where $[0,1]$ denotes the usual interval of real numbers.

Let $f \in M_\infty$. First we define $mn_f(\varphi)$:

$$\begin{aligned}
mn_f(p) &\stackrel{\text{def}}{=} f(p) \quad \text{for } p \in P \\
mn_f(\varphi \wedge \psi) &\stackrel{\text{def}}{=} min\,\{mn_f(\varphi), mn_f(\psi)\} \\
mn_f(\neg\varphi) &\stackrel{\text{def}}{=} 1 - mn_f(\varphi) \\
mn_f(\varphi \vee \psi) &\stackrel{\text{def}}{=} max\,\{mn_f(\varphi), mn_f(\psi)\} \\
mn_f(\varphi \to \psi) &\stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } mn_f(\varphi) \leqslant mn_f(\psi) \\ 1 - \big(mn_f(\varphi) - mn_f(\psi)\big), & \text{else.} \end{cases}
\end{aligned}$$

For any $f \in M_\infty, \ \varphi \in F_\infty$,

$$\begin{aligned}
mng_\infty(\varphi, f) &\stackrel{\text{def}}{=} \langle x \in [0,1] \ : \ x \leqslant mn_f(\varphi)\rangle; \\
f \models_\infty \varphi \quad &\stackrel{\text{def}}{\iff} \quad mng_f(\varphi) = [0,1].
\end{aligned}$$

With this the logic

$$\mathcal{L}_\infty \stackrel{\text{def}}{=} \langle F_\infty, M_\infty, mng_\infty, \models_\infty \rangle$$

is defined.

Even in intuitionistic logic we have $\models \neg(\varphi \wedge \neg\varphi)$. However, in $\mathcal{L}_\infty$ this is not so, the truth value of $(\varphi \wedge \neg\varphi)$ can be as high as 1/2. So in a sense, $\mathcal{L}_\infty$ tolerates contradictions (and by a cheap joke, we could call it "dialectical" because of this, but we will not do so). Also $(\varphi \leftrightarrow \neg\varphi)$ can be valid in some of our models. This again cannot happen even in intuitionistic logic. Further, $mn_f(\varphi) \geq 1/2$ is expressible as $(\neg\varphi \to \varphi)$, hence if we would want to have a new validity relation $\models_1$, where $f \models_1 \varphi$ iff $mn_f(\varphi) \geq 1/2$, then we can express this new $\models_1$ by $f \models_1 \varphi$ iff $mng_\infty(\neg\varphi \to \varphi) = mng_\infty(\varphi \to \varphi)$. We do not look into this new $\models_1$ any more, we only use it as an example of definability of $\models_1$ from *mng* without identifying truth with a greatest meaning or even with a single meaning.

$\mathcal{L}_\infty$ is strongly nice since we can define $\varepsilon_0(\varphi) \stackrel{\text{def}}{=} (\varphi \to \varphi)$, $\delta_0(\varphi) \stackrel{\text{def}}{=} \varphi$ and $(\varphi \Delta_0 \psi) \stackrel{\text{def}}{=} (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Remark 7.0.62. If we omit $\wedge$ from the connectives then we will need $\Delta_0 \stackrel{\text{def}}{=}$ "$\to$" and $\Delta_1 \stackrel{\text{def}}{=}$ "$\leftarrow$". If we replaced $f \models_\infty \varphi \Leftrightarrow mn_f(\varphi) = 1$ by $f \models_\infty \varphi \Leftrightarrow mn_f(\varphi) > 0.9$ then we would loose niceness. However, our logic would still remain semi-nice as described in chapter 5.

Exercises 7.0.63. (1) Try to define logics similar to $\mathcal{L}_\infty$ but perhaps with more intuitive appeal to you.

(2) Prove that the intuitionistic tautology $\big(\varphi \wedge (\varphi \to \psi)\big) \to \psi$ is not valid in $\mathcal{L}_\infty$. Change the semantics in order to make this valid.

(3) Show that $\mathcal{L}_\infty$ is strongly nice.

(4) Obtain a new logic $\mathcal{L}_\mathbb{Q}$ from $\mathcal{L}_\infty$ by executing the following modifications in the definition. Replace [0,1] with the set $\mathbb{Q}$ of rational numbers everywhere. Define $mn_f(\neg\varphi) \stackrel{\text{def}}{=} -mn_f(\varphi)$. Redefine the meaning of "$\to$" as follows:

$$mn_f(\varphi \to \psi) \stackrel{\text{def}}{=} mn_f(\psi) - mn_f(\varphi)\,,$$

and let $mng_\mathbb{Q}(\varphi, f) \stackrel{\text{def}}{=} \{x \in \mathbb{Q} : x \leqslant mn_f(\varphi)\}$. Change the definition of $f \models_\infty \varphi$ to the following:

$$f \models_\mathbb{Q} \varphi \stackrel{\text{def}}{\Longleftrightarrow} 0 \in mng_{\shortparallel}(\varphi, f)\,.$$

The rest remains unchanged.

(4.1) Investigate the logic $\mathcal{L}_\mathbb{Q}$! Compare it with $\mathcal{L}_\infty$.

(4.2) Prove that $mng_\mathbb{Q}(\neg\varphi \vee \psi, f) \neq mng_\mathbb{Q}(\varphi \to \psi, f)$, for some model f. Prove that $\models_\mathbb{Q} (p_1 \vee \neg p_1)$.

(4.3) Prove that $\not\models_{\mathbb{Q}} p_0 \to (p_1 \vee \neg p_1)$. (This property is aimed at by relevance logic, the idea being, roughly, that the formulas p_0 and $(p_1 \vee \neg p_1)$ have no common atomic formulas, hence they are not relevant to each other, so they cannot "relevantly imply" each other.)

(4.4) Prove that $\models_{\mathbb{Q}} (\varphi \to \varphi)$.

(4.5) Prove that $\models_{\mathbb{Q}} \varphi$ iff $(\forall f \in M_{\mathbb{Q}})\ mng_{\mathbb{Q}}(\varphi, f) = mng_{\mathbb{Q}}\big((\varphi \to \varphi) \vee \varphi, f\big)$.

Definition 7.0.64. (Relevance Logic $\mathcal{L}_r$) We obtain a new logic $\mathcal{L}_r$ from $\mathcal{L}_\infty$ by executing the following modifications in the definition. Replace $[0,1]$ with the set $\mathbb{Q}$ of rational numbers everywhere. Define $mn_f(\neg\varphi) \stackrel{\text{def}}{=} -mn_f(\varphi)$. Redefine the meaning of "$\to$" as follows:

$$mn_f(\varphi \to \psi) \stackrel{\text{def}}{=} \begin{cases} \max\{-mn_f(\varphi), mn_f(\psi)\}, & \text{if } mn_f(\varphi) \leqslant mn_f(\psi) \\ min\{mn_f(\varphi), -mn_f(\psi)\}, & \text{else.} \end{cases}$$

The rest is exactly as in Ex. 7.0.63 (4).

Now, Relevance Logic is

$$\mathcal{L}_r = \langle F_r, M_r, mng_r, \models_r \rangle .$$

We note that logic $\mathcal{L}_r$ is also called *R-Mingle (RM)* in the literature.

Exercises 7.0.65. (1) Compare $\mathcal{L}_r$, $\mathcal{L}_{\mathbb{Q}}$ and $\mathcal{L}_\infty$! Compare them with $\mathcal{L}_S$. What are the most striking differences?

(2) Prove that $\models_r (\varphi \to \varphi)$, and $\models_r (\varphi \vee \neg\varphi)$.

(3) Prove that $\not\models_r (p_0 \to (p_1 \vee \neg p_1))$. Compare with what we said about Relevance Logic in Ex. 7.0.63 (4)!

(4) Prove that $(\models_r \varphi) \iff [mng_{\mathbb{Q}}(\varphi, f) = mng_{\mathbb{Q}}(\varphi \to \varphi, f)$, for all $f \in M_{\mathbb{Q}}]$.

(5) Check what happens if we replace $\mathbb{Q}$ with $\mathbb{Z}$ (the set of integers) or with the interval $[-n, n]$ for some $n \in \omega$.

(6) Prove that in $\mathcal{L}_r$ we have

$$[f \models \varphi \text{ and } f \models \psi] \not\Longrightarrow mng_r(\varphi, f) = mng_r(\psi, f).$$

Compare with Def. 3.1.3!

(7) Prove that $\models_r (\varphi \to \psi) \to (\neg\varphi \vee \psi)$ but $\not\models_r (\neg\varphi \vee \psi) \to (\varphi \to \psi)$.

(8) Compare the $\{\wedge, \vee, \neg\}$-fragment of $\mathcal{L}_r$ with that of $\mathcal{L}_S$! (Prove e.g. that for φ of this fragment, $(\models_r \varphi \Rightarrow \models_S \varphi)$). ... Go on comparing!)

Next we define Partial Logics ($\mathcal{L}_P$). Partial logics are designed to express the fact that in certain situations, certain statements may be meaningless. For example, the statement "the integer 2 is of pink color" may be meaningless in certain situations. If φ is meaningless then so is $\neg\varphi$. Also, according to the Copenhagen interpretation of quantum mechanics, in certain situations certain statements are meaningless, e.g. asking for the exact location of a particle in a situation where the particle has only a probability distribution of locations is meaningless.

Definition 7.0.66. (Partial Logic, $\mathcal{L}_P$) Connectives of $\mathcal{L}_P$ are: $\wedge$, $\vee$, $\neg$, N, where the new kind of formula $N(\varphi)$ intends to express that φ is either meaningless or false ("It is not the case that φ" or perhaps "It is not the fact that φ"). (N is a very strong negation.)

- The set of formulas F_P is obtained from F_S by adding the new unary connective N.
- The class M_P of models is

  $$M_P \stackrel{\text{def}}{=} \left\{ f : f \in {}^P\{0, 1, 2\} \right\} .$$

  Here 0, 1, 2 are intended to denote the truthvalues "false", "true" and "undefined", respectively.
- If $2 \notin \{mng_P(\varphi, f), mng_P(\psi, f)\}$, then mng_P of $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $\neg\varphi$ is defined as in the case of $\mathcal{L}_S$. Else (if 2 is one of the meanings) then mng_P of $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $\neg\varphi$ is 2 (so all three are the same and they all are 2).

  $$mng_P(N\varphi, f) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } mng_P(\varphi, f) = 1 \\ 1, & \text{otherwise.} \end{cases}$$

  $$f \models_P \quad \text{iff} \quad mng_P(\varphi, f) = 1 .$$

  With this, $\mathcal{L}_P = \langle F_P, M_P, mng_P, \models_P \rangle$ is defined.

  $\mathcal{L}_P$ above is a quite important logic. It was introduced by Prior and was further investigated by I. Ruzsa (cf. e.g. [67]).

Exercises 7.0.67. (1) Prove that $\mathcal{L}_P$ is a nice logic. (Hint: Use $\varepsilon_0(\varphi) \stackrel{\text{def}}{=} N(\varphi \wedge \neg\varphi)$, $\delta_0(\varphi) \stackrel{\text{def}}{=} \varphi$. Then use $\varphi \Delta_0 \psi \stackrel{\text{def}}{=} N\neg(\varphi \leftrightarrow \psi) \wedge (u(\varphi) \leftrightarrow u(\psi))$, where $u(\varphi) \stackrel{\text{def}}{\Longleftrightarrow} N(\varphi) \wedge N(\neg\varphi)$. Here $u(\varphi)$ means that φ is undefined [or meaningless]).

(2) Try to characterize $\mathsf{Alg}_{\models}(\mathcal{L}_P)$ and $\mathsf{Alg}_m(\mathcal{L}_P)$. How many non-isomorphic algebras are there in $\mathsf{Alg}_m(\mathcal{L}_P)$?

(3) Try to invent the partial version of our more sophisticated logics, e.g. of $\mathcal{L}_{S5}$ (or the others). (Warning: This might take too much time, because there are too many logics. So try one or two and then try to develop an "intuition" that you probably could do the rest.)

Chapter 8

Algebras of relations

To be written later.

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