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On the Graph of Large Distances

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Abstract. For a set S of points in the plane, let $d_1 > d_2 > \cdots$ denote the different distances determined by S. Consider the graph $G(S, k)$ whose vertices are the elements of S, and two are joined by an edge iff their distance is at least d_k. It is proved that the chromatic number of $G(S, k)$ is at most 7 if $|S| \geq \text{const}\, k^2$. If S consists of the vertices of a convex polygon and $|S| \geq \text{const}\, k^2$, then the chromatic number of $G(S, k)$ is at most 3. Both bounds are best possible. If S consists of the vertices of a convex polygon then $G(S, k)$ has a vertex of degree at most $3k-1$. This implies that in this case the chromatic number of $G(S, k)$ is at most $3k$. The best bound here is probably $2k+1$, which is tight for the regular $(2k+1)$-gon.

Introduction

Let S be a set of n points in the plane. Let us denote by $d_1 > d_2 > \cdots$ the different distances determined by these points, and by n_i, the number of distances equal to d_i.

The number of distinct distances leads to interesting questions. A 40-year-old conjecture of Erdős [4, worth \$500] implies that the number of distinct distances determined by n points is at least $cn/(\log n)^{1/2}$ (if true, this is best possible apart from the value of c, as shown by the set of lattice points inside a circle). The case when the set S consists of the vertices of a convex polygon behaves better. Erdős conjectured and Altman [1], [2] proved that the number of distances determined by the vertices of a convex n-gon is at least $\lfloor n/2 \rfloor$, which is of course achieved for the regular n-gon.

The numbers n_i also lead to many difficult problems. Erdős [3] observed that each distance occurs at most $O(n^{3/2})$ times and showed that in the set of lattice points inside an appropriate circle, the same distance may occur $n^{1+c/(\log\log n)}$ times. The upper bound has since been improved to $O(n^{4/3})$ by Spencer *et al.* [6]. For a survey of some related problems and results see Moser and Pach [12].

The situation is rather different in the case when S consists of the vertices of a convex n-gon. Erdős and L. Moser conjectured that in a convex n-gon every distance can occur at most cn times. This conjecture is still unsettled. A recent (unpublished) construction of P. Hajnal [11] shows that the same distance may occur about $9n/5$ times. Even if we do not insist on strict convexity, the best construction known (a chain of regular triangles) gives the same distance only $2n-3$ times.

The situation changes again if we consider the largest distance only. Hopf and Pannwitz [5] and Sutherland [7] proved that the maximum distance among n points occurs at most n times, i.e., $n_1 \le n$ (here, of course, the convex and nonconvex cases do not differ). Vesztergombi [8], [9] showed that $n_2 \le 4n/3$ in the convex case and $n_2 \le 3n/2$ in the general case, and these bounds are tight. More generally, she determined all homogeneous linear inequalities that hold for n, n_1, and n_2. She also observed that $n_k \le 2kn$.

Denote by $G(S, k)$ the graph on vertex set S obtained by joining x to y if their distance is at least d_k. Altman's result mentioned above is equivalent to saying that in the convex case, $G(S, k)$ does not contain a complete $(2k+2)$-gon. In this paper we study the chromatic number of this graph. We prove that if $n > n_0(k)$ then the chromatic number $\chi(G(S, k))$ is at most 7, and give a construction for which the equality holds for arbitrarily large n. Obviously without the assumption $n > n_0(k)$ the theorem is not true, since if we take the vertices of the regular $(2k+1)$-gon as our set of points then $\chi(G(S, k)) = 2k+1$.

If we assume that S is the vertex set of a convex polygon then we can prove an even stronger result: for $n > n_1(k)$ the chromatic number $\chi(G(S, k))$ is at most 3. The problem of determining the largest possible value of the chromatic number of $G(S, k)$ for a given k (both in the convex and nonconvex case, without any assumption on the number of points) turns out quite difficult and we have only a partial answer. We conjecture that if S is the set of vertices of a convex polygon then the chromatic number of $G(S, k)$ is at most $2k+1$. This is best possible (if true) as shown by the regular $(2k+1)$-gon. This conjecture would generalize the result of Altman mentioned above. Perhaps in the convex case there always exists an x_i such that the degree of x_i is at most $2k$. We prove the weaker result that for the vertex set S of a convex polygon there exists an x_i such that the degree of x_i is at most $3k-1$. From this it follows that the number of edges in $G(S, k)$ is at most $3kn$, and that its chromatic number is at most $3k$.

The results of Vesztergombi mentioned above imply that the number of edges in $G(n, 2)$ is at most $2n$. One may conjecture that the number of edges in $G(n, k)$ is at most kn. Our result verifies this conjecture up to a constant factor and shows that the conjecture of Erdős and Moser is valid in the average for the "large" distances. Let us mention the related conjecture of Erdős that in a convex n-gon there is always a vertex x_i such that the number of distinct distances from x_i is at least $n/2$.

It would be nice if in the nonconvex case the maximum of the chromatic number of $G(S, k)$ for fixed k were also equal to the largest complete graph which can be contained in some $G(S, k)$. If the above-mentioned conjecture of Erdős is true, then the largest complete graph contained in $G(S, k)$ has $O(k(\log k)^{1/2})$ vertices. We can prove that the chromatic number is at most ck^2. A bound of the form $k^{1+\varepsilon}$ will not come out easily since so far we could not even prove that $G(S, k)$ does not contain a complete graph on $k^{1+\varepsilon}$ vertices.

In the one-dimensional case these problems are trivial. For large n, $G(S, k)$ is bipartite and, for any n, the chromatic number of $G(S, k)$ is at most $k+1$, which can of course be achieved.

The following problem might be of interest. Let $x_1, \ldots, x_n$ be n points in the plane and $l_1, \ldots, l_k$, k arbitrary distances. Two points are joined by an edge if their distance is one of the l_i's. Denote by $f(k)$ the maximum possible chromatic number of this graph. Could this again be the size of the largest complete graph contained in such a graph?

1. The "Nonconvex" Case

We start with a simple lemma.

Lemma 1.1. *Let C be a circle with center c and radius r, and let T be a set of points on the circle such that c is in the convex hull of T. Then for each point $p \neq c$ of the plane, there is a point $t \in T$ with $d(p, t) > r$.*

Now we prove the main theorem of this section.

Theorem 1.2. *If $n \geq n_2(k) = 18k^2$, then $\chi(G(S, k)) \leq 7$.*

Proof. Let $q \in S$ be a point of maximum degree in $G(S, k)$. Consider the circle C with smallest radius r containing $S' = S - \{q\}$. If $r < d_k$ then we can cut the disc bounded by C into six pieces with diameter less than d_k. This yields a 6-coloration of $G(S, k) - q$, and using a seventh color for q we are done.

So suppose that $r \geq d_k$. Obviously, the convex hull of $C \cap S'$ contains the center c of C. So we can choose a subset T of $C \cap S'$ with $|T| \leq 3$ such that the convex hull of T contains c. Hence, by Lemma 1.1, every point in S is connected to some point in T. So T contains a point of degree more than $6k^2$, and hence by its choice, q has degree greater than $6k^2$. Now among the neighbors of q, there are more than $2k^2$ which are connected to the same point $t \in T$.

But note that these points must lie on k concentrical circles about q as well as on k concentrical circles about t. These two families of circles have at most $2k^2$ intersection points, a contradiction. □

Now we give a construction which shows that this upper bound for the chromatic number is sharp.

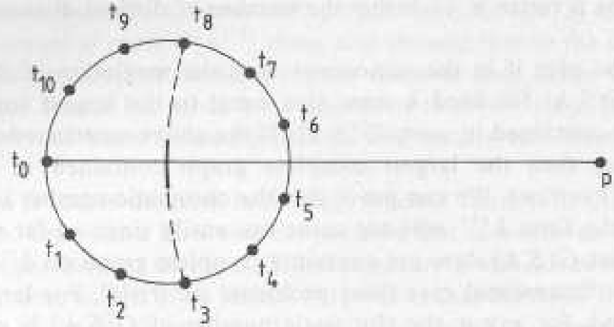

Fig. 1

Take a regular 11-gon with vertices t_i $(i=0,\ldots,10)$ on a circle with radius 1 and center O. Take the point p on the half-line t_0O for which $d(O,p)=d(t_3,p)$ holds (see Fig. 1). Draw a very short arc around p going through O and place the remaining points of S on this arc. Let us consider in this setting the 10 largest distances. These will be the six different distances $d(p,t_i)$ between p and the rest of the points, and the four largest chords in the regular 11-gon. One can easily check that the t_i's need six colors and p needs a seventh color.

The threshold $n_2(k)$ in the theorem is sharp as far as the order of magnitude goes. In fact, let us modify the previous construction as follows. We construct the 11-gon and the point p as before, but now we also add a further point p' obtained by rotating p about O by 90°. Let us draw $k-23$ concentrical circles about p as well as about p' with radii very close to $d(O,p)$, and let us add the $(k-23)^2$ intersection points of these circles inside the 11-gon. This way we get a set S with $\approx k^2$ points such that the chromatic number of $G(S,k)$ is 8.

It would be interesting to determine the threshold for $|S|$(as a function of k) where the chromatic number of $G(S,k)$ becomes bounded. This is related to the following question: given $t\geq 3$, what is the largest s such that $G(S,k)$ can contain a complete bipartite graph $K_{t,s}$? A recent construction of Elekes [10] shows that, for each fixed t, s can be as large as c_tk^2.

2. The "Convex" Case

In this section we deal with the case when S is a set of vertices of a convex n-gon P (briefly, the "convex" case). The convexity of S gives a natural cyclic ordering of the points, so throughout the proofs we refer to this ordering. Before stating the main results of this section we make some simple observations.

Lemma 2.1. *Suppose that $x_1, x_2, x_3, x_4 \in S$ (in this counterclockwise order) and*

$$d(x_1,x_2)\geq d_k, \qquad d(x_2,x_3)\geq d_k, \qquad d(x_3,x_4)\geq d_k.$$

Then for each $y\in S$ between x_4 and x_1, at least one of the distances $d(x_i,y)$ is greater than d_k.

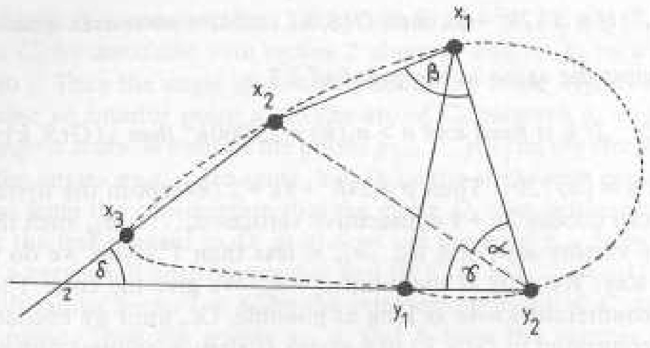

Fig. 2

Proof. Since the angle x_1yx_4 is less than 180° (because S is a convex set), at least one of the angles x_iyx_{i+1} (for $i=1,2,3$) is less than 60°. Hence (x_i, x_{i+1}) cannot be the largest side of the triangle x_iyx_{i+1}, from which the lemma follows. □

Lemma 2.2. *Suppose that x_1, x_2, x_3, y_1, and y_2 are five vertices of S in this counterclockwise order, and assume that $d(x_1,x_2)\geq d_k$, $d(x_2,x_3)\geq d_k$, and $d(x_1,y_1)\leq d(x_1,y_2)$. Then $d(y_2,x_2)\geq d_k$.*

Proof. If the semiline x_2x_3 does not intersect the semiline y_2y_1 then the assertion is obvious. So assume that these similines intersect in a point z as in Fig. 2. Also assume, by way of contradiction, that $d(y_2,x_2)<d_k$. Now the angle $x_1y_2x_2=\alpha$ is greater than the angle $y_2x_1x_2=\beta$, because the lengths of the opposite sides of the triangle $y_2x_1x_2$ are in this order. Similarly, in the triangle y_2x_2z, the angle $x_2y_2z=\gamma$ is larger than the angle $x_2zy_2=\delta$. On the other hand, since the angle x_1x_2z is less than 180°, the sum of the other angles in the convex quadrangle $y_2zx_2x_1$ must be more than 180°, which means that $180°<\beta+(\alpha+\gamma)+\delta<2(\alpha+\gamma)$. But then the angle $x_1y_2y_1=\alpha+\gamma$ is obtuse and, hence, it is the largest angle in the triangle $x_1y_2y_1$. This contradicts our assumption that $d(x_1,y_1)\leq d(x_1,y_2)$. □

Lemma 2.3. *Suppose that $x_1, x_2, x_3, x_4 \in S$ (in this counterclockwise order) and*

$$d(x_1,x_2)\geq d_k, \qquad d(x_2,x_3)\geq d_k, \qquad d(x_3,x_4)\geq d_k.$$

Then the number of verticles of S between x_4 and x_1 is at most $12k^2+4k$.

Proof. By Lemma 2.1, each vertex between x_4 and x_1 is connected in $G(S,k)$ to at least one of the x_i's. By Lemma 2.2, there are at most k vertices between x_4 and x_1 which are connected in $G(S,k)$ to a given x_i but no other x_j. On the other hand, all points which are connected to both x_i and x_j $(1\leq i<j\leq 4)$ lie on k circles about x_i as well as on k circles about x_j, so their number is at most $2k^2$. This gives the bound in the lemma. □

Lemma 2.4. *If $n > 12k^2 + 8k$ then $G(S, k)$ contains no convex quadrilateral.*

Proof. Almost the same as the proof of 2.3. □

Theorem 2.5. *If k is fixed and $n > n_1(k) = 25\,000k^2$ then $\chi(G(S, k) \leq 3$.*

Proof. Let $p = \lfloor n/720 \rfloor$. Then $p > 24k^2 + 8k + 2$ (except in the trivial case when $k = 1$). We can choose $2p + 1$ consecutive vertices $a_0, \ldots, a_{2p}$ such that the angle between the vectors a_0a_1 and $a_{2p-1}a_{2p}$ is less than 1°. Now we do the coloring the greedy way. We start at the point $t_1 = a_p$. We give the color 1 to the points in S going counterclockwise as long as possible, i.e., until we encounter a vertex t_2 which is connected in $G(S, k)$ to a vertex t_1' already colored with color 1. Now starting at t_2 go on using color 2 until it is impossible, i.e., until we encounter a vertex t_3 connected to a vertex t_2' already colored with color 2. Going on with color 3, we either complete a 3-coloring of G, or else we find, similarly as before, vertices t_4 and t_3' connected in $G(S, k)$. Now we show that we can choose $x_1 = t_1'$, $x_2 \in \{t_2, t_2'\}$, $x_3 \in \{t_3, t_3'\}$, and $x_4 = t_4$ so that $d(x_1, x_2) \geq d_k$, $d(x_2, x_3) \geq d_k$, and $d(x_3, x_4) \geq d_k$. If $t_2 = t_2'$ and $t_3 = t_3'$ then this is obvious.

Assume that $t_2 \neq t_2'$. Now in the convex quadrangle $t_1't_2't_2t_3$ the sum of the lengths of the opposite edges (t_1', t_2') and (t_2, t_3) is at least $2d_k$, so at least one diagonal must be of length at least d_k. We choose x_2 accordingly, and similarly we choose x_3.

So we have the same kind of configuration as in Lemma 2.3. Thus by Lemma 2.3 there are at most $12k^2 + 4k$ vertices between x_1 and x_4. This in particular implies that $x_1 = a_i$ and $x_4 = a_j$ where

$$p - 12k^2 - 4k \leq i \leq p < j \leq p + 12k^2 + 4k + 1.$$

One of the pairs (x_1, x_3) and (x_2, x_4), say the former, is also connected in $G(S, k)$.

Now the angle $x_2x_1a_{i+1}$ cannot be larger than 91°, or else the segments x_2a_{i+1}, $x_2a_{i+2}, \ldots, x_2a_{i+k}$ are monotone increasing and all greater than d_k, which is impossible. Similarly, the angle $a_{i-1}x_1x_3$ is less than 91° and hence the angle $x_2x_1x_3$ is less than 2°. Let, e.g., $d(x_1, x_2) > d(x_1, x_3)$. Hence it is easy to deduce using the cosine theorem that $d(x_1, x_2) \geq 1.9d_k$. Hence

$$d(a_{2p}, x_2) \geq \sin(x_2x_1a_{2p})d(x_1, x_2) \geq (\sin 88°)(1.9d_k) \geq 1.8d_k.$$

But then relabeling a_{2p} by x_1 we get a contradiction at Lemma 2.3. □

Again, one can ask if the threshold const $\cdot k^2$ is best possible. The source of this value is Lemma 2.3, where we use that two families of k concentric circles cannot have more that $O(k^2)$ points of intersection. It may seem that the additional information that the points considered are vertices of a convex polygon would exclude most of the intersection points. But this is not the case; we can construct a set S, consisting of the vertices of a convex polygon, such that $|S| >$ const $\cdot k^2$ and $G(S, k)$ contains a K_4 (and hence its chromatic number is larger than 3). In particular, two families of k concentric circles will have const $\cdot k^2$ points of intersection among the vertices of the convex polygon.

Let us sketch this construction. Let $a=(0,0)$, $b=(1,0)$, $c=(3,0)$, and $d=(-1,0)$. Let C_0 be the circle with radius 2 about b, and let p_0 be a point on C_0 very close to c. Then the angle dp_0c is 90°, hence the angle ap_0c is acute. Hence we can choose an interior point p_1 on the arc of C_0 between p_0 and c such that the angle ap_0p_1 is acute. We define the points $p_2, \ldots, p_{k-1}$ on the circle C_0 similarly so that all the angles ap_ip_{i+1} are acute. Let D_i be the circle with center a through p_i. It follows from the construction that the circle D_i does not contain p_{i+1} in its interior but the line tangent to D_i at p_i does not separate p_{i+1} from a.

Let ε be a very small positive number and let C_i $(i=0, \ldots, k-1)$ be the circle about b with radius $2-i\varepsilon$. Let p_{ij} be the intersection point of C_i and D_j in the upper half-plane. Then the points p_{ij}, a, and b form the vertices of a convex polygon and a, b, $p_{0,0}$, and $p_{k-1,k-1}$ form a complete quadrilateral in $G(S, 2k+2)$.

Next we derive a bound on the chromatic number of $G(S, k)$ without the hypothesis that $|S|$ is large. First, let us define the following. Let xy be an edge of $G(S, k)$. Let x_1 be the clockwise neighbor of x and y_1 be the counterclockwise neighbor of y. If $d(x_1, y) > d(x, y)$ we say that the edge x_1y *covers* the edge xy. Similarly, if $d(x, y_1) > d(x, y)$ we say that the edge xy_1 *covers* the edge xy. Starting from any edge xy, let us select an edge $x'y'$ covering it, then an edge $x''y''$ covering $x'y'$, etc. In at most $k-1$ steps we must get stuck (by the definition of $G(S, k)$). Let x_0y_0 be the edge for which we could not find any edge covering it. We call x_0y_0 a *majorant* of xy. Note that in this case the angles formed by x_0y_0 and the two edges of the polygon entering x_0 and y_0 from the side opposite to xy must be acute. It is also clear that the arcs x_0x and yy_0 contain at most $k-1$ sides of P together.

The following proposition will not be used directly, but it seems worth formulating.

Proposition 2.6. *Let x_1, x_2, x_3, and x_4 be four vertices of P (in this cyclic order) and assume that (x_1, x_2) and (x_3, x_4) are two edges of $G(S, k)$. Then either between x_2 and x_3 or between x_4 and x_1 are no more than $2k-2$ sides of P (see Fig. 3).*

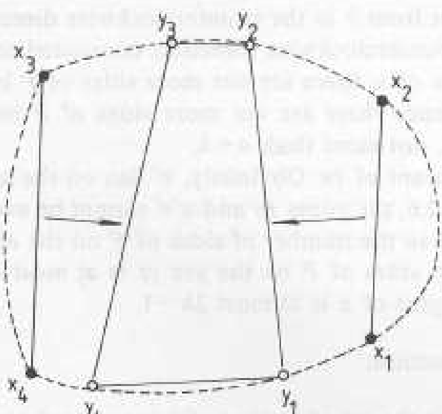

Fig. 3

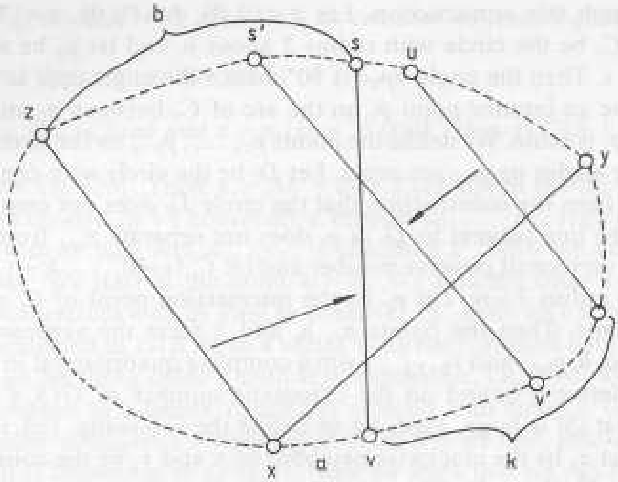

Fig. 4

Proof. Assume that the conclusion does not hold, and let y_1y_2 be a majorant of x_1x_2 and y_3y_4 be a majorant of x_3x_4. Then these majorants are also noncrossing and y_1, y_2, y_3, and y_4 appear in this cyclic order on the polygon. Moreover, from the above remarks it follows that all angles of the convex quadrangle $y_1y_2y_3y_4$ are acute. This is clearly impossible. □

Theorem 2.7. *If S is the set of vertices of a convex polygon then the graph $G(S, k)$ has a point of degree at most $3k-1$.*

Proof. Choose $x \in S$ and let y and z be the first vertices of S in the counterclockwise and clockwise directions, respectively, that are connected to x. Choose x so that the number of points between x and y is maximal (see Fig. 4).

Let sv be a majorant of zx. (It is possible that $v = x$ or $s = z$). Suppose there are a points between x and v and b points between s and z, then $a + b \leq k - 1$. Let t be the kth point from x in the counterclockwise direction, and let u be the first vertex in the counterclockwise direction connected to t in $G(S, k)$. Then because of the choice of x, there are not more sides of P between t and u than between x and y. Hence there are not more sides of P between y and u than between x and t, i.e., not more than $a + k$.

Let $v's'$ be a majorant of tu. Obviously, v' lies on the arc vt. Just like in the proof of Proposition 2.6, the edges sv and $v's'$ cannot be avoiding. Hence s must be on the arc us' and so the number of sides of P on the arc us is at most $k-1$. Hence the number of sides of P on the arc yz is at most $(a+k)+(k-1)+b \leq 3k-2$. Hence the degree of x is at most $3k-1$. □

We obtain by induction:

Corollary 2.8. *If S is the set of vertices of a convex polygon, then the number of edges in $G(S, k)$ is at most $(3k-1)n$.* □

Moreover, we obtain from Theorem 2.7 by deleting a vertex with minimum degree and using induction:

Corollary 2.9. *If S is the set of vertices of a convex polygon then the chromatic number of $G(S, k)$ is at most $3k$.* □

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