PROBLÈMES OUVERTS
THEORIE ANALYTIQUE DES POLYNÔMES ET ANALYSE HARMONIQUE

Groupe de travail organisé par
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Foreword. The “Groupe de travail” *théorie analytique des polynômes et analyse harmonique* has met once a month during the academic year 2006-2007. Its last meeting, on June 28, 2007, has been devoted to open problems. This document contains texts on problems that have been exposed on this occasion.

Avant-propos : le groupe de travail *théorie analytique des polynômes et analyse harmonique* s’est réuni une fois par mois à l’Institut Henri Poincaré au cours l’année académique 2006-2007. Sa dernière session a été consacrée à la présentation de problèmes ouverts. Les textes réunis ici font suite à des exposés faits ce jour-là.

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Multivariate polynomials and complexity lower bounds

Many kinds of complexities are studied in computer science, and for most of them, finding lower bounds is a major challenge. A common method to do so is known as the "polynomial method". What this method tells is that to each problem you can associate a multivariate real polynomial P satisfying some constraints, and the complexity of any solution to this problem must be at least the degree of P.

You may then expect that the constraints on P are such that its degree must be high. Usually P is defined on an exponential number of variables and is constrained only on $\{0;1\}^n$, so the analysis is quite difficult. In some cases though, the problems are symmetrical enough so that we can reduce the number of variables and get something more interesting. Let me now introduce an example, which I think is characteristic of the kind of question that arise.

We are in the ideal case when P is univariate. Actually P is constrained on $\{1;\ldots;n\}$. More precisely, we have the following result, known in our community as "Nisan-Szegedy lemma", although the ideas can be traced back at least to [1] and [4].

Lemma 1 (Noam Nisan and Mario Szegedy, [2]). *Let P be a polynomial satisfying the following properties:*

(1) *for every $i \in \{1;\ldots;n\}$, $|P(i)| \leq M$, and*
(2) *there exists a real number $x \in [0;n]$ such that $|P'(x)| \geq c$.*

Then $\deg(P) \geq \sqrt{\frac{cn}{c+2M}}$.

Actually, this is almost a direct consequence of Markov's theorem; there exists also a more elaborate lemma of the same sort, see [3].

Now, things are not always that easy. Notably, another question naturally arises, which I have little idea how to study.

Let us fix $\varepsilon \in [0;1]$. This ε plays the part of the admissible error in computation.

Let S_n be the simplex of dimension n, that is the set of all $(n+1)$-tuples $(x_1, ..., x_{n+1})$ such that for every i, $x_i \geq 0$, and $\sum_{i=1}^{n+1} x_i = 1$. Now, suppose P is a real polynomial in the x_i's, such that $1 - \varepsilon \leq P\left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right) \leq 1$, and for all nonnegative integers $d_1, \ldots, d_{n+1}$ such that $\sum_{i=1}^{n+1} d_i = n+1$ and at least one of them is 0, we have $0 \leq P\left(\frac{d_1}{n+1}, \ldots, \frac{d_{n+1}}{n+1}\right) \leq \varepsilon$. In other words, at the center of the simplex, P is near 1, and on the other "integer points" of the simplex — i.e. points in $\frac{1}{n+1}\mathbb{Z}^{n+1}$ — P is near 0.

Problem 2. *Find (asymptotic) lower bounds on the degree of P.*

What I know for sure from quantum computing theory is that such polynomials exist of degree at most $\mathcal{O}\left(n^{2/3}\right)$, though I admit I would have difficulties writing them down.

I would be rather pleased with a $\Omega\left(n^{2/3}\right)$ lower bound.

That wouldn't prove anything new, actually, but that would be a good start, as there are rather close problems that could probably be solved in the same way.

Unfortunately, concerning this particular problem, numerical simulations convinced me that such a high lower bound on the degree is out of question, the minimum degree of such polynomials seeming to be more like logarithmic in n, though I'm not sure of anything here.

Anyway, the more general question is: do tools exist that would help us lower bounding the degrees of multivariate polynomials that are constrained on the integer points of some simple symmetrical convex sets, like in this example the simplex?

References

[1] H. Ehlich and K. Zeller. Schwankung von polynomen zwischen Gitterpunkten. *Math. Z.*, **86**: 41–44, 1984.

[2] Noam Nisan and Mario Szegedy. On the degree of boolean functions as real polynomials. In *STOC '92: Proceedings of the twenty-fourth annual ACM symposium on Theory of computing*, pages 462–467, New York, NY, USA, 1992. ACM Press.

[3] Ramamohan Paturi. On the degree of polynomials that approximate symmetric boolean functions (preliminary version). In *STOC '92: Proceedings of the twenty-fourth annual ACM symposium on Theory of computing*, pages 468–474, New York, NY, USA, 1992. ACM Press.

[4] T. J. Rivlin and E. W. Cheney. A comparison of uniform approximations on an interval and a finite subset thereof, *SIAM J. on Numerical Analysis*, **3**(2): 311–320, June 1966.

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Problems on covering the plane by rotations of certain small sets

The problem we present here is a very concrete one, but it resists.

Problem 3 (Iosevich, Kolountzakis, Matolcsi)**.** *Let* $0 < \epsilon < 1/4$ *and let* $E := \{(x,y) \ : \ x \in \mathbb{Z}, \ y \in \mathbb{R}\} + B_\epsilon(0)$*. For any* $\theta \in \mathbb{R}$ *denote* R_θ *the rotation of* $\mathbb{R}^2$ *bz* θ *around the origin. Is there a finite set of angles* $\theta_1, \dots, \theta_n$ *such that*

$$\bigcup_{j=1}^{n} R_{\theta_j} E$$

covers the plane?

The authors of [1] started out from a question by Sz. Révész, motivated in turn by distance set results of Erdős, Kolountzakis etc.

We know that distances attained between points of sets X of sufficient size already cover a full halfline $[r_0, \infty)$. Note that distances between points of X are just the absolute values (lengths) of elements in the *difference set* $D := X - X := \{x - x' \ : \ x, x' \in X\}$ of the given set.

Also, we know that the difference sets of say plane sets of positive uniform asymptotic upper density have a positive uniform lower density, and thus it is not difficult to see that *finitely many translates* cover the whole plane.

Now the question was if this holds also for *rotations*: once a plane set X is given with positive uniform asymptotic upper density, we consider rotations and would like to see if a finite number of such rotated copies suffices to cover the whole plane, (apart from a small neighborhood of the origin, which may always be left uncovered). If so, we obtain a much stronger statement, easily implying the presence of all (sufficiently large) distances between points of the original set X.

It turns out that finitely many rotations do not always suffice. E.g. if we take the integer lattice $\mathbb{Z}^2$ and draw a small disk around each lattice point, say of radius $\varepsilon < 1/4$, then no finite set of rotations suffice (as the authors showed).

Nevertheless, there are also positive results, achieving the aim of strengthening the distance theorem.

A particularly simple, yet unclarified situation is the one in the problem.

One might try to prove that the answer is to the negative by showing that in any such finite set of rotations of E *any* line $y = \alpha x$ which is not parallel to any of the strips cannot be covered. This amounts to covering the real line by finitely many dilates of the function $f(x) = \sum_{n\in\mathbb{Z}} \chi_{(-\epsilon,\epsilon)}(x-n)$. This is indeed possible, for any $\epsilon > 0$, so this approach to the open problem above fails.

References

[1] Alex Iosevich, Mihail N. Kolountzakis, Mate Matolcsi, Covering the plane by rotations of a lattice arrangement of disks, arXiv:math/0611800.

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Questions on multivariate polynomials, distances and pseudo-metrics

The present set of problems are a selection of interrelated problems arising in connection of several joint works involving several people, in particular Len Bos, Dan Burns, Norman Levenberg, Sione Ma'u, Szilárd Révész and Shayne Waldron.

1. Let K be a convex body in $\mathbb{R}^N$. We know from the work of Baran that for a multivariate polynomial $P \in \mathbb{R}[x_1, \ldots, x_N]$ on $\mathbb{R}^N$, the directional derivative at a point $x \in K^o$ in direction of y can be estimated as

$$|D_y P(x)| \leq \deg P \sqrt{\|P\|^2_{C(K)} - P(x)^2} \cdot \liminf_{t\to 0+} \frac{V_K(x+it)}{t},$$

where V_K is the Siciak-Zaharjuta extremal function of K.

One can define the so-called *Baran pseudometric*,

$$\delta_B(x,y) := \lim_{t\to 0+} \frac{V_K(x+it)}{t}.$$

The limit in δ_B always exists, but it is a highly nontrivial, recently proved fact, shown in [1]. Moreover, as we have shown, this is a *continuous* function of $x \in K^o$ and $y \in \mathbb{R}^N$.

One can also define the so-called Markov pseudometric $\delta_M(x,y)$ as on p. 6 of our paper [1] – at least for K a symmetric convex body – and, in this case, $\delta_M = \delta_B$ (in particular it is continuous).

Problem 4 (Annoying problem!). *Is δ_M a pseudometric for general (nonsymmetric) K? That is, is it u.s.c. (i.e. upper semicontinuous)?*

Problem 5 (Generalized annoying problem). *More generally, are δ_M and δ_B pseudometrics for K = closure of a domain in $\mathbb{R}^N$?*

For example, it is not clear/known if the limit in the definition of the directional derivative exists in this generality – perhaps a generalized definition may be needed – but again the issue is u.s.c. for each of δ_M and δ_B.

2. One can define pseudodistances d_M and d_B associated to the pseudometrics δ_M and δ_B by "integrating".

Furthermore, for any compact set K in $\mathbb{R}^N$ one can define also the so-called *Dubiner pseudodistance* d_D. This is a Caratheodory-type pseudodistance defined for $a, b \in K$ via

$$d_D(a,b) := \sup_p \frac{|\cos^{-1}(p(a)) - \cos^{-1}(p(b))|}{\deg p},$$

where the sup is taken over all polynomials $p \in \mathbb{R}[x_1, \ldots, x_N]$ with $||p||_K \leq 1$, $\deg p \geq 1$ (i.e., not identically vanishing polynomials p mapping K into the interval $[-1, 1]$; hence the Caratheodory-type situation).

Conversely, one can form a "Dubiner pseudometric" δ_D by "differentiation". It follows from Baran's work and some "general nonsense" :) on pseudometrics/pseudodistances that for "K = closure of a domain in R^N for which the definitions make sense", $d_D \leq d_M \leq d_B$ and $\delta_D = \delta_M \leq \delta_B$.

Furthermore, if K is a centrally symmetric convex body, all these pseudometrics coincide: $\delta_D = \delta_M = \delta_B$, and for the pseudodistances, $d_D \leq d_M = d_B$, see [2].

Problem 6. *Is $d_D = d_M$ for a centrally symmetric convex body?*

3. In the definition of δ_M, for each $n \in \mathbb{N}$, one can define $\delta_M^{(n)}$ by taking the sup over polynomials of degree at most n. Then *for a centrally symmetric convex body*, $\delta_M = \delta_M^{(1)}$, i.e., one needs only degree one polynomials.

Problem 7. *Is $\delta_M = \delta_M^{(1)}$ for the simplex in $\mathbb{R}^2$?*

Now, with all this background, the "big" question(s):

Problem 8. *Is $\delta_M = \delta_B$ for the simplex?*

It was shown in [1] that if the answer is to the affirmative, then the same holds for *all convex bodies in* $\mathbb{R}^2$.

Problem 9. *Is $\delta_M = \delta_B$ for all convex bodies in $\mathbb{R}^N$?*

References

[1] D. Burns, N. Levenberg, S. Ma'u and Sz. Révész, Monge-Ampère Measures for Convex Bodies and Bernstein-Markov Type Inequalities, arXiv:0705.1095v1.
[2] L. Bos, N. Levenberg and S. Waldron, Pseudometrics, distances, and multivariate polynomial inequalities, submitted for publication.
[3] D. Burns, N. Levenberg, S. Ma'u, Pluripotential theory for convex bodies in $\mathbb{R}^N$, *Math. Zeitschrift* **250** (2005), no. 1, 91-111.
[4] D. Burns, N. Levenberg, S. Ma'u, Exterior Monge-Ampère Solutions, arXiv:math.CV/0607643.

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Integral Concentration of idempotents modulo a prime

The problem of p-concentration on the torus for idempotent polynomials has been considered first in [1], [2], [4], [6]. We use the notation $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ for the torus. Then $e(t) := e^{2\pi it}$ is the usual exponential function adjusted to interval length 1, and we denote e_h the function $e(hx)$. For obvious reasons of being convolution idempotents, the set

$$\mathcal{P} := \left\{ \sum_{h\in H} e_h \ : \ H \subset \mathbb{N}, \ \#H < \infty \right\} \tag{1}$$

is called the set of *(convolution-)idempotent exponential (or trigonometric) polynomials*, or just *idempotents* for short. The p-concentration problem comes from the following definition.

Definition 10. *Let $p > 0$. We say that there is p-concentration if there exists a constant $c > 0$ so that for any symmetric measurable set E of positive measure one can find an idempotent $f \in \mathcal{P}$ with*

$$\int_E |f|^p \geq c \int_{\mathbb{T}} |f|^p. \tag{2}$$

The main theorem of [3] can be stated as:

Theorem 11 (Anderson, Ash, Jones, Rider, Saffari). *There is p-concentration for all $p > 1$.*

The same authors conjectured that the result fails to be true for $p = 1$. We disproved this conjecture in our recent paper [5], where we also gave estimates on best possible constants.

However, some concentration does fail for $p = 1$. This is related to the same problem on the finite groups $G_q := \mathbb{Z}/q\mathbb{Z}$, which identify with the grids $\mathbb{G}_q := \{k/q; k = 0, 1, \cdots, q-1\}$ contained in the torus. Let us give some definitions.

Let q be a prime number. We still denote by $e(x) := e^{2\pi ix/q}$ the exponential function adapted to the group G_q and by e_h the function $e(hx)$. Again the set

$$\mathcal{P}_q := \left\{ \sum_{h\in H} e_h \ : \ H \subset \{0, \cdots, q-1\} \right\} \tag{3}$$

is called the set of *idempotents* on G_q.

We then adapt the definition of p-concentration to the setting of G_q.

Definition 12. *Let $p > 0$. We say that there is uniform (in q) p-concentration for G_q if there exists a constant $c > 0$ so that for each prime number q one can find an idempotent $f \in \mathcal{P}_q$ with*

$$2|f(1)|^p \geq c \sum_{k=0}^{q-1} |f(k)|^p. \tag{4}$$

Moreover, the supremum of all such constants c will be denoted as c_p, and called the level of p-concentration.

Here we can formulate a discrete analogue of the problem in [2, 3].

Problem 13. *Does q-uniform concentration fail for $p = 1$?*

In order to solve the 2-concentration problem on the torus and answer a question from [1], Déchamps-Gondim, Piquard-Lust and Queffélec [6, 7] have considered the concentration problem on G_q, proving the precise value

$$c_2 = \sup_{0 \leq x} \frac{2 \sin^2 x}{\pi x} = 0.46 \cdots . \tag{5}$$

Moreover, they obtained $c_p \geq 2(c_2/2)^{p/2}$ for all $p > 2$. The last assertion is an easy consequence of the increase of ℓ^p norms, and we have, in general,

$$c_p \geq 2(c_{p'}/2)^{p/p'} \tag{6}$$

for $p > p'$.

Let us also mention that they considered the same problem for the class of positive definite polynomials, that is

$$\mathcal{P}_q^+ := \left\{ \sum_{h \in H} a_h e_h \ : \ a_h \geq 0, h \in \{0, \cdots, q-1\} \right\}. \tag{7}$$

We say that there is uniform p-concentration on G_q for the class of positive definite polynomials if there exists some constant c such that (4) holds for some $f \in \mathcal{P}_q^+$. We denote by c_p^+ the level of p-concentration for the class of positive definite polynomials.

With these notations, it has been proved in [6] that $c_2^+ = 1/2$. Since the class of positive definite polynomials is stable by taking products, it follows that, for all even integers $2k$,

$$c_{2k} \leq c_{2k}^+ \leq 1/2.$$

It is easy to see that there is uniform p-concentration on G_q for all $p > 1$, using Dirichlet kernels. This has been used in our paper [5], where the discrete problem under consideration here has been largely studied, at least for p an even integer. Let us come back to our main point, that is the case $p = 1$. Using the recent results of B. Green and S. Konyagin [8], we are able to answer negatively in this case.

All the results summarize in the following theorem, which gives an almost complete answer to the p-concentration problem under consideration, except for the best constants, which are not known for $p \neq 2$.

Theorem 14. *For all $1 < p < \infty$ we have uniform p-concentration on G_q. We have c_2 given by (5), then $0.495 < c_4 \leq 1/2$. For all $p > 2$, we have $c_p > 0.483$. On the other hand for $p \leq 1$ we do not have (uniform in q) p-concentration.*

As far as necessary upper bounds for c_p are considered, since the polynomials f with positive coefficients have their maximum at 0, we have the trivial upper bound $c_p \leq 2/3$. Moreover, for p an even integer, we have seen that $c_p \leq 1/2$. Finally, we can use (6) to improve the bound 2/3 between two even integers.

Proof of the negative result for $p = 1$. Assume that there exists some constant c. Let $f = \sum_{h \in H} e_h$ be an idempotent for which (4) holds. We claim that H may be assumed having cardinality $\leq q/2$. Indeed, H is certainly not the whole set $\{0, \cdots, q-1\}$, since the corresponding idempotent is q times the Dirac mass at 0. Moreover, the idempotent $\widetilde{f}$, having spectrum ${}^cH$, takes the same absolute values as f outside 0, while its value at 0 is $q -$ Card H. So, if Card $H > q/2$, also $\widetilde{f}$ satisfies (4).

So, let $r :=$ Card $H \leq q/2$. We have by assumption $\sum_{k=0}^{q-1} |f(k)| \leq \frac{2}{c} r$. So the function

$$g := r^{-1} \left(f - r\delta_0 \right),$$

which is 0 at 0, has ℓ^1 norm bounded by $\frac{2}{c} + 1$, while its Fourier coefficients are equal to $1/r - 1/q$ (r of them), or $-1/q$, since the delta function has all Fourier coefficients equal to $1/q$. But, according to Theorem 1.3 of [8], we should have $q \min_k |\hat{g}(k)|$ tending to 0 when q tends to ∞. Note that the Fourier transform here is replaced by the inverse Fourier transform in [8], which is the reason for multiplication by q. This gives a contradiction, and allows to conclude for the fact that there is not uniform 1-concentration. □

The proof of the other results summarized in the theorem may be found in [5]. In particular, for $p > 2$, it is easy to adapt the proofs there to see that c_p^+ is bounded below by the given value 0.483. Moreover, one can construct a random *idempotent* polynomial that satisfies the same estimate with positive probability.

We leave as an open question the improvement of these lower bounds. Also, one may ask about the best asymptotic integral concentration ($p = 1$) for q tending to ∞.

Problem 15. *Denote by $c_1(q)$ the sup of all admissible constants in* (4) *for given fixed q. Determine $\gamma := \liminf_{q \to \infty} \log(1/c_1(q)) / \log \log q$.*

Using the full strength of the result of [8], the constant c in the proof of Theorem 14 may be chosen uniformly bounded below in q by the inverse of $(\log q)^\alpha$, with α less than 1/3 (that is, the proof by contradiction shows that $c > 1/\log^\alpha q$ is not possible, hence $\gamma \geq 1/3$). Using the improvements of Sanders in [9], α can probably be taken less than 1/2. On the other hand the Dirichlet kernel exhibits $c_1(q) \geq C/\log q$, i.e. $\gamma \leq 1$. This leaves open the question if α can be taken 1, or anything less than 1.

References

[1] B. Anderson, J. M. Ash, R. L. Jones, D. G. Rider, B. Saffari, Inégalités sur des sommes d'exponentielles, *C. R. Acad. Sci. Paris Ser. I. Math*, **296** (1983), 899.

[2] B. Anderson, J. M. Ash, R. L. Jones, D. G. Rider, B. Saffari, L^p-norm local estimates for exponential sums, *C. R. Acad. Sci. Paris Ser. I. Math*, **330** (2000), 765-769.

[3] B. Anderson, J. M. Ash, R. L. Jones, D. G. Rider, B. Saffari, *Ann. Inst. Fourier*, to appear.

[4] J. M. Ash, R. L. Jones, B. Saffari, Inégalités sur des sommes d'exponentielles, *C. R: Acad. Sci. Paris Ser. I. Math*, **296** (1983), 273-285.

[5] A. Bonami, Sz. Révész, Integral concentration of idempotent trigonometric polynomials with gaps, `arXiv:0707.3023`, 2007.
[6] M. Déchamps-Gondim, F. Lust-Piquard, H. Queffélec, Estimations locales de sommes d'exponentielles, *C. R: Acad. Sci. Paris Ser. I. Math*, **297** (1983), 153-157.
[7] M. Déchamps-Gondim, F. Lust-Piquard, H. Queffélec, Estimations locales de sommes d'exponentielles, *Publ. Math. Orsay* 84-01, No.1 (1984), 1-16.
[8] Green, B., Konyagin, S., On the Littlewood problem modulo a prime, *Canadian J. of Math.* , to appear, arXiv:math.CA/0601565v2, 2006.
[9] Sanders, T., The Littlewood-Gowers problem, *J. Anal. Math.* , to appear, arXiv:math.CA/0605522, 2006.

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Flat polynomials with weakly lacunary spectra

Let f be a 2π-periodic function of analytic type, and an absolutely convergent Taylor series. It is well-known, and easy, that f may be written as $g * h$, the convolution product of two H^2 functions. Can those functions be chosen to be moreover continuous? Using the kahane-Katznelson-de Leuw theorem as improved by Kisliakov, we can show that one of them can be continuous, and even with a uniformly convergent Fourier series. Moreover, if f has a lacunary Taylor series in the sense of Hadamard, we can show with Calado [1] that the answer is yes. If the Taylor series of f is slightly better than absolutely convergent, we can still show, using random methods, that the answer is yes. But we are unable to decide the general case.

References

[1] B.Calado, *Inegalité de Bohr pour les séries entières et les séries de Dirichlet et factorisation par convolution des fonctions continues périodiques*, Thése de l'Université Paris 11, 29 Novembre 2006.

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Convolution squares and rectangles

It is well-known (using the van der Corput lemmas or the Rudin-Shapiro sequence) that there exist trigonometric polynomials $P(t) = \sum_0^N a_n \exp(int)$ for which the sum of moduli of the coefficients a_n is bigger (up to a constant) than $N^{1/2}$ the sup norm of P. If the spectrum of P is too lacunary, for example a Sidon, or a p-Sidon set, this is no longer the case. Yet, this spectrum can be asymptotically of density zero, using an obvious modification of the construction of Rudin and Shapiro (multiply by z^{3^n} instead of z^{2^n} at the n-step. But what happens for "concrete" weakly lacunary sets, typically the set of squares? The random method allows gaps between the sup norm and the Wiener-norm of size $N^{1/2}/\log^{1/2} N$, can we get rid of the logarithmic factor, or at least of part of it? For the set of primes, using a combinatorial lemma of Spencer, we can replace log by log log, showing in passing that the set of primes is not "stationary" in the sense of G.Pisier. But for the set of squares, we are unable to decide whether it is stationary or not.

References

[1] P. Lefevre, D. Li, H. Queffélec, L. Rodriguez-Piazza Lacunary sets and function spaces with finite cotype, *J.Functional.Analysis* **188** (2002), 272-291.
[2] P. Lefevre On some properties of the class of stationary sets, *Colloq.Math.* **76** (1998), 1-18.

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Bernstein inequality for multivariate polynomials

In recent years we have seen a number of quite good estimates on derivatives of multivariate polynomials P under condition of controlling the maximum norm of P on say a convex, or a symmetric convex body of $\mathbb{R}^N$ (Sarantopoulos, Kroó, Révész, Baran). There were a few lectures on the subject here, so I do not want to repeat, also see [2] and [4].

The key problem if the otherwise converging estimates are really sharp, will be presented as part of Norm Levenberg's presentation, so that I also leave here.

The following simple-looking question is related to lower estimations, that is, sharpness questions of the Bernstein problem.

Problem 16. *Let $\Delta \subset \mathbb{R}^2$ be any triangle, with its inscribed circle denoted by $\mathcal{C}$. Determine*

$$M(\Delta) := \sup_{\substack{P\in\mathcal{P}_n \\ \|P|_{\mathcal{C}}\|=1}} \left\{ \inf \|Q(x,y)\|_{C(\Delta)} : \ Q|_{\mathcal{C}} = P|_{\mathcal{C}}, \ Q \in \mathcal{P}_n \right\}.$$

Equivalently, determine

$$M(\Delta) = \sup_{\substack{T\in\mathcal{T}_n \\ \|T\|_{\mathbb{T}}=1}} \left\{ \inf \|Q(x,y)\|_{C(\Delta)} \ : \ Q(\cos t, \sin t) = T(t) \right\}.$$

Clearly, knowing the minimax type quantity $M(\Delta)$, we can then determine, by suitable affine transformations, the same quantities for any pair of triangles and inscribed ellipses $\mathcal{E}$: we just have to consider the affine transformation which takes $\mathcal{E}$ to a circle.

The strongest possible hypothesis would be $M(\Delta) = 1 + o(1)$, when $n \to \infty$, for all triangles.

My interest in the question comes from the following. Estimating from above the directional derivative of a polynomial Q, say of norm 1 on a convex body on $K \subset \mathbb{R}^d$, at a point $x \in K^o$ and in a direction y, we consider an *inscribed ellipse* $\mathcal{E} \subset K$, and estimate the derivative by considering $P|_{\mathcal{E}}$, which then has a derivative along the curve. This is then used to estimate $|D_yP(x)|$.

The arising estimates have the form

$$|D_yP(x)| \leq \deg P \sqrt{\|P\|^2_{C(K)} - P(x)^2} \cdot G_K(x,y),$$

where $G_K(x,y)$ are constants only depending on the geometry, i.e. the (in a certain sense maximal) inscribed ellipse $\mathcal{E}$, but independent from the polynomials and even from the degrees.

That is, the estimation separates the effects of geometry and analysis, giving the degree and the so-called "Bernstein-Szegő factor" (the squareroot) as the result of the analysis, plus another geometry-related quantity.

These Bernstein-type estimates are conjecturally best possible, at least when the degrees are not restricted, but we consider all polynomials of all degrees. To show this one needs to show that the estimates are sharp: and thus that once restricting to $\mathcal{E}$ or $\mathcal{C}$, we do not loose anything. That is aimed at by the question.

Of course, it may well happen that for some polynomials P or T the extension increases the norm, while for others it does not. So if $M(\Delta)$ is large, it still may happen that in the case when the trigonometrical Bernstein inequality is sharp – when $T(t) = \cos(n(t-t_0))$ – then the extension has small norm.

I would say that the minimax problem of determining $M(\Delta)$ is certainly of some degree of difficulty and of independent interest, too.

References

[1] L. Milev and Sz. gy. Révész, Bernstein's inequality for multivariate polynomials on the standard simplex, *J. Inequal. Appl.* (2005), no. 2, 145-163.

[2] Sz. gy. Révész, A comparative analysis of Bernstein type estimates for the derivative of multivariate polynomials, *Ann. Polon. Math.* **88** (2006), no. 3, 229-245.

[3] Sz. Gy. Révész and Y. Sarantopoulos, A generalized Minkowski functional with applications in approximation theory, *J. Convex Analysis* **11** (2004), no. 2, 303-334.

[4] D. Burns, N. Levenberg, S. Ma'u and Sz. Gy. Révész, Monge-Ampère Measures for Convex Bodies and Bernstein-Markov Type Inequalities, `arXiv:0705.1095v1`.

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Rearrangement of Fourier series

If $f \in C(\mathbb{T})$ has the Fourier series,

$$f \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx = \sum_{n=0}^{\infty} A_n(x),$$

then the series may as well diverge in the natural uniform norm of $C(\mathbb{T})$. There are various ways to remedy this, the most well-known now being (Cesaro-) summation (Fejér's Theorem), or pointwise convergence (Carleson). But if we stick both to the original terms $A_k(x)$ of the series, i.e. don't allow weights, and also we stick to convergence in maximum norm, then there is another possibility to try to represent the function: to rearrange the order of the summation.

Problem 17 (Ulyanov, [5])**.** *Is it possible to find for any $f \in C(\mathbb{T})$ a permutation $\nu : \mathbb{N} \to \mathbb{N}$ so that*

$$S_n^{\nu}(f) := \sum_{k=0}^{n} A_{\nu(k)} \longrightarrow f \qquad \text{in } \|\cdot\|_\infty \ ?$$

Note that the rearrangement must depend on the function, as otherwise general theorems [2] exclude this possibility: no uniformly bounded orthonormal system is a system of uniform convergence on $[0, 1]$.

Theorem 18. *Let $f \in C(\mathbb{T})$ have the Fourier series as above. Then there exist $\nu : \mathbb{N} \to \mathbb{N}$ and $n_k \to \infty$ so that at least $S^{\nu}_{n_k}(f) \longrightarrow f$.*

See [3]. After this, the problem can be made "finite": take the trigonometric polynomials $T_k := S^{\nu}_{n_{k+1}}(f) - S^{\nu}_{n_k}(f)$, and as $\|T_k\|_\infty \to 0$, try to rearrange terms within T_k with controlling all the partial sums of the new rearrangement. So denote the symmetric group of permutations $\{0, 1, \ldots, N\} \leftrightarrow \{0, 1, \ldots, N\}$ as S_N: then the finite problem can be formulated as

Problem 19. *Determine*

$$C(N) := \min\Big\{ C : \forall T \in \mathcal{T}_N \ \exists \sigma \in S_N \ \textit{with} \max_{n=1,\ldots,N} \| \sum_{k=1}^{n} A_{\sigma(k)} \| \le C \|T\| \Big\}.$$

Obviously, $C(N)$ is nondecreasing.

Proposition 20. *The affirmative answer to Ulyanov's Problem 17 is equivalent to the assertion $C(N) \le C$, with an absolute constant C in Problem 19.*

I could obtain $C(N) \le C\sqrt{\log N}\sqrt{\sum_{n=1}^{N} a_n^2 + b_n^2}$, so in other words $C(N) \ll \sqrt{\log N}\|T\|_2$. I follows that $C(N) \le \sqrt{\log N}\ \|T\|_\infty$.

The best known result to date is due to Sergey Konjagin [1].

Theorem 21. *We have $C(N) \ll \log\log N \ \|T\|_\infty$.*

References

[1] S. V. Konyagin, Rearrangements of trigonometric series and trigonometric polynomials, *Real Anal. Exchange* **29** (2003/04), no. 1, 323–334.

[2] A. M. Olevskii, *Fourier series with respect to general orthogonal systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete **86**, Springer-Verlag, New York-Heidelberg, 1975.

[3] Sz. Gy. Révész, Rearrangement of Fourier series, *Journal of Approximation Theory*, **60** (1990), 101-121.

[4] Sz. Gy. Révész, Rearrangement of Fourier series and Fourier series whose terms have random signs, *Acta Math. Hung.* **63** (1994), 395-402.

[5] P. L. Ulyanov, Solved and unsolved problems of the theory of trigonometric and orthogonal series (in Russian), *Uspehi Mat. Nauk.*, 19 (1964), 3-69.

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Coefficient estimates and extremal problems for nonnegative and positive definite functions

Once I gave here a lecture on the Landau Extremal problem. Actually, a related lecture I gave already in 1993 in Orsay. The problem I want to recall now is the estimation of coefficients, and thus the function value at 0, of positive definite functions.

Recall that say an even $f \in C(\mathbb{T})$ is positive definite, iff $a_k \geq 0$ in its Fourier (cosine) series expansion $f = \sum_n a_n \cos nx$. Note that by positive definiteness, $\|f\|_\infty = f(0) < \infty$, and $f \in A(\mathbb{T})$, the series is absolutely convergent.

The Landau Extremal Problem led to the investigation of the following extremal function, see [3].

We put for any $a \in \mathbb{R}$

$$\mathcal{F}(a) := \Big\{ f \in C(\mathbb{T}) : f(x) = 1 + a\cos x + \sum_{k=2}^{\infty} a_k \cos kx, \quad f(x) \geq 0 \ (\forall x), \quad a_k \geq 0 \quad (k \in \mathbb{N}) \Big\}$$

and denote

$$\alpha(a) := \inf \big\{ f(0) : f \in \mathcal{F}(a) \big\}.$$

Problem 22. *What is the domain of definition $\mathcal{D} = \mathcal{D}(\alpha)$ of α, that is, what is the set of $a \in \mathbb{R}$ with $\mathcal{F}(a) \neq \emptyset$?*

It is clear that $\mathcal{D} = (A, 2)$ or $[A, 2)$: and clearly for $|a| \geq 2$ we have $0 \leq \int_{\mathbb{T}} f(x)(1 \pm \cos x)dx = 1 \pm 2a$, moreover, in case of equality $f \equiv 0$ wherever $1 \pm \cos x \neq 0$, so if it is a continuous function, then $f \equiv 0$ and $\mathcal{F}(\pm 2) = \emptyset$. There exists some estimates for the value of A: $-\sqrt{3} \leq A \leq -\sqrt{2}$, but its value is not known. To me it became of relevance to estimate $\alpha(a)$ when a is large, close to 2.

Theorem 23. *We have*

$$2 \leq \liminf_{a \to 2-} \alpha(a)\sqrt{2-a} \leq \limsup_{a \to 2-} \alpha(a)\sqrt{2-a} \leq \frac{4\pi}{3\sqrt{3}} < 2.4184\,.$$

To prove that $\alpha(a) \geq \sqrt{\frac{2+a}{2-a}}$, it was key to prove the following sharp lemma.

Lemma 24. *Let $\mu \in BM(\mathbb{T})$ be an even nonnegative measure with Fourier series*

$$d\mu(x) \sim 1 + \sum_{k=1}^{\infty} a_k \cos kx.$$

If $0 \leq a_1 \leq 2$ and $k \leq \pi / \arccos\left(\frac{a_1}{2}\right)$, then we have

$$a_k \geq 2\cos\left(k \arccos\left(\frac{a_1}{2}\right)\right).$$

Moreover, if equality occurs for any particular k in the above range, then equality holds true for all $k \in \mathbb{N}$ and $\mu = \nu_z := \pi(\delta_z + \delta_{-z})$ with $z = \arccos\left(\frac{a_1}{2}\right)$ and δ_z the Dirac mass at z.

The lemma is sharp, and the lower estimation is nice, but clearly the extremal case – that of ν_z – is neither in $C(\mathbb{T})$, nor is positive definite: coefficients oscillate. Therefore, one expects even better lower estimates on $\alpha(a)$ if positive definiteness is somehow fully utilized. That is, a sharp positive definite version of Lemma 24 is to be found.

Problem 25. *Let $f \in C(\mathbb{T})$ be an even, nonnegative and positive definite function with Fourier series*

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx.$$

Find, for arbitrary prescribed value $0 < a < 2$, and arbitrary $k \in \mathbb{N}$, the value of $\min_{f \in \mathcal{F}(a)} a_k$ *or* $\min\{a_k \ : \ f \geq 0, \int_{\mathbb{T}} f = 1, a_1(f) \geq a, a_j(f) \geq 0 \ (j \in \mathbb{N})\}$.

Note that it is not guaranteed, that the found extremal values of a_k will be attained by the very same function for all k. Nevertheless, the hope is, as in the simply nonnegative case, that the estimates "fit together" and lead to the exact asymptotic determination of $\alpha(a)$.

References

[1] Sz. Gy. Révész, Minimization of maxima of nonnegative and positive definite cosine polynomials with prescribed first coefficients, *Acta Sci. Math. (Szeged)*, **60** (1995), 589-608. MR 96k:42001.

[2] Sz. Gy. Révész, The Least Possible Value at Zero of Some Nonnegative Cosine Polynomials and Equivalent Dual Problems, *J. Fourier Anal. & Appl.*, **Kahane Special Issue** (1995), 485-508. MR 97c:42002.

[3] Sz. Gy. Révész, On some extremal problems of Landau, *Serdica Math. J.*, **33** (2007), 125-162.

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Turán-Erőd type reverse Markov inequalities on convex domains on the plane in L^p

On the complex plane polynomials of degree n admit a Markov inequality[1] $\|p'\|_K \leq c_K n^2 \|p\|_K$ on all convex, compact $K \subset \mathbb{C}$. Here the norm $\|\cdot\| := \|\cdot\|_K$ denotes sup norm over values attained on K.

In 1939 Paul Turán studied converse inequalities of the form $\|p'\|_K \geq c_K n^A \|p\|_K$. Clearly such a converse can hold only if further restrictions are imposed on the occurring polynomials p. Turán assumed that all zeroes of the polynomials belong to K. So denote the set of complex (algebraic) polynomials of degree (exactly) n as $\mathcal{P}_n$, and the subset with all the n (complex) roots in some set $K \subset \mathbb{C}$ by $\mathcal{P}_n^{(0)}(K)$. The (normalized) quantity under our study is thus the "inverse Markov factor"

$$M_n(K) := \inf_{p \in \mathcal{P}_n^{(0)}(K)} M(p) \qquad \text{with} \qquad M := M(p) := \frac{\|p'\|_K}{\|p\|_K} \,. \tag{8}$$

Theorem 26. (Turán, [5, p. 90]**).** *If $p \in \mathcal{P}_n(D)$, where D is the unit disk, then we have*

$$\|p'\|_D \geq \frac{n}{2} \|p\|_D \,. \tag{9}$$

[1]Namely, to each point z of K there exists another $w \in K$ with $|w - z| \geq \operatorname{diam}(K)/2$, and thus application of Markov's inequality on the segment $[z, w] \subset K$ yields $|p'(z)| \leq (1/\operatorname{diam}(K)) n^2 \|p\|_K$.

Theorem 26 is best possible, as is shown by $1 + z^n$.

Analogous results were developed for the one dimensional case of the unit interval $I := [-1, 1]$: here the sharp constants were found by Erőd [1].

In $\mathbb{C}$ there is a great variety of convex sets, not only the disk and the interval are available. After development initiated by Erőd [1], finally the right order of magnitude and even the geometric dependence of the involved constants, were clarified.

Theorem 27. *Let $K \subset \mathbb{C}$ be any bounded convex domain. Then for all $p \in \mathcal{P}_n^{(0)}(K)$ we have*

$$\frac{\|p'\|_K}{\|p\|_K} \geq C(K)n \qquad \text{with} \qquad C(K) = 0.0003\frac{w(K)}{d^2(K)} \ . \tag{10}$$

Theorem 28. *Let $K \subset \mathbb{C}$ be any compact, connected set with diameter d and minimal width w. Then for all $n > n_0 := n_0(K) := 2(d/16w)^2 \log(d/16w)$ there exists a polynomial $p \in \mathcal{P}_n^{(0)}(K)$ of degree exactly n satisfying*

$$\|p'\| \leq \ C'(K) \ n \ \|p\| \qquad \text{with} \qquad C'(K) := 600 \ \frac{w(K)}{d^2(K)} \ . \tag{11}$$

The key to Theorem 26 was the following observation, which had already been present implicitly in [5] and [1] and was later formulated explicitly in [2, Proposition 2.1].

Lemma 29. (Turán). *Assume that $z \in \partial K$ and that there exists a disc D_R of radius R so that $z \in \partial D_R$ and $K \subset D_R$. Then for all $p \in \mathcal{P}_n^{(0)}(K)$ we have*

$$|p'(z)| \geq \frac{n}{2R}|p(z)| \ . \tag{12}$$

So, Turán immediately remarks that with this lemma one can as well compare the L^p-norms of p and of p' on the boundary ∂D, resulting in exactly similar estimates. For the unit interval the L^p case was worked out by Zhou [6].

Problem 30. *Determine the order of oscillation of P' in L^p norm for $P \in \mathcal{P}_n^{(0)}(K)$.*

References

[1] J. Erőd, Bizonyos polinomok maximumának alsó korlátjáról, *Mat. Fiz. Lapok* **46** (1939), 58-82 (in Hungarian). See also `JFM 65.0324.02` and `Zbl 0021.39505`. A full English translation has appeared as Erőd, János, "On the lower bound of the maximum of certain polynomials" - Archives section in *East J. Approx.* **12** No. 4 (2006), 477-499.

[2] N. Levenberg and E. Poletsky, Reverse Markov inequalities, *Ann. Acad. Fenn.* **27** (2002), 173-182.

[3] Sz. Gy. Révész, Turán-type converse Markov inequalities for convex domains on the plane, *J. Approx. Theory*, **141** No. 2 (August 2006), 162–173.

[4] Sz. Gy. Révész, On a paper of Erőd and Turán-Markov inequalities for non-flat convex domains, *East J. Approx.* **12** No. 4 (2006), 451–467.

[5] P. Turán, Über die Ableitung von Polynomen, *Compositio Math.* **7** (1939), 89-95.

[6] S. P. Zhou, Some remarks on Turán's inequality III: the completion, *Anal. Math.* **21** (1995), 313-318.

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Quelques questions ouvertes concernant l'irregularite des series trigonometriques lacunaires

The first example of lacunary Fourier series whose smoothness was exactly determined were the Weierstrass functions

$$\mathcal{W}_{a,b}(x) = \sum_{n=1}^{\infty} a^n \cos(b^n x), \tag{13}$$

for $a < 1$ and $b > 1$. Their Hölder exponent is constant and takes everywhere the value $-\log a / \log b$. The simplest proof of the everywhere irregularity amounts to perform a "wavelet transform" of $\mathcal{W}_{a,b}$, i.e. consider the functions $C_m = \mathcal{W}_{a,b} * \psi(a^{-m}\cdot)$ where $\hat{\psi}$ is a C^∞ function supported in $[1/b, b]$ and satisfying $\hat{\psi}(1) = 2$; on one hand, an explicit computation shows that the modulus of this convolution product is constant and equal to a^{-m}; on the other hand, using the vanishing moments of ψ, one shows that, if f is $C^\alpha(x_0)$, then $|C_m(x_0)| \leq Cb^{-m\alpha}$; the result immediately follows.

The idea of considering the convolution product of a Fourier series with a function whose Fourier transform is sharp enough to "select" only one frequency of the series at a time is at least implicit in the various results that this idea to more and more general settings of Fourier series with "gaps" in the frequencies.

Definition 31. *Let $f : \mathbb{R}^d \to \mathbb{C}$, be a locally bounded function, $x_0 \in \mathbb{R}^d$ and $\alpha \geq 0$; $f \in C^\alpha(x_0)$ if there exist $R > 0$, $C > 0$, and a polynomial P of degree less than α such that*

$$\text{if } |x - x_0| \leq R, \quad \text{then } |f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \tag{14}$$

The Hölder exponent of f at x_0 is $h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}$.

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$. We will consider series of the form

$$f(x) = \sum_{n\in\mathbb{N}} a_n e^{i\lambda_n \cdot x}.$$

Definition 32. *The gap sequence associated with (λ_n) is the distance θ_n between λ_n and its closest neighbour. The second order gap sequence $(\omega_n)_{n\in\mathbb{N}}$ is the distance between λ_n and its second closest neighbour.*

The sequence (λ_n) is separated if $\inf_n \theta_n > 0$. We will assume in the following that the sequence (λ_n) is a finite union of separated sequences and that (a_n) is bounded. The following statement summarizes the irregularity results for lacunary Fourier series, see [5].

Proposition 33. *Let f be given by (). Let x_0 be a given point of $\mathbb{R}^d$, $\alpha > 0$ and assume that f belongs to L^∞. If $f \in C^\alpha(x_0)$, then there exists C' which depends only on α such that*

$$\forall n \in \mathbb{N} \quad \text{if } |\lambda_n| \geq \theta_n, \qquad \text{then} \qquad |a_n| \leq \frac{CC'}{\theta_n^\alpha}.$$

Let $\alpha > 1$; if $f \in C^\alpha(x_0)$, then there exists C' which depends only on α such that

$$\forall n \in \mathbb{N} \quad \textit{if } |\lambda_n| \geq \omega_n, \qquad \textit{then} \qquad |a_n| \leq \frac{CC'}{(\omega_n)^{\alpha-1}\theta_n};$$

in both cases, C is the constant that appears in (14).

This is indeed an everywhere irregularity result: For instance, if $H = \sup\{\alpha : (33) \text{ holds}\}$, Proposition 33 implies that the Hölder exponent of f is everywhere smaller than H.

In 1962, G. Freud considered sequences (λ_n) composed of integers satisfying Hadamard's lacunarity condition $\lambda_{n+1}/\lambda_n \geq C > 1$ for n large enough, see [2]: In that case, he showed that, if $f \in L^1(T)$, then the Hölder exponent of f is constant. In 1965, M. Izumi, S.-I. Izumi and J.-P. Kahane obtained the first part of Proposition 33 in the case where the λ_n are integers, f is continuous and belongs to $C^\alpha(x_0)$, see [3]. In 1977 J. Pesek considered the case of periodic multiple Fourier series, i.e. where $\lambda_n \in \mathbb{Z}^d$, see [7]: Assume that $f \in L^1(T^d)$, $f \in C^\alpha(x_0)$ and let $\omega \in (0,1)$; he showed that, if $\forall n, \quad \theta_n \geq C|\lambda_n|^\omega$, then $a_n = O\left(|\lambda_n|^{-\omega\alpha}\right)$ (the assumptions of Proposition 33 actually do not require the lacunarity condition to hold uniformly for all the λ_n). In 2006, J. Dixmier, J.-P. Kahane and J.-L. Nicolas proved the first statement of Proposition 33 when the λ_n do not necessarily belong to $\mathbb{Z}^d$, but form a separated sequence and satisfy the following multi-dimensional Hadamard-type condition, see [1]: $\exists C > 0$ such that $\theta_n \geq C|\lambda_n|$. J. Pesek showed the optimality of the first part of Proposition 33 when $\alpha < 1$. Actually, he expected his optimality result to be true without this limitation. Surprisingly, when $\theta_n = o(\omega_n)$ and $\alpha > 1$, it is not the case, as shown by the second part of the proposition, see [5].

One can easily prove the optimality of Proposition 33 when $0 \leq \alpha \leq 2$. It is clearly not optimal for larger values of α, so that a first open problem is to obtain a formula in the spirit of (33) that would be optimal for all values of α.

Another open problem is to understand the optimality of these criteria when not only the sequence λ_n is given, but also the order of magnitude of the a_n. Let us give a simple example: Suppose that a_n is a sequence of coefficients satisfying

$$(15) \qquad \exists C, C' > 0 \quad \text{such that} \qquad \forall n \geq 1, \quad \frac{C}{n^2} \leq |a_n| \leq \frac{C'}{n^2},$$

and let

$$(16) \qquad g(t) = \sum_{n=1}^{\infty} a_n \sin(n^2 t).$$

Propositions 33 yields that the Hölder exponent of f is everywhere smaller than 2. In the case where $a_n = 1/n^2$, the largest Hölder exponent of the function

$$(17) \qquad \sum_{n=1}^{\infty} \frac{\sin(n^2 t)}{n^2},$$

is 3/2, see [4]; however, it is not known if this is best possible; i.e. does there exist a sequence (a_n) satisfying (15) and such that the Hölder exponent of g at some points is larger than 3/2? Can it be as large as 2? (As should be expected if Proposition 33 is optimal in that case.)

We now consider possible consequences to the multifractal analysis of lacunary Fourier series. Loosely speaking, the general idea behind all results concerning lacunary Fourier series is that, under specific lacunarity conditions, local properties are the same everywhere. A result of this type which concerns local Hölder regularity was obtained by P. B. Kennedy in 1956, see [6]: He proved that, if $\lambda_{n+1} - \lambda_n \to \infty$, then the *uniform* Hölder regularity of f on a small interval is the same as everywhere. Of course, we know that this result cannot be sharpened into a result concerning pointwise regularity (except in the extreme case of Hadamard lacunarity) since, for instance, the Hölder exponent of (17) takes all values in $[1/2, 3/4] \cup \{3/2\}$, see [4]. However, another statement of this flavour is compatible with all known results concerning particular lacunary Fourier series. Let us recall the notion of *Hölder-homogeneous function.*

Definition 34. *Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a locally bounded function. Let $\mathcal{O}$ be an open nonempty set. The spectrum of singularities of f on $\mathcal{O}$ is the function*

$$d_f^{\mathcal{O}}(H) = dim\{x \in \mathcal{O} : h_f(x) = H\},$$

where dim denotes the Hausdorff dimension.

A function f is Hölder-homogeneous if the functions $d_f^{\mathcal{O}}(H)$ do not depend on the open nonempty set $\mathcal{O}$.

Though it is not stated explicitely in [4], it is shown there that Riemann's non-differentiable function (17) actually is Hölder-homogeneous. An open problem is to determine if this is a general property of lacunary series, under some appropriate lacunarity condition.

References

[1] J. Dixmier, J.-P. Kahane and J.-L. Nicolas *Un exemple de non-dérivabilité en géométrie du triangle,* preprint (2006).

[2] G. Freud, *On Fourier series with Hadamard gaps*, Studia Scient. Math. Hung. Vol. 1 pp. 87–96 (1966).

[3] M. Izumi, S.-I. Izumi and J.-P. Kahane, *Théorèmes élémentaires sur les séries de Fourier lacunaires,* J. Anal. Math. Vol. 14 pp. 235–246 (1965).

[4] S. Jaffard, *The spectrum of singularities of Riemann's function.* Rev. Mat. Iber. Vol. 12 N. 2 pp. 441–460 (1996).

[5] S.Jaffard, *Gabor-wavelet transform and Hölder irregularity,* to appear in Proc. HAAT Conference (2007).

[6] P. B. Kennedy, *Fourier series with gaps,* Quart. J. Math. Vol. 7 pp. 224–230 (1956).

[7] J. D. Pesek *One point regularity properties of multiple Fourier series with gaps*, Illin. J. Math. *Vol. 21* , pp. 871–882 (1977).

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Quelques questions ouvertes sur les racines de l'unité et les ensembles de Sidon

Soit $T = \{z \in \mathbb{C}, |z| = 1\}$ le cercle unité, $\mathbb{Q}$ l'ensemble des nombres rationnels, $e^{2\pi i\mathbb{Q}}$ l'ensemble de toutes les racines de l'unité.

Pour tout entier $n \geq 2$, notons T_n l'ensemble des racines n-ièmes de l'unité. Il est bien connu que le cardinal maximal d'un ensemble indépendant sur $\mathbb{Q}$ de T_n est donné par la fonction d'Euler $\phi(n)$ et que l'ensemble

$$A_0 = \{e^{2\pi ik/n},\ 0 \leq k \ < \ \phi(n)\}$$

est indépendant (dans l'espace vectoriel $\mathbb{C}$ sur le corps $\mathbb{Q}$). Rappelons que si

$$n = \prod_{1\leq i\leq r} p_i^{\alpha_i},\ p_i \text{ premier},\ \alpha_i \in \mathbb{N}^*, 1 \leq i \leq r,$$

alors

$$\phi(n) = \prod_{1\leq i\leq r} p_i^{\alpha_i - 1}(p_i - 1).$$

Il est facile de voir que T_n est une réunion de M_n ensembles indépendants, où $M_n = \left[\frac{n}{\phi(n)}\right] + 1$. Par contre, l'ensemble $e^{2\pi i\mathbb{Q}}$ n'est pas une réunion finie d'ensembles indépendants, puisque la suite $(M_n)_{n\geq 1}$ n'est pas bornée (car $\displaystyle\sum_{p \text{ premier}} 1/p = \infty$).

En analyse harmonique, la notion de quasi-indépendance, plus large que celle d'indépendance, est fort utile pour l'étude des ensembles lacunaires.

Un sous-ensemble E d'un groupe additif Γ *est* **quasi-indépendant** *si pour toute partie finie A de E,*

$$\sum_{\gamma\in A} \epsilon_\gamma \gamma = 0,\ (\epsilon_\gamma)_{\gamma\in A} \in \{-1, 0, 1\}^A \quad \Rightarrow \quad \epsilon_\gamma = 0,\ \text{pour tout } \gamma \in A.$$

Lorsque Γ est le groupe additif $\mathbb{C}$, notons $\psi(n)$ le cardinal maximal d'un sous-ensemble quasi-indépendant de T_n, pour $n \geq 2$. C. C. Graham et L. T. Ramsey, dans un article à paraitre [GR], montrent que $\psi(n)$ a des propriétés semblables à celles de $\phi(n)$. Mais l'étude de $\psi(n)$ est déroutante, car on ne dispose pas des mêmes outils que pour celle de $\phi(n)$, et bien des questions restent à élucider.

Question 35. *La suite* $(\frac{\psi(n)}{\phi(n)})_{n\geq 1}$ *est-elle bornée ?*

Question 36. *La suite* $(\frac{n}{\psi(n)})_{n\geq 1}$ *est-elle bornée ?*

Question 37. *Calculer* $\psi(n)$*, pour* $n \geq 2$ *(en dehors de quelques cas connus).*

Pour percevoir l'intérêt de ces questions pour l'analyse harmonique, il faut rappeler la principale question encore ouverte sur la structure des ensembles de Sidon. Ces ensembles lacunaires, dont la théorie s'est beaucoup developpée dans les années 1970 et 1980, peuvent être définis par de nombreuses conditions équivalentes d'analyse fonctionnelle. Mais depuis les travaux de Gilles Pisier [P], suivis d'une version combinatoire de Jean Bourgain [B] (voir aussi [LQ], pour l'ensemble de ces travaux et d'autres résultats sur les ensembles de Sidon), il est possible d'en donner une définition en termes de "richesse" des parties quasi-indépendantes contenues dans l'ensemble :

Le sous-ensemble Λ *du groupe discret* Γ *est un ensemble de Sidon si et seulement s'il existe un entier positif K tel que toute partie finie A de* Λ *contient un sous-ensemble quasi-indépendant B tel que* $|B| \geq |A|/K$.

Le premier exemple connu d'ensemble de Sidon doit être la suite d'Hadamard d'entiers $\{3^n, n \geq 1\}$ (ensemble quasi-indépendant). On a ensuite montré que toute réunion finie d'ensembles quasi-indépendants d'un groupe discret Γ est un ensemble de Sidon. La réciproque est une question toujours ouverte (si on exclut les groupes dont tous les éléments sont d'ordres inférieures à un entier donné). Une des difficultés pour avancer dans la connaissance des ensembles de Sidon est le manque d'exemples nouveaux significatifs. Dans [GR] les auteurs, en jouant sur la structure multiplicative du cercle T et la structure additive de $\mathbb{C}$, donnent un exemple intéressant :

Théorème A. [GR] *Soit P un ensemble de nombres premiers et W le sous-ensemble multiplicatif de $\mathbb{N}$ engendré par P. Soit*

$$E = \{e^{2\pi ia/m}, a \in \mathbb{Z}, m \in W\}$$

Alors E est une réunion finie d'ensembles indépendants si et seulement si on a $\sum_{p\in P} 1/p < \infty$ et, dans ce cas, E est un ensemble de Sidon, réunion de $M = [\prod_{p\in P} \frac{p}{p-1}] + 1$ ensembles indépendants.

On a vu que $e^{2\pi i\mathbb{Q}}$ n'est pas une réunion finie d'ensembles indépendants. Cela pose deux questions :

Question 38. *L'ensemble $e^{2\pi i\mathbb{Q}}$ est-il une réunion finie d'ensembles quasi-indépendants ?*

Question 39. *L'ensemble $e^{2\pi i\mathbb{Q}}$ est-il un ensemble de Sidon ?*

Une réponse positive à la Question 35 entrainerait une réponse négative aux Questions 38 et 39. Une réponse positive à la Question 36 pourrait être une étape pour avoir des réponses positives aux Questions 38 et 39. La Question 37 ne semble pas facile, dans [GR] les auteurs ont dû faire appel aux ordinateurs pour exhiber certains ensembles quasi-indépendants de cardinal maximal. Ci-dessous nous faisons une synthèse des résultats connus sur $\psi(n), n \geq 2$. D'autres résultats, en particulier de critères pour tester la quasi-indépendance d'un sous-ensemble de $e^{2\pi i\mathbb{Q}}$, se trouvent dans [GR].

Quelques propriétés de la fonction ψ

La premier théorème montre que la fonction ψ ressemble à fonction d'Euler ϕ et que pour la calculer on peut se restreindre aux entiers n impairs, sans facteur carré.

Théorème B. [GR] *Pour tout entier $n \geq 2$ on a les propriétes suivantes.*

(a) Si le nombre premier p divise n, alors $\psi(pn) = p\psi(n)$.

(b) Si p est premier alors pour tout entier $k \geq 1$, $\psi(p^k) = \phi(p^k) = p^{k-1}(p-1)$.

(c) Si n est impair alors $\psi(2n) = \psi(n)$.

(d) Si le nombre premier p ne divise pas n, alors $\psi(pn) \geq (p-1)\psi(n)$.

Le théorème suivant montre que les différences entre ψ et ϕ interviennent à partir de trois facteurs premiers.

Théorème C. [GR] *(a) Si l'entier n a exactement deux facteurs premiers distincts, $\psi(n) = \phi(n)$.*

(b) Si $n = 15p$, où $p \geq 7$ est un nombre premier, alors $\psi(n) = \phi(n) + 4$.

Pour $n = 105 = 15 \times 7$, $\phi(n) = 48$ et $\psi(n) = 52$. Dans [GR], les auteurs font appel à un ordinateur pour exhiber un sous-ensemble quasi-indépendante de T_{105}.

Pour finir, faisons deux remarques [DP] sur la fonction ψ :

Remarque 1. La structure d'espace vectoriel fait défaut pour l'étude de ψ. Par exemple, un ensemble quasi-indépendant maximal (pour l'inclusion) de T_n n'a pas nécessairement cardinal maximal $\psi(n)$.

Remarque 2. On a $\limsup_{n\to\infty}(\psi(n) - \phi(n)) = \infty$. En effet, pour $p > 7$ premier, $\psi(105p) - \phi(105p) \geq (p-1)(\psi(105) - \phi(105)) = 4(p-1)$, d'après le Théorème B, condition (d), et le Théorème C, condition (b).

References

[1] [B] Jean Bourgain, Sidon sets and Riesz products, *Ann. Inst. Fourier* **35**, 1985, 137-148.
[2] [DP] Myriam Déchamps et Daniel Perrin, *discussion informelle.*
[3] [GR] L. Thomas Ramsey and Colin C. Graham, Planar sidonicity and quasi-independence for multiplicative subgroups of the roots of unity, *Pacific Journal of Math.*, **225**, N° 2, juin 2006.
[4] [LQ] Daniel Li et Hervé Queffélec, Introduction à l'étude des espaces de Banach, *Société Math. de France 2004, Cours Spécialisés 12.*
[5] [P] Gilles Pisier, Arithmetic Characterisations of Sidon sets, *Bull. Amer. Math. Soc.*, **8**, 1983, 87-89.