Representations of Graphs and Orthogonal Latin Square Graphs

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ABSTRACT

We define graph representations modulo integers and prove that any finite graph has a representation modulo some integer. We use this to obtain a new, simpler proof of Lindner, E. Mendelsohn, N. Mendelsohn, and Wolk's result that any finite graph can be represented as an orthogonal latin square graph.

Let G be a graph with vertices $v_1, \ldots, v_r$ and let n be a natural number. We say that G is representable modulo n if there exist distinct integers $a_1, \ldots, a_r$, $0 \le a_i < n$, satisfying $(a_i - a_j, n) = 1$ if and only if v_i is adjacent to v_j. We call $\{a_1, \ldots, a_r\}$ a representation of G modulo n and n the order of the representation. If $\{a_1, \ldots, a_r\}$ is a representation of G modulo n then so is $\{ba_1 + c, \ldots, ba_r + c\}$, where $(b, n) = 1$ and addition and multiplication are performed modulo n.

We will show that any graph is representable modulo some positive integer. The proof will require the following lemma.

Lemma. For any positive integer m there exist distinct primes $p_1, \ldots, p_m$ such that for all pairs A,B of disjoint nonempty subsets of $\{p_1, \ldots, p_m\}$, $(\Pi\{p_i: p_i \in A\} - \Pi\{p_j: p_j \in B\}, p_1 p_2 \ldots p_m) = 1$.

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Proof. Suppose that we have chosen a prime $p_1 > 3^m$ and further suppose that we have chosen primes $p_1 < \ldots < p_s$, $s < m$, to satisfy the conditions of the lemma. We note that in choosing the next prime the only restriction is that p_{s+1} cannot be congruent to a/b modulo p_i, where a and b are square free products using the primes $p_1, \ldots, p_s$ and $a \neq 1$. Thus for each i, $i = 1, \ldots, s$, there are at most $3^{s-1} < 3^m < p_1 < p_i$ residue classes to be avoided. Thus for some nonzero residue class modulo $p_1 \ldots p_s$ any choice of a prime in this residue class would be a valid choice for the next prime. As we can also guarantee that this residue class is nonzero modulo p_i, for $i = 1, \ldots, s$, a theorem of Dirichlet guarantees the existence of an infinite number of primes in such a residue class, enabling us to extend our selection. The proof follows inductively. ∎

We are now in a position to prove our main theorem.

Theorem. Any finite graph can be represented modulo some positive integer.

Proof. Let G be a graph with vertices $v_1, \ldots, v_r$. Form a new graph G' by adjoining an isolated vertex v_0 to G. Let $e_1, \ldots, e_m$ be the edges of the complement of G' and let $p_1, \ldots, p_m$ be primes satisfying the conditions of the lemma. Then for $i = 1, \ldots, r$ set $a_i = \Pi\ \{p_j\colon e_j$ incident with v_i in the complement of $G'\}$ and set $n = p_1 p_2 \ldots p_m$. Then $\{a_1, \ldots, a_r\}$ is a representation of G modulo n. ∎

Lindner et al. [5] defined an orthogonal latin square graph to be a graph, all of whose vertices are latin squares of the same order, adjacency being orthogonality. We now give a more elementary proof of their theorem [5, Theorem 1] that any finite graph is realizable as an orthogonal latin square graph.

Corollary. Any finite graph can be realized as an orthogonal latin square graph.

Proof. Let G be a finite graph with vertices $v_1, \ldots, v_r$ and adjoin to G a vertex v_0 joined to each of $v_1, \ldots, v_r$. Let $\{a_0, \ldots, a_r\}$, $a_0 = 0$, be a representation of $G \cup \{v_0\}$ modulo n. Define a matrix L_k whose ijth entry is $a_k i + j$, addition and multiplication modulo n. Then $L_1, \ldots, L_r$ are latin squares and L_k is orthogonal to $L_{k'}$ if and only if v_k is adjacent to $v_{k'}$. Thus G is realized as an orthogonal latin square graph. ∎

Remarks

1. Lindner et al. actually proved that any finite graph can be realized as an orthogonal latin square graph using idempotent latin squares. This extra condition is a by-product of the construction used in their proof and does not strengthen the result.
2. This paper was motivated by Lindner et al.'s paper and by the study of orthomorphism graphs of groups, orthogonal latin square graphs in

which all the latin squares are obtained from the Cayley table of a group by permuting rows. We have in fact shown that any finite graph can be represented as an orthomorphism graph of some cyclic group. For more information on orthomorphism graphs, see Evans [1,2,3], Johnson, Dulmage, and Mendelsohn [4], and Mendelsohn and Wolk [6].

3. It should be noted that for a graph to be representable modulo n it is necessary although not sufficient for the smallest prime divisor of n to be at least as large as the clique number of the graph. This is in marked contrast to the situation for orthogonal latin square realizations, in which only finitely many latin square orders are ruled out for a given graph [5, Theorem 3].
4. Let G be a graph obtained from K_4 by removing the edges of a path of length 2. Then we have shown that G has a representation whose order is the product of 6 primes, each greater than 3^6. However $\{0, 1, 3, 5\}$ is a representation of G modulo 15, the smallest possible order for a representation of this graph. This suggests the problem of determining for a given graph the smallest order possible for its representation.
5. A graph representation modulo n is a subset of $\{0, \ldots, n-1\}$ and any subset of $\{0, \ldots, n-1\}$ can be thought of as a graph representation modulo n. What relationship exists between properties of graphs and properties of the integer sets representing them?

References

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