PARTITIONS INTO PARTS WHICH ARE UNEQUAL AND LARGE.

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1. Introduction. Let us denote by $p(n)$ the number of partitions of n, by $q(n)$ the number of partitions of n into unequal parts (or into odd parts), by $r(n, m)$ the number of partitions of n into parts $\geq m$, and by $\rho(n, m)$ the number of partitions of n into unequal parts $\geq m$.

In [Erd] two of us gave the following asymptotic relation

$$(1) \qquad \rho(n, m) = (1 + o(1)) \frac{q(n)}{2^{m-1}} \quad , \qquad m = o(n^{1/5})$$

and in [Dix], a quite different result is given for $r(n, m)$, for $m = O(n^{1/4})$

$$r(n, m) = (m-1)! \left(\frac{\pi}{\sqrt{6n}}\right)^{m-1} p(n) \ (1 + O(m^2/\sqrt{n})) .$$

Using a tauberian theorem, J. Herzog (cf. [Her]) has proved, for $m = O(n^{3/8} (\log n)^{1/4})$:
$\log r(n, m) = \pi\sqrt{2n/3} - (1/2)\, m \log n + m \log m - (1 + \log(\sqrt{6}/\pi) + O(n^{1/4}\sqrt{\log n})$.

The aim of this paper is to prove the following three theorems.

Theorem 1. For all $n \geq 1$, and m , $1 \leq m \leq n$, we have

$$(i) \qquad \frac{1}{2^{m-1}} q(n) \leq \rho(n, m) \leq \frac{1}{2^{m-1}} q(n + \frac{m(m-1)}{2})$$

and

$$(ii) \qquad \rho(n, m) \leq \frac{1}{2^{m-2}} q\left(n + \left[\frac{m(m-1)}{4}\right]\right)$$

where $[x]$ is the integral part of x.

Theorem 2. When n tends to infinity, and $m = o\left(\frac{n}{\log n}\right)^{1/3}$, we have

$$\rho(n, m) = (1 + o(1)) \frac{1}{2^{m-1}} q\left(n + \left[\frac{m(m-1)}{4}\right]\right).$$

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Theorem 3. For fixed ε, with $0 < \varepsilon < 10^{-2}$ and for $m = m(n)$, $1 \le m \le n^{3/8 - \varepsilon}$, and $n \to +\infty$, the relation

$$\rho(n, m) = (1 + o(1)) \frac{q(n)}{\prod_{1 \le j \le m-1} \left(1 + \exp\left(-\frac{\pi j}{2\sqrt{3n}}\right)\right)}$$

holds.

The proof of Theorem 1 is very simple and elementary, and gives immediately (1) when $m = o(n^{1/4})$, with the classical asymptotic estimation of $q(n)$ (cf. (2) and Lemma 3 below).

The proof of Theorem 2 follows the same idea as the proof of theorem 1, but with sharper estimations.

The proof of Theorem 3 is analytic, and uses Cauchy's formula for the generating function of $\rho(n, m)$, which was already used to prove (1). It follows easily from Lemma 3 below that Theorem 3 implies Theorem 2..

At the end of the paper, a table of $\rho(n, m)$ is given. It has been calculated with the recurrence formula $\rho(n, m) = \rho(n, m+1) + \rho(n-m, m+1)$ and $\rho(n, m) = 1$ for $m \ge n/2$.

We shall also need the following asymptotic formula of $q(n)$:

(2) $$q(n) \sim \frac{1}{4\,(3n^3)^{1/4}} \exp(\pi\sqrt{n/3}).$$

Actually, it is possible to give a more precise expansion, using the result of Hardy and Ramanujan (cf. [Har] and [Hua]) :

$$q(n) = \frac{1}{\sqrt{2}} \frac{d}{dn} J_0\left(i\pi\sqrt{\frac{1}{3}(n + \frac{1}{24})}\right) + O\left(\exp(\frac{\pi}{3}\sqrt{n/3})\right) .$$

By the classical results $J'_0(z) = -J_1(z)$ and $I_1(z) = -i J_1(iz)$ on Bessel's functions, the main term of $q(n)$ is equal to

$$\frac{\pi}{2\sqrt{6}\sqrt{n + 1/24}} I_1\left(\frac{\pi}{\sqrt{3}}\sqrt{n + 1/24}\right) .$$

For $m \ge 1$, let us define

$$a_m = \frac{(-1)^m}{2^{3m} m!} \prod_{j=1}^{m} (4 - (2j-1)^2).$$

We have

$$a_1 = -\frac{3}{8} \quad ; \quad a_2 = -\frac{15}{128} \quad ; \quad a_3 = -\frac{105}{1024}.$$

For a real z tending to $+\infty$, we have the asymptotic expansion (cf. [Wat], p. 203) :

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}} \left(1 + \sum_{m=1}^{M} a_m z^{-m} + O(z^{-M-1})\right) ,$$

and if we set $c = \frac{\pi}{\sqrt{3}}$, $\lambda = 1/24$ and

$$g_M(\lambda, t) = \left(\exp\left(c\frac{(1 + \lambda t^2)^{1/2} - 1}{t}\right)\right) \left((1 + \lambda t^2)^{-\frac{3}{4}} + \sum_{m=1}^{M} \frac{a_m}{c^m} t^m (1 + \lambda t^2)^{-\frac{m}{2} - \frac{3}{4}}\right)$$

then the function g_M is analytic in t in a neighbourhood of 0, thus it has a Taylor expansion

$$g_M(\lambda, t) = 1 + \sum_{m=1}^{M} b_m t^m + O(t^{M+1}).$$

We conclude that

$$(3) \qquad q(n) = \frac{1}{4\,(3n^3)^{1/4}} \left(\exp\left(\frac{\pi}{\sqrt{3}} \sqrt{n}\right) \right) \left(1 + \sum_{m=1}^{M} b_m n^{-m/2} + O\left(n^{-\frac{M+1}{2}}\right) \right).$$

The first coefficients b_m have been calculated by the algebraic computer system MACSYMA :

$$b_1 = \frac{\pi}{48\sqrt{3}} - \frac{3\sqrt{3}}{8\pi} = -0.16896 \qquad b_2 = \frac{\pi^2}{13824} - \frac{5}{128} - \frac{45}{128\,\pi^2} = -0.07397$$

$$b_3 = \frac{\pi^3}{1990656\sqrt{3}} - \frac{35\,\pi}{36864\sqrt{3}} + \frac{35\sqrt{3}}{2048\,\pi} - \frac{315\sqrt{3}}{1024\,\pi^3} = -0.009475$$

$$b_4 = \frac{\pi^4}{1146617856} - \frac{7\,\pi^2}{1769472} + \frac{105}{65536} + \frac{315}{16384\,\pi^2} - \frac{42525}{32768\pi^4} = -0.009812.$$

We are pleased to thank J.P. Massias for calculating both the table of $\rho(n, m)$ and the asymptotic expansion of q.

2. Proof of Theorem 1.

Setting $q(0) = \rho(0, m) = 1$, we shall consider generating functions :

$$\sum_{n \geq 0} q(n)\, x^n = \prod_{n \geq 1} (1 + x^n)$$

and

$$(4) \qquad \sum_{n \geq 0} \rho(n, m)\, x^n = \prod_{n \geq m} (1 + x^n).$$

Let us define

$$(5) \qquad P_{m-1}(x) = \prod_{k=1}^{m-1} (1 + x^k) = \sum_{k=0}^{m(m-1)/2} q(k, m-1)\, x^k.$$

We observe that $q(k, m-1) \geq 0$ and that

$$\sum_{k=0}^{m(m-1)/2} q(k, m-1) = 2^{m-1}.$$

We now write

$$\sum_{n=0}^{\infty} q(n)\, x^n = \left(\sum_{n=0}^{\infty} \rho(n, m)\, x^n \right) \left(\sum_{k=0}^{m(m-1)/2} q(k, m-1)\, x^k \right)$$

and

$$(6) \qquad q(n) = \sum_{k=0}^{m(m-1)/2} q(k, m-1)\, \rho(n-k, m)$$

where we set $\rho(n, m) = 0$ for $n < m$ and $n \neq 0$. Now, it is easy to see that ρ is non decreasing in n, therefore, $\rho(n-k, m) \leq \rho(n, m)$, and then (6) gives $q(n) \leq 2^{m-1} \rho(n, m)$. In the same way,

$$q\left(n+\frac{m(m-1)}{2}\right) = \sum_{k=0}^{m(m-1)/2} q(k, m-1)\ \rho\left(n-k+\frac{m(m-1)}{2}, m\right)$$
$$\geq 2^{m-1} \rho(n, m)$$

and this achieves the proof of (i). To prove (ii), we set $M = [\,(m(m-1)/4\,]$ and get

$$q(n+M) = \sum_{k=0}^{m(m-1)/2} q(k, m-1)\ \rho(n-k+M, m)$$
$$\geq \rho(n, m) \sum_{k=0}^{M} q(k, m-1) \geq 2^{m-2} \rho(n, m).$$

3. Proof of Theorem 2.

We first need a few lemmas :

Lemma 1. For $0 \leq u \leq 1/2$, we have

(i) $-\log(1-u) \leq u + u^2$.

For $m \geq 3$ and $0 \leq u \leq 1$, we have

(ii) $(1-u)^m \geq 1 - mu + \frac{m(m-1)}{2} u^2 - \frac{m(m-1)(m-2)}{6} u^3$.

Proof : (i) is easy. To prove (ii) use Taylor's formula for the function $u \mapsto (1-u)^m$.

Lemma 2. Let $q(r, m-1)$ be defined by (5). If $m \geq 3$, R is an integer, $0 \leq R \leq \frac{m(m-1)}{4}$ and $t = \frac{m(m-1)}{4} - R$, then we have

$$\sum_{r=0}^{R} q(r, m-1) \leq 2^{m-1} \exp\left(-\frac{3t^2}{m^3}\right).$$

Proof : For $x \in [1/2, 1]$ we set

$$P = P(x, R, m) = x^{-R} \prod_{r=1}^{m-1} (1 + x^r).$$

and $x = 1 - u$. So we have $0 \leq u \leq 1/2$ and

$$\log P = -R\log(1-u) + (m-1)\log 2 + \sum_{r=1}^{m-1} \log\left(1 + \frac{(1-u)^r - 1}{2}\right)$$
$$\leq -R\log(1-u) + (m-1)\log 2 + \sum_{r=1}^{m-1} \frac{(1-u)^r - 1}{2}$$

$$= \begin{cases} -R\log(1-u) + (m-1)\log 2 + \dfrac{(1-u)-(1-u)^m}{2u} - \dfrac{m-1}{2}, & \text{if } u > 0; \\ (m-1)\log 2, \text{ if } u = 0. \end{cases}$$

Using Lemma 1, (i) and (ii), we obtain that

$$\log P \le (m-1)\log 2 + Ru + Ru^2 - \frac{m(m-1)}{4}u + \frac{m(m-1)(m-2)}{12}u^2$$

$$\le (m-1)\log 2 - tu + \frac{m^3}{12}u^2$$

because $R \le \frac{m(m-1)}{4}$. We now choose $u = \frac{6t}{m^3}$. As $0 \le t \le \frac{m^2}{4}$, we have

$0 \le u \le \frac{3}{2m} \le \frac{1}{2}$ for $m \ge 3$, and we obtain that $\log P \le (m-1)\log 2 - 3t^2/m^3$.

The lemma follows from this inequality, because

$$\sum_{r=0}^{R} q(r, m-1) \le \sum_{r=0}^{R} q(r, m-1)\frac{x^r}{x^R} \le P.$$

Lemma 3. When $n \to +\infty$ and $h = o(n^{3/4})$, we have

$$q(n+h) \sim q(n) \exp\left(\frac{Ah}{\sqrt{n}}\right),$$

where $A = \pi/(2\sqrt{3}) = 0.9069...$

Proof: From (2) we have

$$q(n+h) \sim \frac{1}{4(3(n+h)^3)^{1/4}} \exp(2A\sqrt{n+h})$$

and

$$\sqrt{n+h} = \sqrt{n} + \frac{h}{2\sqrt{n}} + O\left(\frac{h^2}{n^{3/2}}\right) = \sqrt{n} + \frac{h}{2\sqrt{n}} + o(1).$$

Proof of Theorem 2. We first assume that $m \equiv 0$ or $1 \mod 4$, in order that $m(m-1)/4$ should be an integer. The case $m \equiv 2$ or $3 \mod 4$ can be treated similarly. We then choose R and t as in lemma 2, and set

$$R' = \frac{m(m-1)}{2} - R = \frac{m(m-1)}{4} + t.$$

We cut the summation in (6) into three parts :

$$S_1 = \sum_{r=0}^{R-1} \; ; \quad S_2 = \sum_{r=R}^{R'} \; ; \quad S_3 = \sum_{r=R'+1}^{m(m-1)/2},$$

and we shall prove that S_1 and S_3 are $o(q(n))$, if we choose conveniently t.

First of all, it is easy to see that $S_3 \le S_1$. Then we consider

$$S_1 = \sum_{r=0}^{R-1} \rho(n-r, m)\, q(r, m-1).$$

We set $s = \frac{m(m-1)}{4} - r$. Theorem 1, (ii) gives :

$$\rho(n-r, m) \le \frac{1}{2^{m-2}} q(n+s)$$

and by Lemma 3,

$$\rho(n-r, m) \ll \frac{1}{2^{m-1}}\, q(n)\, \exp\left(\frac{As}{\sqrt{n}}\right)$$

By Lemma 2,

$$q(r, m-1) \le 2^{m-1} \exp\left(-\frac{3s^2}{m^3}\right),$$

thus

$$S_1 \ll q(n) \sum_{s=t+1}^{m(m-1)/4} \exp\left(\frac{As}{\sqrt{n}} - \frac{3s^2}{m^3}\right) \ll q(n) \sum_{s \ge t+1} \exp\left(-\frac{3}{m^3}\left(s - \frac{m^3 A}{6\sqrt{n}}\right)^2\right)$$

and, if we choose $t > \dfrac{m^3 A}{6\sqrt{n}}$, we shall obtain that

$$S_1 \ll q(n) \cdot \int_t^{+\infty} \exp\left(-\frac{3}{m^3}\left(u - \frac{m^3 A}{6\sqrt{n}}\right)^2\right) du$$

$$\ll q(n) \frac{m^3}{6\left(t - \dfrac{m^3 A}{6\sqrt{n}}\right)} \exp\left(-\frac{3}{m^3}\left(t - \frac{m^3 A}{6\sqrt{n}}\right)^2\right).$$

Choosing

$$t = \left[\frac{m^3 A}{6\sqrt{n}} + m^{3/2}\sqrt{\log n}\right] \tag{7}$$

implies $S_1 = o(q(n))$ and $S_2 = (1 + o(1))\, q(n)$.

From the definition of S_2, we see that

$$S_2 \le \rho(n-R, m) \sum_{r=0}^{m(m-1)/2} q(r, m-1) = 2^{m-1} \rho(n-R, m),$$

and

$$S_2 \ge \rho(n-R', m) \sum_{r=R}^{R'} q(r, m-1)$$

$$\ge \rho(n-R', m)\, 2^{m-1}\left(1 - 2\exp\left(-\frac{3(t+1)^2}{m^3}\right)\right)$$

by Lemma 2. This implies that

$$\frac{q(n+R)}{2^{m-1}}(1 + o(1)) \le \rho(n, m) \le \frac{q(n+R')}{2^{m-1}}(1 + o(1))$$

and Theorem 2 follows from Lemma 3, just observing that the hypothesis and (7) imply that $t = o(\sqrt{n})$.

4. Proof of Theorem 3.

Let $k = m - 1 \ge 1$ and $q_k(n) = \rho(n, m)$.

Let us observe that the relation

$$1 + \sum_{n=1}^{\infty} q_k(n)\, w^n = \prod_{v=k+1}^{\infty} (1 + w^v)$$

holds for $|w| < 1$. Cauchy's formula gives the representation

$$q_k(n) = \frac{1}{2\pi i} \int_{|w|=r} w^{-n-1} \prod_{v=k+1}^{\infty} (1 + w^v) \, dw$$

for $0 < r < 1$. For Re $z > 0$, let us define $h_k(z)$ by

$$h_k(z) = \prod_{v=k+1}^{\infty} (1 + \exp(-v z)) .$$

Then we may write

$$q_k(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h_k(x + i y) \exp(n x + i n y) \, dy$$

for $x > 0$.

Let C_0 be a sufficiently large constant, further, $1 \le k \le n^{\frac{3}{8} - \varepsilon}$. We choose $x = x_0 = \pi / (2\sqrt{3n})$, $y_1 = n^{-3/4 + \varepsilon/3}$, $y_2 = C_0 x_0$, and it will be convenient to set

$$D = \left\{ \prod_{v=1}^{k} (1 + \exp(-v x_0)) \right\}^{-1} .$$

Observe that, with our choices of x_0 and k, theorem 3 becomes $\rho(n, m) = (1 + o(1)) D q(n)$. We investigate $q_k(n)$ as

$$q_k(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h_k(x_0 + i y) \exp(n x_0 + i n y) \, dy$$

$$= \frac{1}{2\pi} \left\{ \int_{-\pi}^{-y_2} + \int_{-y_2}^{-y_1} + \int_{-y_1}^{y_1} + \int_{y_1}^{y_2} + \int_{y_2}^{\pi} \right\}.$$

For $|y| \le y_2$ (and $n \to +\infty$), we can apply (4.3) - (4.4) of [Erd] and get

$$\prod_{v=1}^{\infty} (1 + \exp(-v(x_0 + i y))) = \exp\left(\frac{\pi^2}{12(x_0 + i y)} - \frac{1}{2} \log 2 + o(1) \right) ,$$

further

$$(8) \qquad \prod_{v=1}^{k} (1 + \exp(-v(x_0 + i y)))^{-1} = D \exp\left(- \sum_{v=1}^{k} \log\left(1 - \frac{1 - \exp(-v i y)}{1 + \exp(v x_0)} \right) \right) .$$

For $|y| \le y_1$, we deduce from (8)

$$\prod_{v=1}^{k} (1 + \exp(-v(x_0 + i y)))^{-1} = D \exp(O(k \, . \, k y_1)) = D \exp(o(1)).$$

Therefore (cf. [Erd], pp. 435-437),

$$\frac{1}{2\pi}\int_{-y_1}^{y_1} \qquad = (1+o(1))\ D\ q(n).$$

Next, for $y_1 \le |y| \le y_2$, it follows from (8) that

$$\left|\prod_{\nu=1}^{k}(1+\exp(-\nu(x_0+iy)))^{-1}\right| = D\exp\left(-\sum_{\nu=1}^{k}\log\left(\left|1-\frac{1-\exp(-\nu i y)}{1+\exp(\nu x_0)}\right|\right)\right).$$

Here,

$$\left|1-\frac{1-\exp(-\nu i y)}{1+\exp(\nu x_0)}\right| \ge \left|1-\frac{i\nu y}{1+\exp(\nu x_0)}\right| - \frac{|1-\exp(-\nu i y)-i\nu y|}{1+\exp(\nu x_0)} \ge$$

$$\ge 1-\frac{1}{2}\cdot\frac{\nu^2 y^2}{1+\exp(\nu x_0)} \ge 1-\frac{\nu^2 y^2}{4} = 1-O(k^2 y_2^2) = 1-o(1).$$

If $y_1 \le |y| \le y'_1 := n^{-9/16}$, then

$$\sum_{\nu=1}^{k} -\log\left(\left|1-\frac{1-\exp(-\nu i y)}{1+\exp(\nu x_0)}\right|\right) \le \sum_{\nu=1}^{k} O(\nu^2 y_1'^2) = O(k^3 y_1'^2) = O(n^{-3\varepsilon}) = o(1).$$

Thus (cf. [Erd], p. 438),

$$\frac{1}{2\pi}\left|\int_{y_1\le|y|\le y'_1}\right| = O(1)\ D\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}-\frac{2\sqrt{3}}{\pi}n^{2\varepsilon/3}\right) = o(1)\ D\ q(n).$$

If $y'_1 \le |y| \le y_2$ $(= C_0 x_0)$ then

$$\left|\prod_{\nu=1}^{k}(1+\exp(-\nu(x_0+iy)))^{-1}\right| \le D\exp(O(k^3 y_2^2)) = D\exp(O(n^{1/8-3\varepsilon})),$$

consequently (cf. [Erd], p. 438),

$$\frac{1}{2\pi}\left|\int_{y'_1\le|y|\le y_2}\right| \le D\exp\left(\frac{\pi^2 x_0}{12(x_0^2+(y'_1)^2)}+n x_0+O(n^{\frac{1}{8}-3\varepsilon})\right).$$

Here,

$$\frac{\pi^2 x_0}{12(x_0^2+y_1'^2)} = \frac{\pi^2}{12 x_0}\cdot\frac{1}{1+\frac{(y'_1)^2}{x_0^2}} = \frac{\pi^2}{12 x_0}\left(1-\frac{(y'_1)^2}{x_0^2}+O\left(\frac{(y'_1)^4}{x_0^4}\right)\right) \le$$

$$\le \frac{\pi^2}{12 x_0}-\frac{\pi^2}{24}\frac{(y'_1)^2}{x_0^3} = \frac{\pi^2}{12 x_0} - c_1 n^{\frac{3}{8}}.$$

Thus,

$$\frac{1}{2\pi}\left|\int_{y'_1 \le |y| \le y_2}\right| \le D\ q(n)\ \exp\left(-c_1 n^{\frac{3}{8}} + O(n^{\frac{1}{8}-3\varepsilon}) + O(\log n)\right) = o(1)\ D\ q(n).$$

Finally, for $C_0 x_0 \le |y| \le \pi$,

$$\frac{1}{2\pi}\left|\int_{C_0 x_0 \le |y| \le \pi}\right| \le q(n)\ \exp(-c_2\sqrt{n})$$

with a suitable positive constant c_2 (cf. [Erd] , pp. 439 - 440).

Since

$$D \le 2^k \le \exp(n^{3/8-\varepsilon}) = o(\exp(c_2\sqrt{n})),$$

Theorem 3 is proved.

Remark. In the same way, one can prove Theorem 3 with the factor

$$1 + O\left(n^{-\frac{1}{4}+\varepsilon}\right) + O\left(m^2\ n^{-\frac{3}{4}+\frac{\varepsilon}{3}}\right)$$

instead of $1 + o(1)$.

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Table of ρ(n , m)

n= \ m=	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	1	1										
3	2	1	1									
4	2	1	1	1								
5	3	2	1	1	1							
6	4	2	1	1	1	1						
7	5	3	2	1	1	1	1					
8	6	3	2	1	1	1	1	1				
9	8	5	3	2	1	1	1	1	1			
10	10	5	3	2	1	1	1	1	1	1		
11	12	7	4	3	2	1	1	1	1	1	1	
12	15	8	5	3	2	1	1	1	1	1	1	1
13	18	10	6	4	3	2	1	1	1	1	1	1
14	22	12	7	4	3	2	1	1	1	1	1	1
15	27	15	9	6	4	3	2	1	1	1	1	1
16	32	17	10	6	4	3	2	1	1	1	1	1
17	38	21	12	8	5	4	3	2	1	1	1	1
18	46	25	15	9	6	4	3	2	1	1	1	1
19	54	29	17	11	7	5	4	3	2	1	1	1
20	64	35	20	12	8	5	4	3	2	1	1	1
21	76	41	24	15	10	7	5	4	3	2	1	1
22	89	48	28	17	11	7	5	4	3	2	1	1
23	104	56	32	20	13	9	6	5	4	3	2	1
24	122	66	38	23	15	10	7	5	4	3	2	1
25	142	76	44	27	17	12	8	6	5	4	3	2
26	165	89	51	31	20	13	9	6	5	4	3	2
27	192	103	59	36	23	16	11	8	6	5	4	3
28	222	119	68	41	26	17	12	8	6	5	4	3
29	256	137	78	47	30	20	14	10	7	6	5	4
30	296	159	91	55	35	23	16	11	8	6	5	4
31	340	181	103	62	39	26	18	13	9	7	6	5
32	390	209	118	71	45	29	20	14	10	7	6	5
33	448	239	136	81	51	34	23	17	12	9	7	6
34	512	273	155	93	58	38	26	18	13	9	7	6
35	585	312	176	105	66	43	29	21	15	11	8	7
36	668	356	201	120	75	49	33	23	17	12	9	7
37	760	404	228	135	84	55	37	26	19	14	10	8
38	864	460	259	154	96	62	42	29	21	15	11	8
39	982	522	294	174	108	70	47	33	24	18	13	10
40	1113	591	332	197	122	79	53	36	26	19	14	10
41	1260	669	375	221	137	88	59	41	29	22	16	12
42	1426	757	425	251	155	100	67	46	33	24	18	13
43	1610	853	478	281	173	111	74	51	36	27	20	15
44	1816	963	538	317	195	125	83	57	40	29	22	16
45	2048	1085	607	356	219	140	93	64	45	33	25	19
46	2304	1219	681	400	245	157	104	71	50	36	27	20
47	2590	1371	764	447	274	174	115	79	55	40	30	23
48	2910	1539	858	502	307	196	129	88	62	44	33	25
49	3264	1725	961	561	342	217	143	97	68	49	36	28
50	3658	1933	1075	628	383	243	160	109	76	54	40	30

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51	4097	2164	1203	701	427	270	177	120	84	60	44	34
52	4582	2418	1343	782	475	301	197	133	93	66	48	36
53	5120	2702	1499	871	529	333	218	147	102	73	53	40
54	5718	3016	1673	972	589	372	243	164	114	81	59	44
55	6378	3362	1863	1081	654	411	268	180	125	89	64	48
56	7108	3746	2073	1202	727	457	297	200	138	98	71	52
57	7917	4171	2308	1336	807	506	329	220	152	108	78	58
58	8808	4637	2564	1483	894	561	364	244	168	119	86	63
59	9792	5155	2847	1645	991	619	401	268	184	130	94	69
60	10880	5725	3161	1825	1098	687	444	297	204	144	104	76
61	12076	6351	3504	2021	1214	757	489	325	223	157	113	83
62	13394	7043	3882	2237	1343	837	540	360	246	173	125	91
63	14848	7805	4301	2476	1485	924	595	395	270	189	136	100
64	16444	8639	4757	2736	1638	1019	655	435	297	208	149	109
65	18200	9561	5260	3023	1809	1122	721	477	325	227	163	119
66	20132	10571	5814	3338	1995	1238	794	526	358	250	179	131
67	22250	11679	6419	3683	2198	1361	872	575	391	272	194	142
68	24576	12897	7083	4060	2422	1498	958	633	429	299	213	155
69	27130	14233	7814	4476	2667	1648	1053	693	470	326	232	169
70	29927	15694	8611	4928	2933	1811	1156	761	515	358	254	185
71	32992	17298	9484	5424	3226	1988	1267	832	562	389	276	200
72	36352	19054	10443	5967	3545	2184	1390	913	616	427	302	219
73	40026	20972	11488	6560	3893	2395	1523	997	672	464	328	237
74	44046	23074	12631	7207	4274	2626	1668	1093	735	508	359	259
75	48446	25372	13884	7917	4691	2880	1827	1194	803	553	390	281
76	53250	27878	15247	8687	5142	3154	1998	1305	876	604	425	306
77	58499	30621	16737	9530	5637	3453	2186	1425	955	656	462	331
78	64234	33613	18366	10449	6175	3780	2390	1558	1043	717	504	362
79	70488	36875	20138	11451	6760	4134	2611	1698	1136	778	546	391
80	77312	40437	22071	12541	7399	4519	2851	1854	1238	849	595	426
81	84756	44319	24181	13732	8095	4941	3114	2021	1349	922	646	461
82	92864	48545	26474	15023	8848	5395	3397	2203	1468	1004	702	502
83	101698	53153	28972	16431	9671	5891	3705	2400	1597	1089	761	542
84	111322	58169	31695	17963	10564	6430	4040	2615	1739	1186	827	590
85	121792	63623	34651	19628	11533	7014	4403	2845	1890	1286	896	637
86	133184	69561	37866	21435	12587	7646	4795	3097	2054	1398	973	692
87	145578	76017	41366	23403	13732	8337	5223	3369	2233	1516	1054	748
88	159046	83029	45163	25535	14971	9080	5683	3662	2424	1646	1142	811
89	173682	90653	49287	27852	16319	9889	6184	3981	2632	1783	1237	875
90	189586	98933	53770	30367	17780	10766	6726	4326	2858	1936	1341	950
91	206848	107915	58628	33093	19361	11715	7312	4697	3100	2096	1450	1024
92	225585	117670	63900	36048	21077	12740	7945	5100	3361	2272	1570	1109
93	245920	128250	69622	39255	22936	13856	8633	5536	3646	2460	1699	1197
94	267968	139718	75818	42725	24945	15056	9373	6004	3950	2664	1837	1295
95	291874	152156	82534	46486	27125	16359	10175	6513	4280	2882	1986	1396
96	317788	165632	89814	50559	29482	17767	11041	7060	4636	3120	2147	1510
97	345856	180224	97690	54965	32029	19289	11977	7651	5019	3373	2319	1627
98	376256	196032	106218	59732	34787	20931	12986	8289	5431	3648	2506	1758
99	409174	213142	115452	64893	37768	22712	14079	8979	5879	3943	2706	1895
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